Chapter 4

Input/Output Behavior

> Research engineer Harold S. Black revolutionized telecommunications by inventing systems that eliminated feedback distortion in telephone calls. The major task confronting the lab at that time was elimination of distortion. After six years of persistence, Black conceived the principles and equations for his negative feedback amplifier in a flash commuting to work aboard the ferry. Basically, the concept involved feeding systems output back to the input as a method of system control.
>
> National Inventors Hall of Fame, `www.invent.org`

Previous chapters have focused on the dynamics of a system with relatively little attention the inputs and outputs. This chapter gives an introduction to input/output behavior for linear systems and shows how a nonlinear system can be approximated locally by a linear model.

4.1 Introduction

In Chapters 2 and 3 we consider construction and analysis of differential equation models for physical systems. We placed very few restrictions on these systems other than basic requirements of smoothness and well-posedness. In this chapter we specialize our results to the case of linear, time-invariant, input/output systems. This important class of systems is one for which a wealth of analysis and synthesis tools are available, and hence it has found great utility in a wide variety of applications.

What Is a Linear System?

Recall that a function $F : \mathbb{R}^p \to \mathbb{R}^q$ is said to be *linear* if it satisfies the following property:

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y) \qquad x, y \in \mathbb{R}^p, \alpha, \beta \in \mathbb{R}. \tag{4.1}$$

This equation implies that the function applied to the sum of two vectors is the sum of the function applied to the individual vectors, and that the results of applying F to a scaled vector is given by scaling the result of applying F to the original vector.

Input/output systems are described in a similar manner. Namely, we wish to capture the notion that if we apply two inputs u_1 and u_2 to a dynamical system and obtain outputs y_1 and y_2, then the result of applying the sum, $u_1 + u_2$, would give $y_1 + y_2$ as the output. Similarly, scaling one of the inputs would give a scaled version of the outputs. Thus, if we apply the input

$$u(t) = \alpha u_1(t) + \beta u_2(t)$$

then the output should be given by

$$y(t) = \alpha y_1(t) + \beta y_2(t).$$

This property is called linear *superposition*; when it holds (and after taking into account some subtleties with initial conditions), we say that the input/output system is linear.

A second source of linearity in the systems we will study is between the transient response to initial conditions and the forced response due to the input. You may recall from the study of ordinary differential equations that the solution to a linear ODE is broken into two components: the homogeneous response, $y_h(t)$, that depends only on initial conditions, and the particular response, $y_p(t)$, that depends only on the input. The complete solution is the some of these two components, $y(t) = y_h(t) + y_p(t)$. As we will see in this chapter, it can be further shown that if we scale the initial conditions by α and the input by β, then the solution will be $y(t) = \alpha y_h(t) + \beta y_p(t)$, just as in the case of a linear function.

As we shall show more formally in the next section, linear ordinary differential equations generate linear input/output systems. Indeed, it can be shown that if a state space system exhibits linear response to inputs and initial conditions, then it can always be written as a linear differential equation.

Where Do Linear Systems Come From?

Before defining linear systems more systematically, we take a moment to consider where linear systems appear in science and engineering examples. We have seen several examples of linear differential equations in the examples of the previous chapters. These include the spring mass system (damped oscillator) and the electric motor.

More generally, many physical systems can be modeled very accurately by linear differential equations. Electrical circuits are one example of a broad class of systems for which linear models can be used effectively. Linear models are also broadly applicable in mechanical engineering, for example as models of small deviations from equilibria in solid and fluid mechanics. Signal processing systems, including digital filters of the sort used in CD and MP3 players, are another source of good examples, although often these are best modeled in discrete time.

In many cases, we *create* systems with linear input/output response through the use of feedback. Indeed, it was the desire for linear behavior that led Harold S. Black, mentioned in the quotation at the beginning of the chapter, to the principle of feedback as a mechanism for generating amplification. Almost all modern single processing systems, whether analog or digital, use feedback to produce linear or near-linear input/output characteristics. For these systems, it is often useful to represent the input/output characteristics as linear, ignoring the internal details required to get that linear response.

For other systems, nonlinearities cannot be ignored if one cares about the global behavior of the system. The predator prey problem is one example of this; to capture the oscillatory behavior of the couple populations we must include the nonlinear coupling terms. However, if we care about what happens near an equilibrium point, it often suffices to approximate the nonlinear dynamics by their local *linearization*. The linearization is essentially an approximation of the nonlinear dynamics around the desired operating point.

