Chapter 5

State and Output Feedback

This chapter describes how feedback can be used shape the local behavior of a system. Both state and output feedback are discussed. The concepts of reachability and observability are introduced and it is shown how states can be estimated from measurements of the input and the output.

5.1 Introduction

The idea of using feedback to shape the dynamic behavior was discussed in broad terms in Section 1.4. In this chapter we will discuss this in detail for linear systems. In particular it will be shown that under certain conditions it is possible to assign the system eigenvalues to arbitrary values by feedback, allowing us to "design" the dynamics of the system.

The state of a dynamical system is a collection of variables that permits prediction of the future development of a system. In this chapter we will explore the idea of controlling a system through feedback of the state. We will assume that the system to be controlled is described by a linear state model and has a single input (for simplicity). The feedback control will be developed step by step using one single idea: the positioning of closed loop eigenvalues in desired locations. It turns out that the controller has a very interesting structure that applies to many design methods. This chapter may therefore be viewed as a prototype of many analytical design methods.

If the state of a system is not available for direct measurement, it is often possible to determine the state by reasoning about the state through our knowledge of the dynamics and more limited measurements. This is done by building an "observer" that uses measurements of the inputs and outputs of a linear system, along with a model of the system dynamics, to estimate the state.

The details of the analysis and designs in this chapter are carried out for systems with one input and one output, but it turns out that the structure of the controller and the forms of the equations are exactly the same for systems with many inputs and many outputs. There are also many other design techniques that give controllers with the same structure. A characteristic feature of a controller with state feedback and an observer is that the complexity of the controller is given by the complexity of the system to be controlled. Thus the controller actually contains a model of the system. This is an example of *the internal model principle* which says that a controller should have an internal model of the controlled system.

5.2 Reachability

We begin by disregarding the output measurements and focus on the evolution of the state which is given by

$$\frac{dx}{dt} = Ax + Bu, \tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, A is an $n \times n$ matrix and B an $n \times 1$ matrix. A fundamental question is if it is possible to find control signals so that any point in the state space can be reached.

First observe that possible equilibria for constant controls are given by

$$Ax + bu_0 = 0$$

This means that possible equilibria lies in a one (or possibly higher) dimensional subspace. If the matrix A is invertible this subspace is spanned by $A^{-1}B$.

Even if possible equilibria lie in a one dimensional subspace it may still be possible to reach all points in the state space transiently. To explore this we will first give a heuristic argument based on formal calculations with impulse functions. When the initial state is zero the response of the state to a unit step in the input is given by

$$x(t) = \int_0^t e^{A(t-\tau)} B d\tau. \tag{5.2}$$

The derivative of a unit step function is the impulse function $\delta(t)$, which may be regarded as a function which is zero everywhere except at the origin and with the property that

$$\int_{\infty}^{\infty} \delta(t)dt = 1.$$

The response of the system to a impulse function is thus the derivative of (5.2)

$$\frac{dx}{dt} = e^{At}B.$$

Similarly we find that the response to the derivative of a impulse function is

$$\frac{d^2x}{dt^2} = Ae^{At}B.$$

The input

$$u(t) = \alpha_1\delta(t) + \alpha_2\dot{\delta}(t) + \alpha\ddot{\delta}(t) + \cdots + \alpha_n\delta^{(n-1)}(t)$$

thus gives the state

$$x(t) = \alpha_1 e^{At}B + \alpha_2 Ae^{At}B + \alpha_3 A^2 e^{At}B + \cdots + \alpha_n A^{n-1}e^{At}B.$$

Hence, right after the initial time $t = 0$, denoted $t = 0+$, we have

$$x(0+) = \alpha_1 B + \alpha_2 AB + \alpha_3 A^2 B + \cdots + \alpha_n A^{n-1}B$$

The right hand is a linear combination of the columns of the matrix

$$W_r = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}. \tag{5.3}$$

To reach an arbitrary point in the state space we thus require that there are n linear independent columns of the matrix W_c. The matrix is called the *reachability matrix.*

An input consisting of a sum of impulse functions and their derivatives is a very violent signal. To see that an arbitrary point can be reached with smoother signals we can also argue as follow. Assuming that the initial condition is zero, the state of a linear system is given by

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t e^{A\tau}Bu(t-\tau)d\tau.$$

It follows from the theory of matrix functions that

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \ldots + A^{n-1}\alpha_{n-1}(\tau)$$

and we find that

$$x(t) = B\int_0^t \alpha_0(\tau)u(t-\tau)\,d\tau + AB\int_0^t \alpha_1(\tau)u(t-\tau)\,d\tau + \\ \dots + A^{n-1}B\int_0^t \alpha_{n-1}(\tau)u(t-\tau)\,d\tau.$$

Again we observe that the right hand side is a linear combination of the columns of the reachability matrix W_r given by (5.3).

We illustrate by two examples.

Example 5.1 *(Reachability of the Inverted Pendulum).* Consider the inverted pendulum example introduced in Example 3.5. The nonlinear equations of motion are given in equation (3.5)

$$\begin{aligned}\frac{dx}{dt} &= \begin{bmatrix} x_2 \\ \sin x_1 + u\cos x_1 \end{bmatrix}\\ y &= x_1.\end{aligned}$$

Linearizing this system about $x = 0$, the linearized model becomes

$$\begin{aligned}\frac{dx}{dt} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x.\end{aligned} \tag{5.4}$$

The dynamics matrix and the control matrix are

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The reachability matrix is

$$W_r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5.5}$$

This matrix has full rank and we can conclude that the system is reachable. This implies that we can move the system from any initial state to any final state and, in particular, that we can always find an input to bring the system from an initial state to the equilibrium.

