CDS 101: Lecture 2.1
System Modeling

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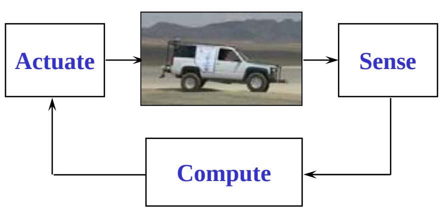
Goals:

- Define what a model is and its use in answering questions about a system
- Introduce the concepts of state, dynamics, inputs and outputs
- Provide examples of common modeling techniques: differential equations, difference equations, finite state automata

Reading:

- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 2
- Advanced: Lewis, *A Mathematical Approach to Classical Control*, Ch 1

Review from last week



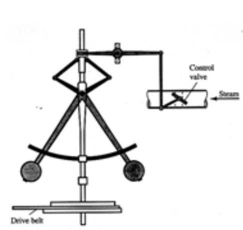
Control =

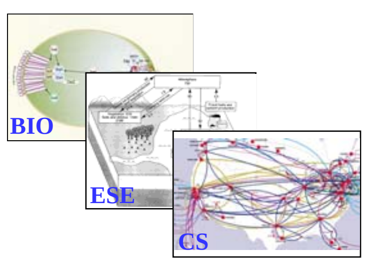
Sensing + Computation + Actuation

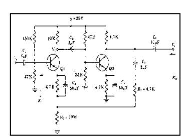
Feedback Principles

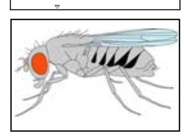
- Robustness to Uncertainty
- Design of Dynamics

Many examples of feedback and control in natural & engineered systems:









Model-Based Analysis of Feedback Systems

Analysis and design based on *models*

- A model provides a *prediction* of how the system will behave
- Feedback can give counter-intuitive behavior; models help sort out what is going on
- For control design, models don't have to be exact: *feedback* provides robustness

Control-oriented models: *inputs* and *outputs*

The model you use depends on the questions you want to answer

- A single system may have many models
- Time and spatial scale must be chosen to suit the questions you want to answer
- Formulate questions *before* building a model

Weather Forecasting

- **Question 1: how much will it rain tomorrow?**
- **Question 2: will it rain in the next 5-10 days?**
- **Question 3: will we have a drought next summer?**

Different questions ⇒ different models

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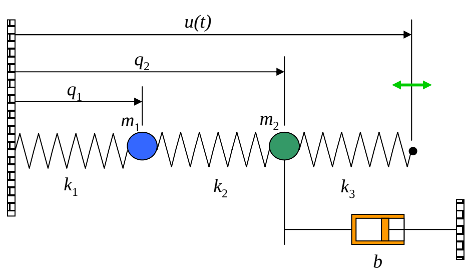
Example #1: Spring Mass System



Applications

- Flexible structures (many apps)
- Suspension systems (eg, "Bob")
- Molecular and quantum dynamics

Questions we want to answer

- How much do masses move as a function of the forcing frequency?
- What happens if I change the values of the masses?
- Will Bob fly into the air if I take that hill at 25 mph?

Modeling assumptions

- Mass, spring, and damper constants are fixed and known
- Springs satisfy Hooke's law
- Damper is (linear) viscous force, proportional to velocity

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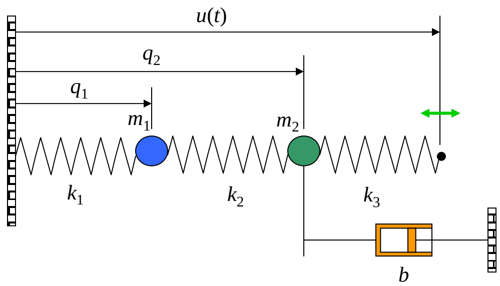
Modeling a Spring Mass System



Model: rigid body physics (Ph 1)

- Sum of forces = mass ∗ acceleration
- Hooke's law: $F = k(x - x_{rest})$
- Viscous friction: $F = b\,v$

$$\begin{aligned} m_1\ddot{q}_1 &= k_2(q_2 - q_1) - k_1 q_1 \\ m_2\ddot{q}_2 &= k_3(u - q_2) - k_2(q_2 - q_1) - b\dot{q}_2 \end{aligned}$$

Converting models to state space form

- Construct a *vector* of the variables that are required to specify the evolution of the system
- Write dynamics as a *system* of first order differential equations:

$$\frac{dx}{dt} = f(x,u) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^p$$

$$y = h(x) \qquad y \in \mathbb{R}^q$$

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m} q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{b}{m}\dot{q}_2 \end{bmatrix}$$

$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

"State space form"