No matter where they come from, the tools of linear systems analysis are a powerful collection of techniques that can be used to better understand and design feedback systems.

4.2 Properties of Linear Systems

In this section we give a more formal definition of linear input/output systems and describe the major properties of this important class of systems.

A linear dynamical system can be represented as

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du,\end{aligned} \tag{4.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ and hence $A \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix and similarly $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times p}$. This is one of the standard models in control. As in the previous chapters, we will usually restrict ourselves to the SISO case by taking $p = q = 1$.

Definitions

Consider a state space system of the form

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u)\end{aligned} \tag{4.3}$$

We will assume that all functions are smooth and that for a reasonable class of inputs (e.g., piecewise continuous functions of time) that the solutions of equation (4.3) exist for all time.

It will be convenient to assume that the origin $x = 0$, $u = 0$ is an equilibrium point for this system ($\dot{x} = 0$) and that $h(0, 0) = 0$. Indeed, we can do so without loss of generality. To see this, suppose that $(x_e, u_e) \neq (0, 0)$ is an equilibrium point of the system with output $y_e = h(x_e, u_e) \neq 0$. Then we can define a new set of states, inputs, and outputs

$$\tilde{x} = x - x_e \qquad \tilde{u} = u - u_e \qquad \tilde{y} = y - y_e$$

and rewrite the equations of motion in terms of these variables:

$$\begin{aligned}\frac{d}{dt}\tilde{x} &= f(\tilde{x} + x_e, \tilde{u} + u_e) =: \tilde{f}(\tilde{x}, \tilde{u}) \\ \tilde{y} &= h(\tilde{x} + x_e, \tilde{u} + u_e) =: \tilde{h}(\tilde{x}, \tilde{u}).\end{aligned}$$

In the new set of variables, we have the the origin is an equilibrium point with output 0, and hence we can carry our analysis out in this set of variables. Once we have obtained our answers in this new set of variables, we simply have to remember to "translate" them back to the original coordinates (through a simple set of additions).

Returning to the original equations (4.3), now assuming without loss of generality that the origin is the equilibrium point of interest, we define the system to be a *linear input/output system* if the following conditions are satisfied:

(i) If $y_{h1}(t)$ is the output of the solution to equation (4.3) with initial condition $x(0) = x_1$ and input $u(t) = 0$ and $y_{h2}(t)$ is the output with initial condition $x(0) = x_2$ and input $u(t) = 0$, then the output corresponding to the solution of equation (4.3) with initial condition $x(0) = \alpha x_1 + \beta x_2$ is

$$y(t) = \alpha y_{h1}(t) + \beta y_{h2}(t).$$

(ii) If $y_h(t)$ is the output of the solution to equation (4.3) with initial condition $x(0) = x_0$ and input $u(t) = 0$, and $y_p(t)$ is the output of the system with initial condition $x(0) = 0$ and input $u(t)$, then the output corresponding to the solution of equation (4.3) with initial condition $x(0) = \alpha x_0$ and input $\beta u(t)$ is

$$y(t) = \alpha y_h(t) + \beta y_p(t).$$

(iii) If $y_1(t)$ and $y_2(t)$ are outputs corresponding to solutions to the system 4.3 with initial conditions $x(0) = 0$ and inputs $u_1(t)$ and $u_2(t)$, respectively, then the solution of the differential equation with initial condition $x(0) = 0$ and input $\delta u_1(t)+\gamma u_2(t)$ has output $\delta y_1(t)+\gamma y_2(t)$.

Thus, we define a system to be linear if the outputs are jointly linear in the initial condition response and the forced response.

The linearity of the outputs with respect to the inputs (and the state) is called the *principle of superposition* and is illustrated in Figure 4.1 for the case where $x(0) = 0$. The basic idea is that if we have two input signals and we add them together, then the output is simply the superposition (sum) of the corresponding output signals. This property of linear systems is a critical one and allows us to use many tools from mathematics to study the stability and performance of such systems.

We now consider a differential equation of the form

$$\dot{x} = Ax + Bu \tag{4.4}$$

where $A \in \mathbb{R}^{n\times n}$ is a square matrix, $B \in \mathbb{R}^n$ is a column vector of length n. (In the case of a multi-input systems, B becomes a matrix of appropriate dimension.) Equation (4.4) is a system of linear, first order, differential equations with input u and state x. We now show that this system is linear system, in the sense described above.