Example 5.2 *(System in Reachable Canonical Form).* Next we will consider a system by in *reachable canonical form*:

$$\frac{dz}{dt} = \begin{bmatrix} -a_1 & -a_2 & \dots & a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u = \tilde{A}z + \tilde{B}u$$

To show that W_r is full rank, we show that the inverse of the reachability matrix exists and is given by

$$\tilde{W}_r^{-1} = \begin{bmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 0 & 1 & a_1 & \dots & a_{n-1} \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{5.6}$$

To show this we consider the product

$$\begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}B \end{bmatrix} W_r^{-1} = \begin{bmatrix} w_0 & w_1 & \cdots & w_{n-1} \end{bmatrix}$$

where

$$\begin{aligned} w_0 &= \tilde{B} \\ w_1 &= a_1\tilde{B} + \tilde{A}\tilde{B} \\ &\vdots \\ w_{n-1} &= a_{n-1}B + a_{n-2}\tilde{A}B + \cdots + \tilde{A}^{n-1}B. \end{aligned}$$

The vectors w_k satisfy the relation

$$w_k = a_k + \tilde{w}_{k-1}$$

and iterating this relation we find that

$$\begin{bmatrix} w_0 & w_1 & \cdots & w_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

which shows that the matrix (5.6) is indeed the inverse of $\tilde{W}_r$.

Systems That Are Not Reachable

It is useful of have an intuitive understanding of the mechanisms that make a system unreachable. An example of such a system is given in Figure 5.1. The system consists of two identical systems with the same input. The intuition can also be demonstrated analytically. We demonstrate this by a simple example.

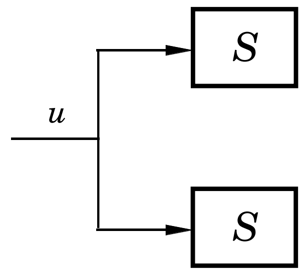

Figure 5.1: A non-reachable system.

Example 5.3 *(Non-reachable System).* Assume that the systems in Figure 5.1 are of first order. The complete system is then described by

$$\frac{dx_1}{dt} = -x_1 + u$$
$$\frac{dx_2}{dt} = -x_2 + u$$

The reachability matrix is

$$W_r = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

This matrix is singular and the system is not reachable. One implication of this is that if x_1 and x_2 start with the same value, it is never possible to find an input which causes them to have different values. Similarly, if they start with different values, no input will be able to drive them both to zero.

Coordinate Changes

It is interesting to investigate how the reachability matrix transforms when the coordinates are changed. Consider the system in (5.1). Assume that the coordinates are changed to $z = Tx$. As shown in the last chapter, that the dynamics matrix and the control matrix for the transformed system are

$$\tilde{A} = TAT^{-1}$$
$$\tilde{B} = TB$$

The reachability matrix for the transformed system then becomes

$$\tilde{W}_r = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} =$$

We have

$$\begin{aligned}
\tilde{A}\tilde{B} &= TAT^{-1}TB = TAB \\
\tilde{A}^2\tilde{B} &= (TAT^{-1})^2TB = TAT^{-1}TAT^{-1}TB = TA^2B \\
&\vdots \\
\tilde{A}^n\tilde{B} &= TA^nB.
\end{aligned}$$

The reachability matrix for the transformed system is thus

$$\tilde{W}_r = \begin{bmatrix}\tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B}\end{bmatrix} = T\begin{bmatrix}B & AB & \dots & A^{n-1}B\end{bmatrix} = TW_r \qquad (5.7)$$

This formula is useful for finding the transformation matrix T that converts a system into reachable canonical form (using $\tilde{W}_r$ from Example 5.2).

5.3 State Feedback

Consider a system described by the linear differential equation

$$\begin{aligned}
\frac{dx}{dt} &= Ax + Bu \\
y &= Cx
\end{aligned} \qquad (5.8)$$

The output is the variable that we are interested in controlling. To begin with it is assumed that all components of the state vector are measured. Since the state at time t contains all information necessary to predict the future behavior of the system, the most general time invariant control law is function of the state, i.e.

$$u(t) = f(x(t))$$

If the feedback is restricted to be a linear, it can be written as

$$u = -Kx + K_r r \qquad (5.9)$$

where r is the reference value. The negative sign is simply a convention to indicate that negative feedback is the normal situation. The closed loop system obtained when the feedback (5.8) is applied to the system (5.9) is given by

$$\frac{dx}{dt} = (A - BK)x + BK_r r \qquad (5.10)$$

It will be attempted to determine the feedback gain K so that the closed loop system has the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n \tag{5.11}$$

This control problem is called the eigenvalue assignment problem or the pole placement problem (we will define "poles" more formally in a later chapter).

Examples

We will start by considering a few examples that give insight into the nature of the problem.

Example 5.4 *(The Double Integrator).* The double integrator is described by

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

Introducing the feedback

$$u = -k_1 x_1 - k_2 x_2 + K_r r$$

the closed loop system becomes

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ K_r \end{bmatrix} r \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} \tag{5.12}$$

The closed loop system has the characteristic polynomial

$$\det \begin{bmatrix} s & -1 \\ k_1 & s + k_2 \end{bmatrix} = s^2 + k_2 s + k_1$$

Assume it is desired to have a feedback that gives a closed loop system with the characteristic polynomial

$$p(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2$$

Comparing this with the characteristic polynomial of the closed loop system we find find that the feedback gains should be chosen as

$$k_1 = \omega_0^2, \quad k_2 = 2\zeta\omega_0$$

To have unit steady state gain the parameter K_r must be equal to $k_1 = \omega_0^2$. The control law can thus be written as

$$u = k_1(r - x_1) - k_2 x_2 = \omega_0^2(r - x_1) - 2\zeta_0\omega_0 x_2$$

In the next example we will encounter some difficulties.

