Chapter 4

Linear Input/Output Systems

This chapter provides an introduction to linear input/output systems, a class of models that is the focus of the remainder of the text.

4.1 Introduction

In Chapters **??** and **??** we consider construction and analysis of differential equation models for physical systems. We placed very few restrictions on these systems other than basic requirements of smoothness and well-posedness. In this chapter we specialize our results to the case of linear, time-invariant, input/output systems. This important class of systems is one for which a wealth of analysis and synthesis tools are avilable, and hence it has found great utility in a wide variety of applications.

What is a linear system?

Recall that a function $F : \mathbb{R}^p \to \mathbb{R}^q$ is said to be *linear* if it satisfies the following property:

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y) \qquad x, y \in \mathbb{R}^p, \alpha, \beta \in \mathbb{R}. \tag{4.1}$$

This equation implies that the function applied to the sum of two vectors is the sum of the function applied to the individual vectors, and that the results of applying F to a scaled vector is given by scaling the result of applying F to the original vector.

Input/output systems are described in a similar manner. Namely, we wish to capture the notion that if we apply two inputs u_1 and u_2 to a dynamical system and obtain outputs y_1 and y_s, then the result of applying the sum, $u_1 + u_2$, would give $y_1 + y_2$ as the output. Similarly, scaling one of the inputs would give a scaled version of the outputs. Thus, if we apply the input

$$u(t) = \alpha u_1(t) + \beta u_2(t)$$

then the output should be given by

$$y(t) = \alpha y_1(t) + \beta y_2(t).$$

When this property is true (and after taking into account some subtleties with initial conditions), we say that the input/output system is linar.

A second source of linearity in the systems we will study is between the transient response to initial conditions and the forced response due to the input. We have already seen in Chapter **??** that if $y_h(t)$ is the homogenous response (due to initial conditions) and $y_p(t)$ is the particular response (due to a given input), then the complete solution is $y(t) = y_h(t) + y_p(t)$. If the system is linear, then we will further show that if we scale the initial conditions by α and the input by β, then the solution will be $y(t) = \alpha y_h(t) + \beta y_p(t)$, just as it is in the case of a function.

As we shall show more formally in the next section, linear ordinary differential equations generate linear input/output systems. Indeed, it can be shown that if a state space system exhibits linear response to inputs and initial conditions, then it can always be written as a linear differential equation.

Where do linear systems come from?

Before defining linear systems more systematically, we take a moment to consider where linear systems appear in science and engineering examples. We have seen several examples of linear differential equations in the examples of the previous chapter. These include the spring mass system, the electric motor, and the double integrator system.

More generally, many physical systems can be modelled very accurately by linear differential equations. Electrical circuits are one example of a broad class of systems for which linear models can be used effectively. Linear models are also broadly applicable in mechanical engineering, as models of the in solid and fluid mechanics. Signal processing systems, including digital filters of the sort used in CD and MP3 players, are another source of good examples, although often these are best modelled in discrete time (as described in more detail in Section 4.4.

In many cases, we *create* systems with linear input/output response through the use of feedback. Indeed, it was the desire for linear behavior that led Black to the principle of feedback as a mechanism for generating amplification. Almost all modern single processing systems, whether analog or digital, use feedback to produce linear or near-linear input/output characteristics. For these systems, it is often useful to represent the input/output characteristics as linear, ignoring the internal details required to get that linear response.

For other systems, nonlinearities cannot be ignored if one cares about the global behavior of the system. The predator prey problem is one example of this; to capture the oscillatory behavior of the couple populations we must include the nonlinear coupling terms. However, if we care about what happens *near an equilibrium point*, it often suffices to approximate the nonlinear dynamics by their local *linearization*. The linearization is essentially an approximation of the nonlinear dynamics around the desired operating point.

No matter where they come from, the tools of linear systems analysis are a powerful collection of techniques that can be used to better understand and design feedback systems.

4.2 Properties of Linear Systems

In this section we give a more formal definition of linear input/output systems and describe the major properties of this important class of systems.

Definitions

Consider a state space system of the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{4.2}$$

We will assume that all functions are smooth and that for a reasonable class of inputs (e.g., piecewise continuous functions of time) that the solutions of equation (4.2) exist for all time.

It will be convenient to assume that the origin $x = 0$, $u = 0$ is an equilibrium point for this system ($\dot{x} = 0$) and that $h(0) = 0$. Indeed, we can do so without loss of generality. To see this, suppose that $(x_e, u_e) \neq (0, 0)$

is an equilibrium point of the system with output $y_e = h(x_e) \neq 0$. Then we can define a new set of states, inputs, and outputs

$$\tilde{x} = x - x_e \qquad \tilde{u} = u - u_e \qquad \tilde{y} = y - y_e$$

and rewrite the equations of motion in terms of these variables:

$$\begin{aligned} \frac{d}{dt}\tilde{x} &= f(\tilde{x} + x_e, \tilde{u} + u_e) =: \tilde{f}(\tilde{x}, \tilde{u}) \\ \tilde{y} &= h(\tilde{x} + x_e) =: \tilde{h}(\tilde{x}). \end{aligned}$$

In the new set of variables, we have the the origin is an equilibrium point with output 0, and hence we can carry our analysis out in this set of variables. Once we have obtained our answers in this new set of variables, we simply have to remember to "translate" them back to the original coordinates (through a simple set of additions).