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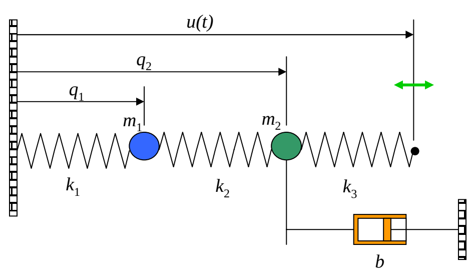
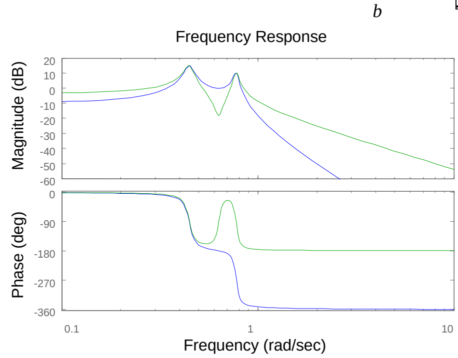
Frequency Response for a Mass Spring System





Steady state frequency response

- Force the system with a sinusoid
- Plot the "steady state" response, after transients have died out
- Plot relative magnitude and phase of output versus input (more later)

Matlab simulation (see handout)

```
function dydt = f(t, y, ...)
u = 0.00315*cos(omega*t);
dydt = [
  y(3);
  y(4);
  -(k1+k2)/m1*y(1) + k2/m1*y(2);
  k2/m2*y(1) - (k2+k3)/m2*y(2)
      - b/m2*y(4) + k3/m2*u ];

t,y] = ode45(dydt,tspan,y0,[], k1,
k2, k3, m1, m2, b, omega);
```

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Modeling Terminology

***State* captures effects of the past**

- independent physical quantities that determines future evolution (absent external excitation)

***Inputs* describe external excitation**

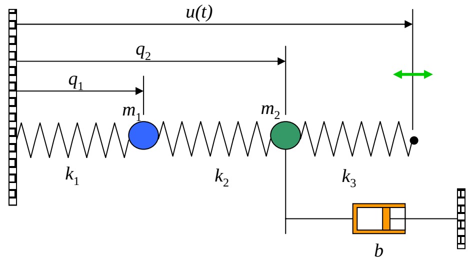
- Inputs are *extrinsic* to the system dynamics (externally specified)

***Dynamics* describes state evolution**

- update rule for system state
- function of current state and any external inputs

***Outputs* describe measured quantities**

- Outputs are function of state and inputs $\Rightarrow$ not independent variables
- Outputs are often *subset* of state



Example: spring mass system

- State: position and velocities of each mass: $q_1, q_2, \dot{q}_1, \dot{q}_2$
- Input: position of spring at right end of chain: $u(t)$
- Dynamics: basic mechanics
- Output: measured positions of the masses: q_1, q_2

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Modeling Properties

Choice of state is not unique

- There may be *many* choices of variables that can act as the state
- Trivial example: different choices of units (scaling factor)
- Less trivial example: sums and differences of the mass positions (HW 2.4)

Choice of inputs and outputs depends on point of view

- Inputs: what factors are *external* to the model that you are building
  - Inputs in one model might be outputs of another model (eg, the output of a cruise controller provides the input to the vehicle model)
- Outputs: what physical variables (often states) can you *measure*
  - Choice of outputs depends on what you can sense and what parts of the component model interact with other component models

Can also have different *types* of models

- Ordinary differential equations for rigid body mechanics
- Finite state machines for manufacturing, Internet, information flow
- Partial differential equations for fluid flow, solid mechanics, etc

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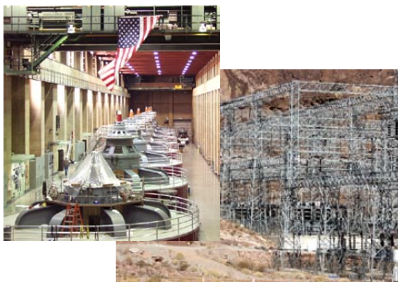
Differential Equations

Differential equations model continuous evolution of state variables

- Describe the rate of change of the state variables
- Both state and time are continuous variables

$$\frac{dx}{dt} = f(x,u)$$
$$y = h(x)$$

Example: electrical power grid



Swing equations

$$\ddot{\delta}_1 + D_1\dot{\delta}_1 = \omega_0\left(P_1 - B\sin(\delta_1-\delta_2) + G\cos(\delta_1-\delta_2)\right)$$
$$\ddot{\delta}_2 + D_1\dot{\delta}_2 = \omega_0\left(P_2 - B\sin(\delta_1-\delta_2) + G\cos(\delta_1-\delta_2)\right)$$

- Describe how generator rotor angles (δ_i) interact through the transmission line (G, B)
- Stability of these equations determines how loads on the grid are accommodated

State: rotor angles, velocities ($\delta_i, \dot{\delta}_i$)
Inputs: power loading on the grid (P_i)
Outputs: voltage levels and frequency (based on rotor speed)