Example 5.5 *(An Unreachable System).* Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + K_r r$$

The closed loop system is

$$\frac{dx}{dt} = \begin{bmatrix} -k_1 & 1 - k_2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} K_r \\ 0 \end{bmatrix} r$$

This system has the characteristic polynomial

$$\det \begin{bmatrix} s + k_1 & -1 + k_2 \\ 0 & s \end{bmatrix} = s^2 + k_1 s = s(s + k_1)$$

This polynomial has zeros at $s = 0$ and $s = -k_1$. One closed loop eigenvalue is thus always equal to $s = 0$ and it is not possible to obtain an arbitrary characteristic polynomial.

This example shows that the eigenvalue placement problem cannot always be solved. An analysis of the equation describing the system shows that the state x_2 is not reachable. It is thus clear that some conditions on the system are required.

The reachable canonical form has the property that the parameters of the system are the coefficients of the characteristic equation. It is therefore natural to consider systems on this form when solving the eigenvalue placement problem. In the next example we investigate the case when the system is in reachable canonical form.

Example 5.6 *(System in Reachable Canonical Form).* Consider a system in reachable canonical form, i.e,

$$\frac{dz}{dt} = \tilde{A}z + \tilde{B}u = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \tag{5.13}$$
$$y = \tilde{C}z = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} z.$$

The open loop system has the characteristic polynomial

$$D_n(s) = \det \begin{bmatrix} s + a_1 & a_2 & \dots & a_{n-1} & a_n \\ -1 & s & & 0 & 0 \\ 0 & -1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & -1 & s \end{bmatrix}.$$

Expanding the determinant by the last row we find that the following recursive equation for the determinant:

$$D_n(s) = sD_{n-1}(s) + a_n.$$

It follows from this equation that

$$D_n(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n$$

A useful property of the system described by (5.13) is thus that the coefficients of the characteristic polynomial appear in the first row. Since the all elements of the B-matrix except the first row are zero it follows that the state feedback only changes the first row of the A-matrix. It is thus straight forward to see how the closed loop eigenvalues are changed by the feedback. Introduce the control law

$$u = -\tilde{K}z + K_r r = -\tilde{k}_1 z_1 - \tilde{k}_2 z_2 - \dots - \tilde{k}_n z_n + K_r r \tag{5.14}$$

The closed loop system then becomes

$$\begin{aligned} \frac{dz}{dt} &= \begin{bmatrix} -a_1 - \tilde{k}_1 & -a_2 - \tilde{k}_2 & \dots & -a_{n-1} - \tilde{k}_{n-1} & -a_n - \tilde{k}_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{bmatrix} z + \begin{bmatrix} K_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} r \\ y &= \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} z \end{aligned} \tag{5.15}$$

The feedback thus changes the elements of the first row of the A matrix, which corresponds to the parameters of the characteristic equation. The closed loop system thus has the characteristic polynomial

$$s^n + (a_l + \tilde{k}_1)s^{n-1} + (a_2 + \tilde{k}_2)s^{n-2} + \dots + (a_{n-1} + \tilde{k}_{n-1})s + a_n + \tilde{k}_n$$

Requiring this polynomial to be equal to the desired closed loop polynomial (5.11) we find that the controller gains should be chosen as

$$\begin{aligned}
\tilde{k}_1 &= p_1 - a_1 \\
\tilde{k}_2 &= p_2 - a_2 \\
&\vdots \\
\tilde{k}_n &= p_n - a_n
\end{aligned}$$

This feedback simply replace the parameters a_i in the system (5.15) by p_i. The feedback gain for a system in reachable canonical form is thus

$$\tilde{K} = \begin{bmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{bmatrix} \tag{5.16}$$

To have unit steady state gain the parameter K_r should be chosen as

$$K_r = \frac{a_n + \tilde{k}_n}{b_n} = \frac{p_n}{b_n} \tag{5.17}$$

Notice that it is essential to know the precise values of parameters a_n and b_n in order to obtain the correct steady state gain. The steady state gain is thus obtained by precise calibration. This is very different from obtaining the correct steady state value by integral action, which we shall see in later chapters. We thus find that it is easy to solve the eigenvalue placement problem when the system has the structure given by (5.13).

The General Case

To solve the problem in the general case, we simply change coordinates so that the system is in reachable canonical form. Consider the system (5.8). Change the coordinates by a linear transformation

$$z = Tx$$

so that the transformed system is in reachable canonical form (5.13). For such a system the feedback is given by (5.14) where the coefficients are given by (5.16). Transforming back to the original coordinates gives the feedback

$$u = -\tilde{K}z + K_r r = -\tilde{K}Tx + K_r r$$

It now remains to find the transformation. To do this we observe that the reachability matrices have the property

$$\tilde{W}_r = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} = T \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = TW_r$$

The transformation matrix is thus given by

$$T = \tilde{W}_r W_r^{-1} \tag{5.18}$$

and the feedback gain can be written as

$$K = \tilde{K}T = \tilde{K}\tilde{W}_r W_r^{-1} \tag{5.19}$$

Notice that the matrix $\tilde{W}_r$ is given by (5.6). The feedforward gain K_r is given by equation (5.17).