Theorem 4.1. *Let $x_{h1}(t)$ and $x_{h2}(t)$ be the solutions of the linear differential equation* (4.4) *with input $u(t) = 0$ and initial conditions $x(0) = x_1$ and*

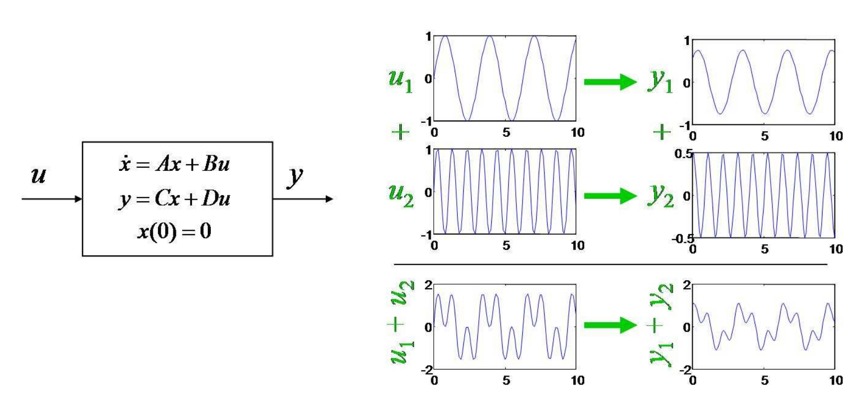

Figure 4.1: Illustration of the principle of superposition.

.

x_2, respectively, and and let $x_{p1}(t)$ and $x_{p2}(t)$ be the solutions with initial condition $x(0) = 0$ and inputs $u_1(t), u_2(t) \in \mathbb{R}$. Then the solution of equation (4.4) with initial condition $x(0) = \alpha x_1 + \beta x_2$ and input $u(t) = \delta u_1 + \gamma u_2$ and is given by

$$x(t) = \big(\alpha x_{h1}(t) + \beta x_{h2}(t)\big) + \big(\delta x_{p1}(t) + \gamma x_{p2}(t)\big).$$

Proof. Substitution. □

It follows that since the output is a linear combination of the states (through multiplication by the row vector C), the system is input/output linear as we defined above. As in the case of linear differential equations in a single variable, we define the solution $x_h(t)$ with zero input as the *homogeneous* solution and the solution $x_p(t)$ with zero initial condition as the *particular* solution.

It is also possible to show that if a system is input/output linear in the sense we have described, that it can always be represented by a state space equation of the form (4.4) through appropriate choice of state variables.

The Matrix Exponential

Although we have shown that the solution of a linear set of input/output differential equations defines a linear input/output system, we have not actually solved for the solution of the system. We begin by considering the homogeneous response, corresponding to the system

$$\dot{x} = Ax \tag{4.5}$$

For the *scalar* differential equation

$$\dot{x} = ax \qquad x \in \mathbb{R}, a \in \mathbb{R}$$

the solution is given by the exponential

$$x(t) = e^{at}x(0).$$

We wish to generalize this to the vector case, where A becomes a matrix.

We define the *matrix exponential* as the infinite series

$$e^S = I + S + \frac{1}{2}S^2 + \frac{1}{3!}S^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}S^k, \tag{4.6}$$

where $S \in \mathbb{R}^{n\times n}$ is a square matrix and I is the $n \times n$ identity matrix. We make use of the notation

$$S^0 = I \qquad S^2 = SS \qquad S^n = S^{n-1}S,$$

which defines what we mean by the "power" of a matrix. Equation (4.6) is easy to remember since it is just the Taylor series for the scalar exponential, applied to the matrix S. It can be shown that the series in equation (4.6) converges for any matrix $S \in \mathbb{R}^{n\times n}$ in the same way that the normal exponential is define for any scalar $a \in \mathbb{R}$.

Replacing S in (4.6) by At where $t \in \mathbb{R}$ we find that

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k,$$

Differentiating this expression with respect to t gives

$$\frac{d}{dt}e^{At} = A + At + \frac{1}{2}A^3t^2 + \cdots = A\sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k = Ae^{At}. \tag{4.7}$$

Multiplying by $x(0)$ from the right we find that $x(t) = e^{At}x(0)$ is the solution to the differential equation (4.5) with initial condition $x(0)$. We summarize this important result as a theorem.

