

CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101/110
Homework Set #3

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Note: In the upper left hand corner of the first page of your homework set, please put the class you are taking (CDS 101, CDS 110) and the number of hours that you spent on this homework set (including reading).

All students should complete the following problems:

1. For each of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable), or unstable. To determine stability, you can either use a phase portrait (if appropriate) or simulate the system using multiple nearby initial conditions to how the state evolves. (Note: if you know how to check stability through the linearization, you can also use this approach.)

- (a) *Nonlinear spring mass.* Consider a nonlinear spring mass system,

$$m\ddot{x} = -k(x - ax^3) - b\dot{x},$$

where $m = 1000$ kg is the mass, $k = 250$ kg/sec² is the nominal spring constant, $a = 0.01$ represents the nonlinear “softening” of the spring, and $b = 100$ kg/sec is the damping coefficient. Note that this is very similar to the spring mass system we have studied in class, except for the nonlinearity.

- (b) *Predator prey ODE.* Use the ODE model described in class,

$$\begin{aligned}\dot{x}_1 &= b_r x_1 - a x_1 x_2 \\ \dot{x}_2 &= b x_1 x_2 - d_f x_2,\end{aligned}$$

with the parameters $b_r = 0.7$, $d_f = 0.5$, $a = 0.007$, $b = 0.0005$.

- (c) *Congestion control of the Internet.* A simple model for congestion control between N computers connected by a router is given by the differential equation

$$\begin{aligned}\dot{x}_i &= -b \frac{x_i^2}{2} + (b_{\max} - b) \\ \dot{b} &= \sum_{i=1}^N x_i - c\end{aligned}$$

where $x_i \in \mathbb{R}$, $i = 1, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_{\max} > 0$ is the maximum buffer size, and $c > 0$ is the capacity of the link connecting the router to the computers. The \dot{x}_i equation represents the control law that the individual computers use to determine how fast to send data across the network (this version is motivated by a protocol called “Reno”) and the \dot{b} equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, x_1 , x_2 , and b), and take $b_{\max} = 1$ Mb and $c = 2$ Mb/sec.

2. (MATLAB/SIMULINK) Consider the cruise control system from Homework Set #1, problem 1. Set the gains of the system to their default values ($K_i = 100$, $K_p = 500$).
- Using `hw1cruise.mdl` from the course home page, plot the step response of the system (from 55 mph to 65 mph) and measure the rise time, overshoot, settling time, and steady state error.
 - Modify the block diagram to allow a sinusoidal reference signal superimposed on top of a commanded reference (so that you get something that oscillates around the nominal speed of 55 m/s). Plot the response of the system to a commanded reference speed that varies sinusoidally between 50 m/s and 60 m/s at a frequency of 1 Hz (about 6 rad/sec). Measure the relative amplitude and phase of the velocity with respect to the commanded input. Your answer should be the ratio of the output amplitude to the input amplitude (after subtracting off the means) and the number of radians of phase “lead” or “lag” between the sinusoids.
 - In most real-life systems, inputs magnitudes are limited by the capabilities of the actuator. A modified version of the cruise controller with input saturation is available from the lecture homepage, with the file name `hw1cruise_sat.mdl` (you can see the saturation by clicking into the vehicle block). Using this model, show that if we increase the amplitude of the desired oscillations sufficiently high, that the response of the system is no longer a pure sinusoid at the desired frequency.

Only CDS 110a students need to complete the following additional problems:

3. Consider a second order system of the form

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2y = u(t)$$

with initial conditions $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$.

- Compute the homogenous solution to this equation ($u(t) = 0$) with initial condition $y_0 = 0$, $\dot{y}_0 = 1$. This is the “impulse response” for this system. Plot the impulse response as a function of time for $\omega_n = 1$, $\zeta = 0.5$.
- Compute the response of the system to a sinusoidal input $u(t) = A \sin(\omega t)$. Your result should be analytical (a formula, like the ones given in lecture) and you should make sure to keep the effects of the initial conditions. Now assuming that the initial conditions have died out (i.e., ignoring the homogeneous part of the solution), plot the “frequency response” of the system. Your answer should be in the form of two plots: the relative amplitude and the relative phase of the output compared to the input, both as a function of frequency. Use a logarithmic scale for the frequency and amplitude, and a linear scale for the phase. (This type of plot is called a “Bode plot”).

Note: you can find this solution worked out in many textbooks. You are encouraged to look for the solution, but make sure that you provide a derivation of your results and that you understand them. (Pretend that this might be the type of thing you were asked on a closed book section of the midterm.)

- Suppose that we now implement a feedback control law of the form

$$u(t) = k_1(y - v(t)) + k_2\dot{y},$$

which is intended to allow us to track a new input $v(t)$ (just like the cruise control example). Compute the frequency response of the closed loop and show that we can set the closed loop natural frequency ω'_n and damping ratio ζ' to arbitrary values by adjusting the gains k_1 and k_2 . Give formulas for the gains in terms of the desired ω'_n and ζ' .

(d) *Optional:* Use the results from this problem to design a cruise control law for the system in problem #2 of last week's homework that has a settling time of 1 second and no overshoot.

4. For each of the systems in the table below, defined in more detail in Problem 1, determine if there exists a Lyapunov function of the given form that proves that the indicated equilibrium point is asymptotically stable. The parameter α should be taken as a free parameter and used as needed to satisfy the conditions of the Lyapunov theorem.

You should try to solve the problem for general parameter values if possible, but if you can't find a general solution in a reasonable amount of time, then you should use the numerical values from Problem 1.

Part	System/equilibrium point	Parameter ranges	Lyapunov function candidate
(a)	Nonlinear spring mass, $x_e = (0, 0)$	$m, k, a, c > 0$	$V = kx^2 + \alpha x\dot{x} + m\dot{x}^2$
(b)	Predator prey ODE, $x_e \neq (0, 0)$	$a, b, b_r, d_f > 0$	$V = (x_1 - x_{e,1})^2 + \alpha(x_2 - x_{e,2})^2$
(c)	Congestion control, $x_e, b_e \neq 0$	$N, b_{\max}, c > 0$	$V = \alpha \sum (x_i - x_{e,i})^2 + (b - b_e)^2$

Note: for some of these systems, the equilibrium point may be asymptotically stable but the Lyapunov function candidate may not allow you to prove stability. This is one of the limitations of Lyapunov stability: you have to *find* a Lyapunov function that proves stability of the system.

Optional: If you are not able to find a Lyapunov function of the given form for an equilibrium point which you showed in problem 1 is asymptotically stable, try to find a Lyapunov function of a more general form that works.

Optional: For those systems in which you are able to find a Lyapunov function, determine whether the given function can also be used to prove whether the system is exponentially stable and whether the system is globally asymptotically stable.