The results obtained can be summarized as follows.

Theorem 5.1 *(Pole-placement by State Feedback). Consider the system given by equation* (5.8)

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx\end{aligned}$$

with one input and one output. If the system is reachable there exits a feedback

$$u = -Kx + K_r r$$

that gives a closed loop system with the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n.$$

The feedback gain is given by

$$\begin{aligned}K &= \tilde{K}T = \begin{bmatrix}p_1 - a_1 & p_2 - a_2 & \ldots & p_n - a_n\end{bmatrix}\tilde{W}_r W_r^{-1}\\ K_r &= \frac{p_n}{a_n}\end{aligned}$$

where a_i are the coefficients of the characteristic polynomial of the matrix A and the matrices W_r and $\tilde{W}_r$ are given by

$$\begin{aligned}W_r &= \begin{bmatrix}B & AB & \ldots & A^{n-1}\end{bmatrix}\\ \tilde{W}_r &= \begin{bmatrix}1 & a_1 & a_2 & \ldots & a_{n-1}\\ 0 & 1 & a_1 & \ldots & a_{n-2}\\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1\end{bmatrix}^{-1}\end{aligned}$$

Remark 5.1 *(A mathematical interpretation).* Notice that the eigenvalue placement problem can be formulated abstractly as the following algebraic problem. Given an $n \times n$ matrix A and an $n \times 1$ matrix B, find a $1 \times n$ matrix K such that the matrix $A - BK$ has prescribed eigenvalues.

Computing the Feedback Gain

We have thus obtained a solution to the problem and the feedback has been described by a closed form solution.

For simple problems it is easy to solve the problem by the following simple procedure: Introduce the elements k_i of K as unknown variables. Compute the characteristic polynomial

$$\det(sI - A + BK).$$

Equate coefficients of equal powers of s to the coefficients of the desired characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1} + p_n.$$

This gives a system of linear equations to determine k_i. The equations can always be solved if the system is observable. Example 5.4 is typical illustrations.

For systems of higher order it is more convenient to use equation (5.19), this can also be used for numeric computations. However, for large systems this is not sound numerically, because it involves computation of the characteristic polynomial of a matrix and computations of high powers of matrices. Both operations lead to loss of numerical accuracy. For this reason there are other methods that are better numerically. In MATLAB the state feedback can be computed by the procedures `acker` or `place`.

5.4 Observability

In Section 5.3 it was shown that it was possible to find a feedback that gives desired closed loop eigenvalues provided that the system is reachable and that all states were measured. It is highly unrealistic to assume that all states are measured. In this section we will investigate how the state can be estimated by using the mathematical model and a few measurements. It will be shown that the computation of the states can be done by dynamical systems. Such systems will be called *observers*.

Consider a system described by

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.20}$$

where x is the state, u the input, and y the measured output. The problem of determining the state of the system from its inputs and outputs will be considered. It will be assumed that there is only one measured signal, i.e. that the signal y is a scalar and that C is a (row) vector.

Observability

When discussing reachability we neglected the output and focused on the state. We will now discuss a related problem where we will neglect the input and instead focus on the output. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= Ax\\ y &= Cx\end{aligned} \tag{5.21}$$

We will now investigate if it is possible to determine the state from observations of the output. This is clearly a problem of significant practical interest, because it will tell if the sensors are sufficient.

The output itself gives the projection of the state on vectors that are rows of the matrix C. The problem can clearly be solved if the matrix C is invertible. If the matrix is not invertible we can take derivatives of the output to obtain

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAx.$$

From the derivative of the output we thus get the projections of the state on vectors which are rows of the matrix CA. Proceeding in this way we get

$$\begin{bmatrix} y \\ \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x \tag{5.22}$$

We thus find that the state can be determined if the matrix

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{5.23}$$

has n independent rows. Notice that because of the Cayley-Hamilton equation it is not worth while to continue and take derivatives of order higher

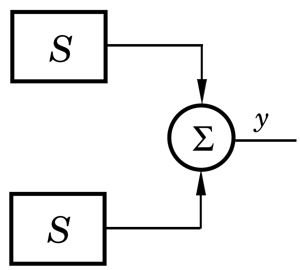

Figure 5.2: A non-observable system.

than $n-1$. The matrix W_o is called the observability matrix. A system is called observable if the observability matrix has full rank. We illustrate with an example.

Example 5.7 *(Observability of the Inverted Pendulum).* The linearized model of inverted pendulum around the upright position is described by (5.4). The matrices A and C are

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The observability matrix is

$$W_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has full rank. It is thus possible to compute the state from a measurement of the angle.

The calculation can easily be extended to systems with inputs. The state is then given by a linear combination of inputs and outputs and their higher derivatives. Differentiation can give very large errors when there is measurement noise and the method is therefore not very practical particularly when derivatives of high order appear. A method that works with inputs will be given the next section.

A Non-Observable System

It is useful to have an understanding of the mechanisms that make a system unobservable. Such a system is shown in Figure 5.2. Next we will consider the system in *observable canonical form*, i.e.

$$\begin{aligned}
\frac{dz}{dt} &= \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 \dots & 0 \end{bmatrix} z + Du
\end{aligned}$$

A straight forward but tedious calculation shows that the inverse of the observability matrix has a simple form. It is given by

$$W_o^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \vdots & & & & \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & 1 \end{bmatrix}$$

This matrix is always invertible. The system is composed of two identical systems whose outputs are added. It seems intuitively clear that it is not possible to deduce the states from the output. This can also be seen formally.