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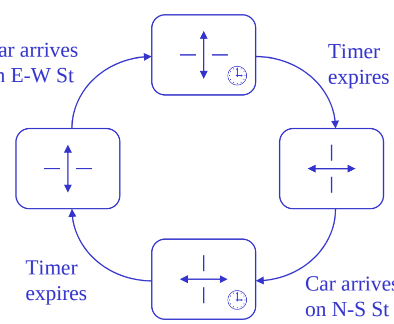
Finite State Machines

Finite state machines model discrete transitions between finite # of states

- Represent each configuration of system as a state
- Model transition between states using a graph
- Inputs force transition between states

Example: Traffic light logic



State: current pattern of lights that are on + internal timers
Inputs: presence of car at intersections
Outputs: current pattern of lights that are on

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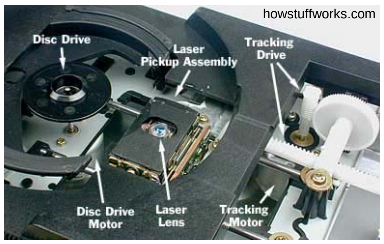
Difference Equations

Difference eqs model discrete transitions between continuous variables

- "Discrete time" description (clocked transitions)
- New state is function of current state + inputs
- State is represented as a *continuous* variable

$$x_{k+1} = f(x_k, u_k)$$
$$y_{k+1} = h(x_{k+1})$$

Example: CD read/write head *controller* (implemented on DSP)



State: estimated center, wobble
Inputs: read head signal
Outputs: commanded motion

Controller operation (every 1/44,100 sec)

- Get analog signal from read head
- Determine the data (1/0) plus estimate the location of the track center
- Update estimate of "wobble"
- Compute where to position disk head for next read (limited by motor torque)

Performance specification

- Keep disk head on track center
- Reject disturbances due to disk shape, shaking and bumps, etc

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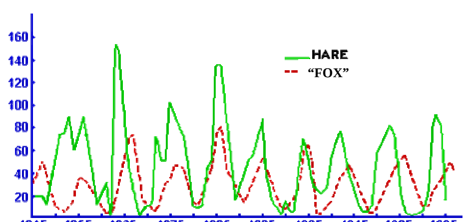
Example #2: Predator Prey



http://www.math.duke.edu/education/ccp/materials/diffeq/predprey/contents.html

Questions we want to answer

- Given the current population of rabbits and foxes, what will it be next year?
- If we hunt down lots of foxes in a given year, what will the effect on the rabbit and fox population be?
- How do long term changes in the amount of rabbit food available affect the populations?

Modeling assumptions

- The predator species is totally dependent on the prey species as its only food supply.
- The prey species has an external food supply and no threat to its growth other than the specific predator.

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Example #2: Predator Prey (2/2)

Discrete Lotka-Volterra model

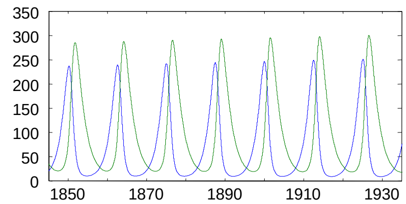
- State
  - R_k # of rabbits in period k
  - F_k # of foxes in period k
- Inputs (optional)
  - u_k amount of rabbit food
- Outputs: # of rabbits and foxes
- Dynamics: Lotka-Volterra eqs

$$R_{k+1} = R_k + b_r(u)R_k - aF_k R_k$$
$$F_{k+1} = F_k - d_f F_k + aF_k R_k$$

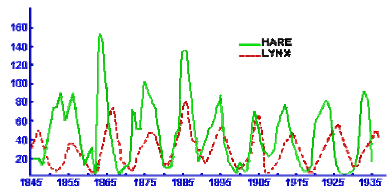
- Parameters/functions
  - $b_r(u)$ rabbit birth rate (per year) (depends on food supply)
  - d_f fox death rate (per year)
  - a interaction term

Matlab simulation (see handout)

- Discrete time model, "simulated" through repeated addition



Comparison with data



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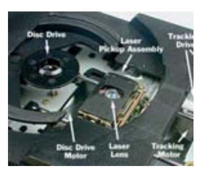
Summary: System Modeling

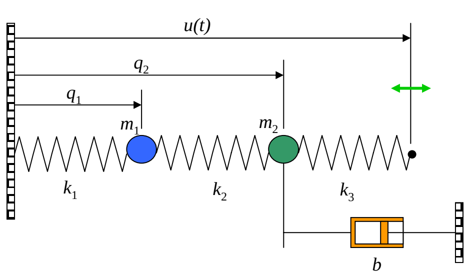
Model = state, inputs, outputs, dynamics

$$\frac{dx}{dt} = f(x,u)$$
$$y = h(x)$$



$$x_{k+1} = f(x_k, u_k)$$
$$y_{k+1} = h(x_{k+1})$$

***Principle*: Choice of model depends on the questions you want to answer**

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  k2/m2*y(1) - (k2+k3)/m2*y(2)
      - b/m2*y(4) + k3/m2*u ];
```

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