Returning to the original equations (**??**), now assuming without loss of generality that the origin is the equilibrium point of interest, we define the system to be a *linear input/output system* if the following conditions are satisfied:

(i) If $y_{h1}(t)$ is the output of the solution to equation (**??**) with initial condition $x(0) = x_1$ and input $u(t) = 0$ and $y_{h2}(t)$ is the output with initial condition $x(0) = x_2$ and input $u(t) = 0$, then the output corresponding to the solutoin of equation (**??**) with initial condition $x(0) = \alpha x_1 + \beta x_2$ is

$$y(t) = \alpha y_{h1}(t) + \beta y_{h2}(t).$$

(i) If $y_h(t)$ is the output of the solution to equation (**??**) with initial condition $x(0) = x_0$ and input $u(t) = 0$, and $y_p(t)$ is the output of the system with initial condition $x(0) = 0$ and input $u(t)$, then the output corresponding to the solution of equation (**??**) with initial condition $x(0) = \alpha x_0$ and input $\delta u(t)$ is

$$y(t) = \alpha y_h(t) + \delta y_p(t).$$

(ii) If $y_1(t)$ and $y_2(t)$ are outputs corresponding to solutions to the system **??** with initial conditions $x(0) = 0$ and inputs $u_1(t)$ and $u_2(t)$, respectively, then the solution of the differential equation with initial condition $x(0) = 0$ and input $\delta u_1(t) + \gamma u_2(t)$ has output $\delta y_1(t) + \gamma y_2(t)$.

Thus, we define a system to be linear if the outputs are jointly linear in the initial condition response and the forced response. This is of course precisely the case for the ordinary differnetial equations that we studied in Chapter **??**, where the solutions were the sum of the homogeneous and particular solutions.

We now consider a differential equation of the form

$$\dot{x} = Ax + Bu \tag{4.3}$$

where $A \in \mathbb{R}^{n \times n}$ is a square matrix, $B \in \mathbb{R}^n$ is a column vector of length n. (In the case of a multi-input systems, B becomes a matrix of appropriate dimension.) Equation (4.3) is a system of linear, first order, differential equations with input u and state x. We now show that this system is linear system, in the sense described above.

Theorem 1. *Let $x_{h1}(t)$ and $x_{h2}(t)$ be the solutions of the linear differential equation* (4.3) *with input $u(t) = 0$ and initial conditions $x(0) = x_1$ and x_2, respectively, and and let $x_{p1}(t)$ and $x_{p2}(t)$ be the solutions with initial condition $x(0) = 0$ and inputs $u_1(t), u_2(t) \in \mathbb{R}$. Then the solution of equation* (4.3) *with initial condition $x(0) = \alpha x_1 + \beta x_2$ and input $u(t) = \delta u_1 + \gamma u_2$ and is given by*

$$x(t) = \big(\alpha x_{h1}(t) + \beta x_{h2}(t)\big) + \big(\delta x_{p1}(t) + \gamma x_{p2}(t)\big).$$

Proof. Substitution. □

It follows that since the output is a linear combination of the states (through multiplication by the row vector C), the system is input/output linear as we defined above. As in the case of linear differential equations in a single variable, we define the solution $x_h(t)$ with zero input as the *homogeneous* solution and the solution $x_p(t)$ with zero initial condition as the *particular* solution.

It is also possible to show that if a system is input/output linear in the sense we have described, that it can always be represented by a state space equation of the form (**??**) through appropriate choice of state variables.

The matrix exponential

Although we have shown that the solution of a linear set of input/output differential equations defines a linear input/output system, we have not actually solved for the solution of the system. We begin by considering the homogeneous response, corresponding to the system

$$\dot{x} = Ax \tag{4.4}$$

Recall that for a *scalar* differential equation

$$\dot{x} = ax \qquad x \in \mathbb{R}, a \in \mathbb{R}$$

the solution is given by the exponential

$$x(t) = e^{at}x(0).$$

We wish to generalize this to the vector case, where A becomes a matrix.

We define the *matrix exponential* as the infinite series

$$e^S = I + S + \frac{1}{2}S^2 + \frac{1}{3!}S^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}S^k, \tag{4.5}$$

where $S \in \mathbb{R}^{n \times n}$ is a square matrix and I is the $n \times n$ identify matrix. We make use of the notation

$$S^0 = I \qquad S^2 = SS \qquad S^n = S^{n-1}S,$$

which defines what we mean by the "power" of a matrix. Equation (**??**) is easy to remember since it is just the Taylor series for the scalar exponential, applied to the matrix S. It can be shown that the series in equation (4.5) convergies for any matrix $S \in \mathbb{R}^{n \times n}$ in the same way that the normal exponential is define for any scalar $a \in \mathbb{R}$.