Theorem 4.2. *The solution to the homogeneous system of differential equation* (4.5) *is given by*

$$x(t) = e^{At}x(0).$$

Notice that the form of the solution is exactly the same as for scalar equations.

The form of the solution immediately allows us to see that the solution is linear in the initial condition. In particular, if x_{h1} is the solution to equation (4.5) with initial condition $x(0) = x_1$ and x_{h2} with initial condition x_2, then the solution with initial condition $x(0) = \alpha x_1 + \beta x_2$ is given by

$$x(t) = e^{At}\big(\alpha x_1 + \beta x_2\big) = \big(\alpha e^{At} x_1 + \beta e^{At} x_2\big) = \alpha x_{h1}(t) + \beta x_{h2}(t)$$

Similarly, we see that the corresponding output is given by

$$y(t) = Cx(t) = \alpha y_{h1}(t) + \beta y_{h2}(t),$$

where y_1 and y_2 are the outputs corresponding to x_{h1} and x_{h2}.

We illustrate computation of the matrix exponential by three examples.

Example 4.1 *(The double integrator).* A very simple linear system that is useful for understanding basic concepts is the second order system given by

$$\begin{aligned} \ddot{q} &= u \\ y &= q. \end{aligned}$$

This system system is called a *double integrator* because the input u is integrated twice to determine the output y.

In state space form, we write $x = (q, \dot{q})$ and

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

The dynamics matrix of a double integrator is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and we find by direct calculation that $A^2 = 0$ and hence

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Thus the homogeneous solution ($u = 0$) for the double integrator is given by

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(0) + t x_2(0) \\ x_2(0) \end{bmatrix} \\ y(t) &= x_1(0) + t x_2(0). \end{aligned}$$

Example 4.2 *(Undamped oscillator).* A simple model for an oscillator, such as the spring mass system with zero damping, is

$$\ddot{q} + kq = u.$$

Setting $k = 1$ and putting the system into state space form, the dynamics matrix for this system is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

We have

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

which can be verified by differentiation:

$$\frac{d}{dt} e^{At} = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Example 4.3 *(Diagonal system).* Consider a diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

The kth power of At is also diagonal,

$$(At)^k = \begin{bmatrix} \lambda_1^k t^k & & & 0 \\ & \lambda_2^k t^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k t^k \end{bmatrix}$$

and it follows from the series expansion that the matrix exponential is given by

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}.$$

The Convolution Integral

We now return to the general input/output case in equation (4.2), repeated here:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned} \tag{4.8}$$

Using the matrix exponential the solution to (4.8) can be written as

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{4.9}$$

To prove this we differentiate both sides and use the property (4.7) of the matrix exponential. This gives

$$\frac{dx}{dt} = Ae^{At}x(0) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau + Bu(t) = Ax + Bu,$$

which proves the result. Notice that the calculation is essentially the same as for proving the result for a first order equation.

It follows from equations (4.8) and (4.9) that the input output relation is given by

$$y(t) = Ce^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{4.10}$$

It is easy to see from this equation that the input/output systems is jointly linear in both the initial conditions and the state: this follows from the linearity of matrix/vector multiplication and integration.

Equation (4.10) is called the *convolution equation* and it represents the general form of the solution of a system of coupled linear differential equations. We see immediately that the dynamics of the system, as characterized by the matrix A, play a critical role in both the stability and performance of the system. Indeed, the matrix exponential describes *both* what happens when we perturb the initial condition and how the system responds to inputs.

Coordinate Changes

The components of the input vector u and the output vector y are unique physical signals, but the state variables depend on the coordinate system chosen to represent the state. The matrices A, B and C depend on the coordinate system but not the matrix D, which directly relates inputs and outputs. The consequences of changing coordinate system will now be investigated. Introduce new coordinates z by the transformation $z = Tx$, where T is a regular (invertible) matrix. It follows from (4.2) that

$$\begin{aligned} \frac{dz}{dt} &= T(Ax + Bu) = TAT^{-1}z + TBu = \tilde{A}z + \tilde{B}u \\ y &= Cx + DU = CT^{-1}z + Du = \tilde{C}z + Du \end{aligned}$$

The transformed system has the same form as (4.2) but the matrices A, B and C are different

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB, \qquad \tilde{C} = CT^{-1}, \qquad \tilde{D} = D \tag{4.11}$$

It is interesting to investigate if there are special coordinate systems that gives systems of special structure.