Coordinate Changes

It is interesting to investigate how the observability matrix transforms when the coordinates are changed. Consider the system in equation (5.21). Assume that the coordinates are changed to $z = Tx$. It follows from linear algebra that the dynamics matrix and the output matrix are given by

$$\begin{aligned}
\tilde{A} &= TAT^{-1} \\
\tilde{C} &= CT^{-1}.
\end{aligned}$$

The observability matrix for the transformed system then becomes

$$\tilde{W}_o = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix}$$

We have

$$\begin{aligned}
\tilde{C}\tilde{A} &= CT^{-1}TAT^{-1} = CAT^{-1}\\
\tilde{C}\tilde{A}^2 &= CT^{-1}(TAT^{-1})^2 = CT^{-1}TAT^{-1}TAT^{-1} = CA^2T^{-1}\\
&\vdots\\
\tilde{C}\tilde{A}^n &= CA^nT^{-1}
\end{aligned}$$

and we find that the observability matrix for the transformed system has the property

$$\tilde{W}_o = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} T^{-1} = W_o T^{-1} \tag{5.24}$$

This formula is very useful for finding the transformation matrix T.

5.5 Observers

For a system governed by equation (5.20), we can attempt to determine the state simply by simulating the equations with the correct input. An estimate of the state is then given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu \tag{5.25}$$

To find the properties of this estimate, introduce the estimation error

$$\tilde{x} = x - \hat{x}$$

It follows from (5.20) and (5.25) that

$$\frac{d\tilde{x}}{dt} = A\tilde{x}$$

If matrix A has all its eigenvalues in the left half plane, the error $\tilde{x}$ will thus go to zero. Equation (5.25) is thus a dynamical system whose output converges to the state of the system (5.20).

The observer given by (5.25) uses only the process input u; the measured signal does not appear in the equation. It must also be required that the system is stable. We will therefore attempt to modify the observer so that the output is used and that it will work for unstable systems. Consider the following observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{5.26}$$

This can be considered as a generalization of (5.25). Feedback from the measured output is provided by adding the term $L(y - C\hat{x})$. Notice that $C\hat{x} = \hat{y}$ is the output that is predicted by the observer. To investigate the observer (5.26), form the error

$$\tilde{x} = x - \hat{x}$$

It follows from (5.20) and (5.26) that

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}$$

If the matrix L can be chosen in such a way that the matrix $A - LC$ has eigenvalues with negative real parts, the error $\tilde{x}$ will go to zero. The convergence rate is determined by an appropriate selection of the eigenvalues.

The problem of determining the matrix L such that $A - LC$ has prescribed eigenvalues is very similar to the eigenvalue placement problem that was solved above. In fact, if we observe that the eigenvalues of the matrix and its transpose are the same, we find that could determine L such that $A^T - C^T L^T$ has given eigenvalues. First we notice that the problem can be solved if the matrix

$$\begin{bmatrix} C^T & A^T C^T & \dots & A^{(n-1)T} C^T \end{bmatrix}$$

is invertible. Notice that this matrix is the transpose of the observability matrix for the system (5.20).

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

of the system. Assume it is desired that the characteristic polynomial of the matrix $A - LC$ is

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_n$$

It follows from Remark 5.1 of Theorem 5.1 that the solution is given by

$$L^T = \begin{bmatrix} p_1 - a_1 & p_2 - a_2 & \dots & p_n - a_n \end{bmatrix} \tilde{W}_o^T W_o^{-T}$$

where W_o is the observability matrix and $\tilde{W}_o$ is the observability matrix of the system of the system

$$\begin{aligned}\frac{dz}{dt} &= \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}\end{aligned}$$

which is the observable canonical form of the system (5.20). Transposing the formula for L we obtain

$$L = W_o^{-1}\tilde{W}_o \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{bmatrix}$$

The result is summarized by the following theorem.

Theorem 5.2 *(Observer design by eigenvalue placement). Consider the system given by*

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx\end{aligned}$$

where output y is a scalar. Assume that the system is observable. The dynamical system

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

with L chosen as

$$L = W_o^{-1}\tilde{W}_o \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{bmatrix} \tag{5.27}$$

where the matrices W_o and $\tilde{W}_o$ are given by

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \ \tilde{W}_o^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \vdots & & & & \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & 1 \end{bmatrix}$$

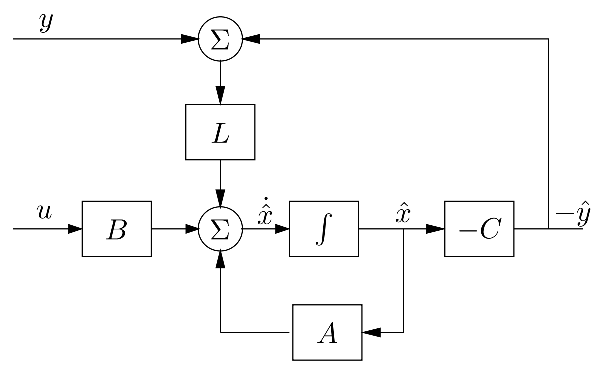

Figure 5.3: Block diagram of the observer. Notice that the observer contains a copy of the process.

Then the observer error $\tilde{x} = x - \hat{x}$ is governed by a differential equation having the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_n$$

Remark 5.2. The dynamical system (5.26) is called an observer for (the states of the) system (5.20) because it will generate an approximation of the states of the system from its inputs and outputs.

Remark 5.3. The theorem can be derived by transforming the system to observable canonical form and solving the problem for a system in this form.

Remark 5.4. Notice that we have given two observers, one based on pure differentiation (5.22) and another described by the differential equation (5.26). There are also other forms of observers.

Interpretation of the Observer

The observer is a dynamical system whose inputs are process input u and process output y. The rate of change of the estimate is composed of two terms. One term $A\hat{x} + Bu$ is the rate of change computed from the model with $\hat{x}$ substituted for x. The other term $L(y - \hat{y})$ is proportional to the difference $e = y - \hat{y}$ between measured output y and its estimate $\hat{y} = C\hat{x}$. The estimator gain L is a matrix that tells how the error e is weighted and distributed among the states. The observer thus combines measurements with a dynamical model of the system. A block diagram of the observer is shown in Figure 5.3

Duality

Notice the similarity between the problems of finding a state feedback and finding the observer. The key is that both of these problems are equivalent to the same algebraic problem. In eigenvalue placement it is attempted to find L so that $A - BL$ has given eigenvalues. For the observer design it is instead attempted to find L so that $A - LC$ has given eigenvalues. The following equivalence can be established between the problems

$$\begin{aligned} A &\leftrightarrow A^T \\ B &\leftrightarrow C^T \\ K &\leftrightarrow L^T \\ W_r &\leftrightarrow W_o^T \end{aligned}$$

The similarity between design of state feedback and observers also means that the same computer code can be used for both problems.

Computing the Observer Gain

The observer gain can be computed in several different ways. For simple problems it is convenient to introduce the elements of L as unknown parameters, determine the characteristic polynomial of the observer $\det(A - LC)$ and identify it with the desired characteristic polynomial. Another alternative is to use the fact that the observer gain can be obtained by inspection if the system is in observable canonical form. In the general case the observer gain is then obtained by transformation to the canonical form. There are also reliable numerical algorithms. They are identical to the algorithms for computing the state feedback. The procedures are illustrated by a few examples.

Example 5.8 *(The Double Integrator).* The double integrator is described by

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

The observability matrix is

$$W_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. the identity matrix. The system is thus observable and the problem can be solved. We have

$$A - LC = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}$$

It has the characteristic polynomial

$$\det A - LC = \det \begin{bmatrix} s + l_1 & -1 \\ -l_2 & s \end{bmatrix} = s^2 + l_1 s + l_2$$

Assume that it is desired to have an observer with the characteristic polynomial

$$s^2 + p_1 s + p_2 = s^2 + 2\zeta\omega s + \omega^2$$

The observer gains should be chosen as

$$\begin{aligned} l_1 &= p_1 = 2\zeta\omega \\ l_2 &= p_2 = \omega^2 \end{aligned}$$

The observer is then

$$\frac{d\hat{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (y - \hat{x}_1)$$

5.6 Output Feedback

In this section we will consider the same system as in the previous sections, i.e. the nth order system described by

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.28}$$

where only the output is measured. As before it will be assumed that u and y are scalars. It is also assumed that the system is reachable and observable. In Section 5.3 we had found a feedback

$$u = -Kx + K_r r$$

for the case that all states could be measured and in Section 5.4 we have presented developed an observer that can generate estimates of the state $\hat{x}$ based on inputs and outputs. In this section we will combine the ideas of these sections to find an feedback which gives desired closed loop eigenvalues for systems where only outputs are available for feedback.

If all states are not measurable, it seems reasonable to try the feedback

$$u = -K\hat{x} + K_r r \tag{5.29}$$

where $\hat{x}$ is the output of an observer of the state (5.26), i.e.

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{5.30}$$

Since the system (5.28) and the observer (5.30) both are of order n, the closed loop system is thus of order $2n$. The states of the system are x and $\hat{x}$. The evolution of the states is described by equations (5.28), (5.29)(5.30). To analyze the closed loop system, the state variable $\hat{x}$ is replace by

$$\tilde{x} = x - \hat{x} \tag{5.31}$$

Subtraction of (5.28) from (5.28) gives

$$\frac{d\tilde{x}}{dt} = Ax - A\hat{x} - L(y - C\hat{x}) = A\tilde{x} - LC\tilde{x} = (A - LC)\tilde{x}$$

Introducing u from (5.29) into this equation and using (5.31) to eliminate $\hat{x}$ gives

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu = Ax - BK\hat{x} + BK_r r = Ax - BK(x - \tilde{x}) + BK_r r\\ &= (A - BK)x + BK\tilde{x} + BK_r r\end{aligned}$$

The closed loop system is thus governed by

$$\frac{d}{dt}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} BK_r \\ 0 \end{bmatrix} r \tag{5.32}$$

Since the matrix on the right-hand side is block diagonal, we find that the characteristic polynomial of the closed loop system is

$$\det(sI - A + BK)\det(sI - A + LC)$$

This polynomial is a product of two terms, where the first is the characteristic polynomial of the closed loop system obtained with state feedback and the other is the characteristic polynomial of the observer error. The feedback (5.29) that was motivated heuristically thus provides a very neat solution to the eigenvalue placement problem. The result is summarized as follows.

Theorem 5.3 *(Pole placement by output feedback). Consider the system*

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx\end{aligned}$$

The controller described by

$$\begin{aligned}u &= -K\hat{x} + K_r r \\ \frac{d\hat{x}}{dt} &= A\hat{x} + Bu + L(y - C\hat{x})\end{aligned}$$

gives a closed loop system with the characteristic polynomial

$$\det(sI - A + BK)\det(sI - A + LC)$$

This polynomial can be assigned arbitrary roots if the system is observable and reachable.