We can also compare the solution of the system in transformed coordinates to that in the original state coordinates. We make use of an important property of the exponential map:

$$e^{TST^{-1}} = Te^{S}T^{-1},$$

which can be verified by substitution in the definition of the exponential map. Using this property, it is easy to show that

$$x(t) = T^{-1}z(t) = T^{-1}e^{\tilde{A}t}Tx(0) + T^{-1}\int_0^t e^{\tilde{A}(t-\tau)}\tilde{B}u(\tau)\,d\tau.$$

From this form of the equation, we see that if it is possible to transform A into a form $\tilde{A}$ for which the matrix exponential is easy to compute, we can use that computation to solve the general convolution equation for the untransformed state x by simple matrix multiplications. This technique is illustrated in the next section.

4.3 Stability and Performance

The special form of a linear system, and its solution through the convolution integral, allow us to analytically solve for the stability and input/output performance properties that were given in Chapter **??**.

Diagonalizable Systems

The easiest class of linear systems to analyze are those whose system matrices are in diagonal form. In this case, the dynamics have the form

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u$$
$$y = \begin{bmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_n \end{bmatrix} x + Du.$$

It is easy to see that the state trajectories for this system are independent of each other, so that we can write the solution in terms of n individual systems

$$\dot{x}_i = \lambda_i x_i + \beta_i u.$$

In particular, if we consider the stability of the system when $u = 0$, we see that the equilibrium point $x_e = 0$ is stable if $lambda_i \leq 0$ and asymptotically stable if $\lambda_i < 0$.

Very few systems are diagonal, but some systems can be transformed into diagonal form via coordinate transformations. One such class of systems is those for which the systems matrix has distinct (non-repeating) eigenvalues. In this case it is possible to find a matrix T such that the matrix TAT^{-1} and the transformed system is in diagonal form. In fact, it turns out that the diagonal elements are precisely the eigenvalues of the original matrix A. We can reason about the stability of the original system by nothing that $x(t) = T^{-1}z(t)$ and so if the transformed system is stable (or asymptotically stable) then the original system has the same type stability.

This analysis can be extended to systems with complex eigenvalues, as we illustrate in the example below.

Example 4.4. Couple spring mass Consider the coupled mass spring system show in Figure 4.2. The input to this system is the sinusoidal motion of the end of rightmost spring and the output is the position of each mass, q_1 and q_2. The equations of motion for the system are given by

$$m_1\ddot{q}_1 = -2kq_1 - b\dot{q}_1 + kq_2$$
$$m_2\ddot{q}_2 = kq_1 - 2kq_2 - b\dot{q}_2 + ku$$

In state-space form, we define the state to be $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)$ and we can

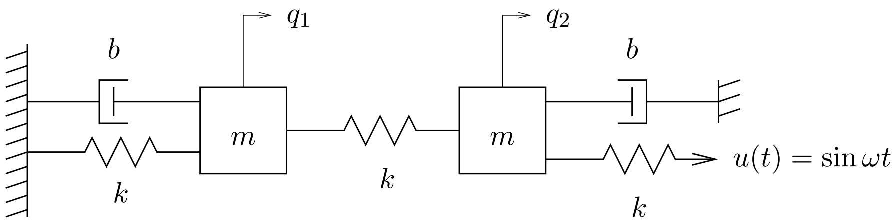

Figure 4.2: Coupled spring mass system.

rewrite the equations as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & -\frac{b}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{bmatrix} u.$$

This is a coupled set of four differential equations and quite difficult to solve in analytical form.

We now define a transformation $z = Tx$ that puts this system into a simpler form. Let $z_1 = \frac{1}{2}(q_1 + q_2)$, $z_2 = \dot{z}_1$, $z_3 = \frac{1}{2}(q_1 - q_2)$ and $z_4 = \dot{z}_3$, so that

$$z = Tx = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} x.$$

Using the coordinate transformations described above (or simple substitution of variables, which is equivalent), we can write the system in the z coordinates as

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3k}{m} & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{2m} \\ 0 \\ -\frac{k}{2m} \end{bmatrix} u.$$

Note that the resulting matrix equations are not diagonal but they are block diagonal. Hence we can solve for the solutions by solving the two sets of second order system represented by the states (z_1, z_2) and (z_3, z_4).