Remark 5.5. Notice that the characteristic polynomial is of order $2n$ and that it can naturally be separated into two factors, one $\det(sI - A + BK)$ associated with the state feedback and the other $\det(sI - A + LC)$ with the observer.

Remark 5.6. The controller has a strong intuitive appeal. It can be thought of as composed of two parts, one state feedback and one observer. The feedback gain K can be computed as if all state variables can be measured.

The Internal Model Principle

A block diagram of the controller is shown in Figure 5.4. Notice that the controller contains a dynamical model of the plant. This is called the internal model principle. Notice that the dynamics of the controller is due to the observer. The controller can be viewed as a dynamical system with input y and output u.

$$\begin{aligned}\frac{d\hat{x}}{dt} &= (A - BK - LC)\hat{x} + Ly \\ u &= -K\hat{x} + K_r r\end{aligned} \tag{5.33}$$

The controller has the transfer function

$$C(s) = K[sI - A + BK + LC]^{-1}L \tag{5.34}$$

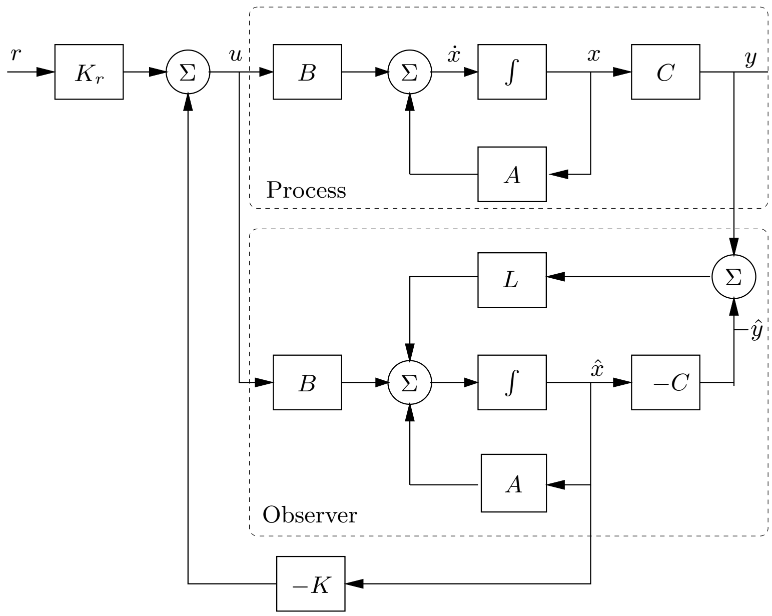

Figure 5.4: Block diagram of a controller which combines state feedback with an observer.

5.7 Integral Action

The controller based on state feedback achieves the correct steady state response to reference signals by careful calibration of the gain L_r and it lacks the nice property of integral control. It is then natural to ask why the the beautiful theory of state feedback and observers does not automatically give controllers with integral action. This is a consequence of the assumptions made when deriving the analytical design method which we will now investigate.

When using an analytical design method, we postulate criteria and specifications, and the controller is then a consequence of the assumptions. In this case the problem is the model (5.8). This model assumes implicitly that the system is perfectly calibrated in the sense that the output is zero when the input is zero. In practice it is very difficult to obtain such a model. Consider, for example, a chemical process control problem where the output is temperature and the control variable is a large rusty valve. The model (5.8) then implies that we know exactly how to position the valve to get a specified outlet temperature—indeed, a highly unrealistic assumption.

Having understood the difficulty it is not too hard to change the model. By modifying the model to

$$\begin{aligned} \frac{dx}{dt} &= Ax + B(u+v) \\ y &= Cx, \end{aligned} \tag{5.35}$$

where v is an unknown constant, we can can capture the idea that the model is no longer perfectly calibrated. This model is called a model with an input disturbance. Another possibility is to use the model

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + v \end{aligned}$$

where v is an unknown constant. This is a model with an output disturbance. It will now be shown that a straightforward design of an output feedback for this system does indeed give integral action. Both disturbance models will produce controllers with integral action. We will start by investigating the case of an input disturbance. This is a little more convenient for us because it fits the control goal of finding a controller that drives the state to zero.

The model with an input disturbance can conveniently be brought into the framework of state feedback. To do this, we first observe that v is an unknown constant which can be described by

$$\frac{dv}{dt} = 0$$

To bring the system into the standard format we simply introduce the disturbance v as an extra state variable. The state of the system is thus

$$z = \begin{bmatrix} x \\ v \end{bmatrix}$$

This is also called state augmentation. Using the augmented state the model (5.35) can be written as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \end{aligned} \tag{5.36}$$

Notice that the disturbance state is not reachable. If the disturbance can be measured, the state feedback is then

$$u = -\tilde{K}z + K_r r = -K_x x - K_v v + K_r r \tag{5.37}$$

The disturbance state v is not reachable. The the effect of the disturbance on the system can, however, be eliminated by choosing $K_v = 1$. If the disturbance v is known the control law above can be interpreted as a combination of feedback from the system state and feedforward from a measured disturbance. It is not realistic to assume that the disturbance can be measured and we will instead replace the states by estimates. The feedback law then becomes

$$u = -K_x\hat{z} + K_r r = -K_x\hat{x} - \hat{v} + K_r r$$

This means that feedback is based on estimates of the state and the disturbance. There are many other ways to introduce integral action.

5.8 A General Controller Structure

So far reference signals have been introduced simply by adding it to the state feedback. A more sophisticated way of doing this is shown by the block diagram in Figure 5.5, where the controller consists of three parts: an observer that computes estimates of the states based on a model and measured process inputs and outputs, a state feedback and a trajectory generator that generates the desired behavior of all states x_m and a feedforward signal u_{ff}. The signal u_{ff} is such that it generates the desired behavior of the states when applied to the system, under ideal conditions of no disturbances and no modeling errors. The controller is said to have *two degrees of freedom* because the response to command signals and disturbances are decoupled. Disturbance responses are governed by the observer and the state feedback and the response to command signal is governed by the feedforward. To get some insight into the behavior of the system let us discuss what happens when the command signal is changed. To fix the ideas let us assume that the system is in equilibrium with the observer state $\hat{x}$ equal to the process state x. When the command signal is changed a feedforward signal $u_{\mathrm{ff}}(t)$ is generated. This signal has the property that the process output gives the desired output $x_m(t)$ when the feedforward signal is applied to the system. The process state changes in response to the feedforward signal. The observer tracks the state perfectly because the initial state was correct. The estimated state $\hat{x}$ will be equal to the desired state x_m and the feedback signal $L(x_m - \hat{x})$ is zero. If there are some disturbances or some modeling errors the feedback signal will be different from zero and attempt to correct the situation.

The controller given in Figure 5.5 is a very general structure. There are many ways to generate the feedforward signal and there are also many dif-

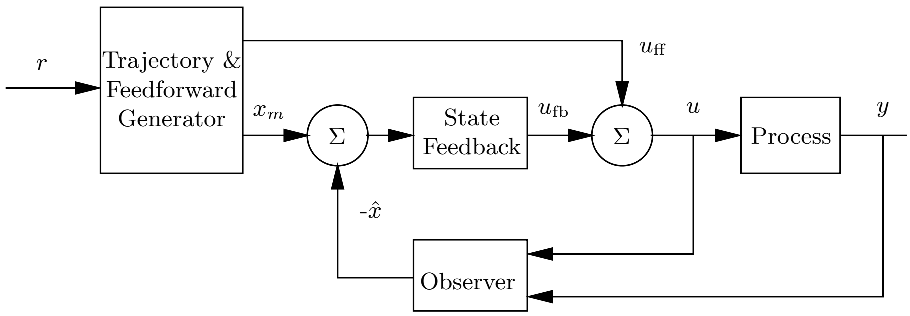

Figure 5.5: Block diagram of a controller based on a structure with two degrees of freedom. The controller consists of a command signal generator, state feedback and an observer.

ferent ways to compute the feedback gains L and the gain L of the observer.

The system in Figure 5.5 is an example of *the internal model principle* which says that a controller should contain a model of the system to be controlled and the disturbances action on the system.

Computer Implementation

The controllers obtained so far have been described by ordinary differential equations. They can be implemented directly using analog computers. Since most controllers are implemented using digital computers we will briefly discuss how this can be done.

The computer typically operates periodically, signals from the sensors are sampled and converted to digital form by the A/D converter, the control signal is computed, converted to analog form for the actuators, as shown in Figure 1.3 on page 5. To illustrate the main principles we consider the controller described by equations (5.29) and (5.30), i.e.

$$
\begin{aligned}
u &= -K\hat{x} + K_r r \\
\frac{d\hat{x}}{dt} &= A\hat{x} + Bu + K(y - C\hat{x})
\end{aligned}
$$

The first equation which only consists of additions and multiplications can be implemented directly in a computer. The second equation has to be approximated. A simple way is to replace the derivative by a difference

$$
\frac{\hat{x}(t_{k+1}) - \hat{x}(t_k)}{h} = A\hat{x}(t_k) + Bu(t_k) + K(y(t_k) - C\hat{x}(t_k))
$$

where t_k are the sampling instants and $h = t_{k+1} - t_k$ is the sampling period. Rewriting the equation we get

$$\hat{x}(t_{k+1} = \hat{x}(t_k) + h\big(A\hat{x}(t_k) + Bu(t_k) + K(y(t_k) - C\hat{x}(t_k))\big). \tag{5.38}$$

The calculation of the state only requires addition and multiplication and can easily be done by a computer. A pseudo code for the program that runs in the digital computer is

```
"Control algorithm - main loop
r=adin(ch1)                  "read setpoint from ch1
y=adin(ch2)                  "read process variable from ch2
u=C*x+Kr*r                   "compute control variable
daout(ch1)                   "set analog output ch1
x=x+h*(A*x+B*u+L*(y-C*x))    "update state estimate
```

The program runs periodically. Notice that the number of computations between reading the analog input and setting th analog output has been minimized. The state is updated after the analog output has been set. The program has one states `x`. The choice of sampling period requires some care.

For linear systems the difference approximation can be avoided by observing that the control signal is constant over the sampling period. An exact theory for this can be developed. Doing this we get a control law that is identical to (5.38) but with slightly different coefficients.

There are several practical issues that also must be dealt with. For example it is necessary to filter a signal before it is sampled so that the filtered signal has little frequency content above $f_s/2$ where f_s is the sampling frequency. If controllers with integral action are used it is necessary to provide protection so that the integral does not become too large when the actuator saturates. Care must also be taken so that parameter changes do not cause disturbances. Some of these issues are discussed in Chapter 10.

5.9 Exercises

1. Consider a system on reachable canonical form. Show that the inverse of the reachability matrix is given by

$$\tilde{W}_r^{-1} = \begin{bmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 0 & 1 & a_1 & \dots & a_{n-1} \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{5.39}$$