Once we have solved the two sets of independent second order equations (described in more detail in the next section), we can recover the dynamics in the original coordinates by inverting the state transformation and writing $x = T^{-1}z$. We can also determine the stability of the system by looking at the stability of the independent second order systems.

Jordan Form

Some matrices with equal eigenvalues cannot be transformed to diagonal form. They can however be transformed to the Jordan form. In this form the dynamics matrix has the eigenvalues along the diagonal. When there are equal eigenvalues there may be ones appearing in the super diagonal indicating that there are coupling between the states.

More specifically, we define a matrix to be in Jordan form if it can be written as

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & \dots & & J_k \end{bmatrix} \quad \text{where} \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_i & 1 \\ 0 & \dots & 0 & 0 & \lambda_i \end{bmatrix}. \tag{4.12}$$

Each matrix J_i is called a *Jordan block* and λ_i for that block corresponds an eigenvalue of J. Every matrix $A \in \mathbb{R}n \times n$ can be transformed into Jordan form with the eigenvalues of A determining λ_i in the Jordan form. Hence we can study the stability of a system by studying the stability of its corresponding Jordan form. We summarize the main results in the following theorem.

Theorem 4.3. *The system*

$$\dot{x} = Ax$$

is asymptotically stable if and only if all eigenvalues of A all have strictly negative real part and is unstable if any eigenvalue of A has strictly positive real part.

The case when one or more eigenvalues has zero real part is more complicated and depends on the Jordan form associated with the dynamics matrix.

Impulse, Step and Frequency Response

The step response of (4.10) is obtained as the output when the initial condition is zero and the output is a unit step. We get

$$h(t) = C \int_0^t e^{A\tau} B d\tau + D. \tag{4.13}$$

The impulse response is the derivative of the step response.

$$g(t) = Ce^{At} B + D\delta(t). \tag{4.14}$$

It is also the output when the initial condition is zero and the input is a delta function. Note that both the step response and the impulse response are invariant to changes in the coordinate system (substitute $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ to verify this).

To compute the frequency response we make use of complex variables and exploit the form of the complex exponential:

$$e^{\alpha+i\beta} = e^{\alpha}(\cos\beta + i\sin\beta).$$

This useful formula allows us to compute the frequency response by letting the input be $u(t) = e^{i\omega t}u_0 = (\cos\omega t + i\sin\omega t)u_0$. The steady state output and state are then $y(t) = e^{i\omega t}y_0$ and $x(t) = e^{i\omega t}x_0$. Inserting these expressions in the differential equation (4.2) we get

$$\begin{aligned} i\omega e^{i\omega t}x_0 &= Ae^{i\omega t}x_0 + Be^{i\omega t}u_0 \\ e^{i\omega t}y_0 &= Ce^{i\omega t}x_0 + De^{i\omega t}u_0. \end{aligned}$$

Solving the first equation for x_0 and inserting in the second equation gives

$$y_0 = \Big(C\big(i\omega I - A\big)^{-1}B + D\Big)u_0.$$

The frequency response is thus

$$G(i\omega) = C\big(i\omega I - A\big)^{-1}B + D. \tag{4.15}$$

The frequency response is invariant to changes in the coordinate system.

4.4 Second Order Systems

One class of systems that occurs frequently in the analysis and design of feedback systems are second order, linear differential equations. Because of their ubiquitous nature, it is useful to apply the concepts of this chapter to that specific class of systems and build more intuition about the relationship between stability and performance. We will first consider a second order system with dynamics

$$\begin{aligned} &\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = \omega_0^2 u \\ &y = q \end{aligned} \tag{4.16}$$

The step responses of systems with different values of ζ are shown in Figure 4.3. The figure shows that parameter ω_0 essentially gives a time scaling.

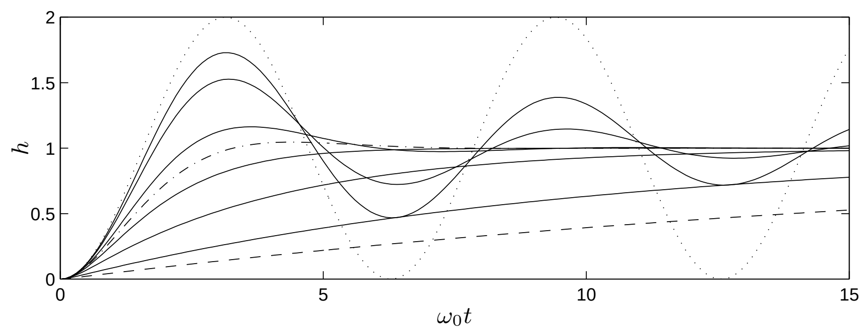

Figure 4.3: Normalized step responses h for the system (4.16) for $\zeta = 0$ (dotted), 0.1, 0.2, 0.5, 0.707 (dash dotted), 1, 2, 5 and 10 (dashed).

The response is faster if ω_0 is larger. The shape of the response is determined by ζ. The step responses have an overshoot of

$$M = \begin{cases} e^{-\pi\zeta/\sqrt{1-\zeta^2}} & \text{for } |\zeta| < 1 \\ 1 & \text{for} \zeta \geq 1. \end{cases} \tag{4.17}$$

For $\zeta < 1$ the maximum overshoot occurs at

$$t_{max} = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}} \tag{4.18}$$

There is always an overshoot if $\zeta < 1$. The maximum decreases and is shifted to the right when ζ increases and it becomes infinite for $\zeta = 1$, when the overshoot disappears. In most cases it is desirable to have a moderate overshoot which means that the parameter ζ should be in the range of 0.5 to 1. The value $\zeta = 1$ gives no overshoot.

It can be shown that the frequency response for this system is given by

$$G(i\omega) = \frac{\omega_0^2}{(i\omega)^2 + 2\zeta\omega_0(i\omega) + \omega_0^2} = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + 2i\zeta\omega_0\omega}.$$

A graphical illustration of the frequency response is given in Figure 4.4. Notice the resonance peak that increases with decreasing ζ. The peak is often characterized by is Q-value where $Q = G(i\omega_0) = 0.5/\zeta$.

4.5 Linearization

Another source of linear system models is through the *approximation* of a nonlinear system by a linear one. These approximations are aimed at

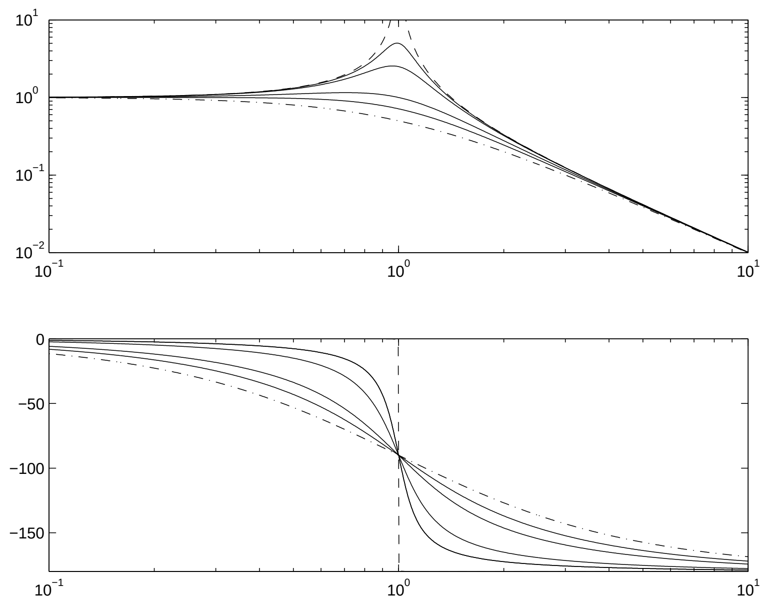

Figure 4.4: Frequency response of a the second order system (4.16). The upper curve shows the gain ratio and the lower curve shows the phase shift. The parameters is Bode plot of the system with $\zeta = 0$ (dashed), 0.1, 0.2, 0.5, 0.7 and 1.0 (dashed-dot).

studying the local behavior of a system, where the nonlinear effects are expected to be small. In this section we discuss how to locally approximate a system by its linearization and what can be said about the approximation in terms of stability.

Jacobian Linearizations of Nonlinear Systems

Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u) \qquad && x \in \mathbb{R}^n, u \in \mathbb{R} \\ y &= h(x, u) \qquad && y \in \mathbb{R} \end{aligned} \tag{4.19}$$

with an equilibrium point at $x = x_e$, $u = u_e$. Without loss of generality, we assume that $x_e = 0$ and $u_e = 0$, although initially we will consider the general case to make the shift of coordinates explicit.

In order to study the *local* behavior of the system around the equilibrium point (x_e, u_e), we suppose that $x - x_e$ and $u - u_e$ are both small, so that nonlinear perturbations around this equilibrium point can be ignored compared with the (lower order) linear terms. This is roughly the same type of argument that is used when we do small angle approximations, replacing $\sin\theta$ with θ and $\cos\theta$ with 1.

In order to formalize this idea, we define a new set of state variables z, inputs v, and outputs w:

$$z = x - x_e \qquad v = u - u_e \qquad w = y - h(x_e, u_e).$$

These variables are all close to zero when we are near the equilibrium point, and so in these variables the nonlinear terms can be thought of as the higher order terms in a Taylor series expansion of the relevant vector fields (assuming for now that these exist).

Example 4.5. Consider a simple scalar system,

$$\dot{x} = 1 - x^3 + u.$$

The point $(x_e, u_e) = (1, 0)$ is an equilibrium point for this system and we can thus set

$$z = x - 1 \qquad v = u.$$

We can now compute the equations in these new coordinates as

$$\begin{aligned} \dot{z} &= \frac{d}{dt}(x-1) = \dot{x} \\ &= 1 - x^3 + u = 1 - (z+1)^3 + v \\ &= 1 - z^3 - 3z^2 - 3z - 1 + v = -3z - 3z^2 - z^3 + v. \end{aligned}$$

If we now assume that x stays very close to the equilibrium point, then $z = x - x_e$ is small and $z \ll z^2 \ll z^3$. We can thus *approximate* our system by a *new* system

$$\dot{z} = -3z + v.$$

This set of equations should give behavior that is close to that of the original system as long as z remains small.

More formally, we define the *Jacobian linearization* of the nonlinear system (4.19) as

$$\begin{aligned} \dot{z} &= Az + Bv \\ w &= Cz + Dv, \end{aligned} \tag{4.20}$$

where

$$
\begin{aligned}
A &= \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_e,u_e)} & B &= \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_e,u_e)} \\
C &= \left.\frac{\partial h(x,u)}{\partial x}\right|_{(x_e,u_e)} & D &= \left.\frac{\partial h(x,u)}{\partial u}\right|_{(x_e,u_e)}
\end{aligned}
\tag{4.21}
$$

The system (4.20) approximates the original system (4.19) when we are near the equilibrium point that the system was linearized about.

It is important to note that we can only define the linearization of a system about an equilibrium point. To see this, consider a polynomial system

$$\dot{x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + u,$$

where $a_1 \neq 0$. There are a family of equilibrium points for this system given by $(x_e, u_e) = (-(a_0 + u_0)/a_1, u_0)$ and we can linearize around any of these. Suppose instead that we try to linearize around the origin of the system, $x = 0$, $u = 0$. If we drop the higher order terms in x, then we get

$$\dot{x} = a_0 + a_1 x + u,$$

which is *not* the Jacobian linearization if $a_0 \neq 0$. The constant term must be kept and this is not present in (4.20). Furthermore, even if we kept the constant term in the approximate model, the system would quickly move away from this point (since it is "driven" by the constant term a_0) and hence the approximation could soon fail to hold.

Local Stability of Nonlinear Systems

Having constructed an approximate model around an equilibrium point, we can now ask to what extent this model predicts the behavior of the original nonlinear system. The following theorem gives a partial answer for the case of stability.

Theorem 4.4. *Consider the system* (4.19) *and let* $A \in \mathbb{R}^{n\times n}$ *be defined as in equation* (4.21)*. If the real part of the eigenvalues of* A *are strictly less than zero, then* x_e *is a locally asymptotically stable equilibrium point of* (4.19)*.*

This theorem proves that *global* uniform asymptotic stability of the linearization implies *local* uniform asymptotic stability of the original nonlinear system. The estimates provided by the proof of the theorem can be used to give a (conservative) bound on the domain of attraction of the origin. Systematic techniques for estimating the bounds on the regions of attraction of equilibrium points of nonlinear systems is an important area of research and involves searching for the “best” Lyapunov functions.

4.6 Exercises