Theorem 2. *The solution to the homogeneous system of differential equation* (4.4) *is given by*

$$x(t) = e^{At}x(0).$$

Proof. Substitute the solution into the differential equation. □

The form of the solution immediately allows us to see that the solution is linear in the initial condition. In particular, if x_{h1} is the solution to equation (4.4) with initial condition $x(0) = x_1$ and x_{h2} with initial condition x_2, then the solution with initial condition $x(0) = \alpha x_1 + \beta x_2$ is given by

$$x(t) = e^{At}\big(\alpha x_1 + \beta x_2\big) = \big(\alpha e^{At}x_1 + \beta e^{At}x_2\big) = \alpha x_{h1}(t) + \beta x_{h2}(t)$$

Similary, we see that the corresponding output is given by

$$y(t) = Cx(t) = \alpha y_{h1}(t) + \beta y_{h2}(t),$$

where y_1 and y_2 are the outputs corresponding to x_{h1} and x_{h2}.

The convolution integral

We now return to the general input output case in equationeq:LinearSystems:linsys, repeated here:

$$\dot{x} = Ax + Bu$$
$$y = Dx$$

Using the matrix exponential the solution to (4.3) can be written as

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{4.6}$$

To prove this we differentiate both sides and use the property (**??**) of the matrix exponential. This gives

$$\frac{dx}{dt} = Ae^{At}x(0) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau + Bu(t) = Ax + Bu$$

which prove the result. Notice that the calculation is essentially the same as for proving the result for a first order equation.

It follows from Equations (4.3) and (4.6) that the input output relation is given by

$$y(t) = Ce^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{4.7}$$

It is easy to see from here that the input/output systems is jointly linear in both the initial conditions and the state: this follows from the linearity of matrix/vector multiplication and integration.

Equation (**??**) is called the *convolution equation* and it represents the general form of the solution of a system of coupled linear differential equations. We see immediately that the dynamics of the system, as characterized by the matrix A play a critical role in both the stability and performance of the system. Indeed, the matrix exponential describes *both* what happens when we perturb the initial condition and how the system responds to inputs.

Stability of linear systems

Impulse, step and frequency response

4.3 Linearization

Another source of linear system models is through the *approximation* of a nonlinear system by a linear one. These approximations are aimed at studying the local behavior of a system, where the nonlinear effects are expected to be small. In this section we discuss how to locally approximate a system by its linearization and what can be said about the approximation in terms of stability.

Jacobian linearizations of nonlinear systems

Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \qquad && x \in \mathbb{R}^n, u \in \mathbb{R} \\ y &= h(x,u) \qquad && y \in \mathbb{R} \end{aligned} \tag{4.8}$$

with an equilibrium point at $x = x_e$, $u = u_e$. Without loss of generality, we assume that $x_e = 0$ and $u_e = 0$, although initially we will consider the general case to make the shift of coordinates explicit.

In order to study the *local* behavior of the system around the equilibrium point (x_e, u_e), we suppose that $x - x_e$ and $u - u_e$ are both small, so that nonlinear perturbations around this equilibrium point can be ignored compared with the (lower order) linear terms. This is roughly the same type of argument that is used when we do small angle approximations, replacing $\sin\theta$ with θ and $\cos\theta$ with 1.

In order to formalize this idea, we define a new set of state variables z, inputs v, and outputs w:

$$z = x - x_e \qquad v = u - u_e \qquad w = y - h(x_e).$$

These variables are all close to zero when we are near the equilibrium point, and so in these variables the nonlinear terms can be thought of as the higher order terms in a Taylor series expansion of the relevant vector fields (assuming for now that these exist).

Example 8. Consider a simple scalar system,

$$\dot{x} = 1 - x^3 + u.$$

The point $(x_e, u_e) = (1, 0)$ is an equlibrium point for this system and we can thuse set

$$z = x - 1 \qquad v = u.$$

We can now compute the equations in these new coordinates as

$$\begin{aligned} \dot{z} &= \frac{d}{dt}(x-1) = \dot{x} \\ &= 1 - x^3 + u = 1 - (z+1)^3 + v \\ &= 1 - z^3 - 3z^2 - 3Z - 1 + v = -3z - 3z^2 - z^3 + v. \end{aligned}$$

If we now assume that x stays very close to the equilibrium point, then $z = x - x_e$ is small and $z \ll z^2 \ll z^3$. We can thus *approximate* our system by a *new* system

$$\dot{z} = -3z + v.$$

This set of equations should give behavior that is close to that of the original system as long as z remains small.

More formally, we define the *Jacobian linearization* of the nonlinear system (4.8) is defined as

$$\begin{aligned} \dot{z} &= Az + Bv \\ \dot{w} &= Cz + Dv, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} A &= \left.\frac{df(x,u)}{dx}\right|_{(x_e,u_e)} & B &= \left.\frac{df(x,u)}{du}\right|_{(x_e,u_e)} \\ C &= \left.\frac{dh(x,u)}{dx}\right|_{(x_e,u_e)} & D &= \left.\frac{dh(x,u)}{du}\right|_{(x_e,u_e)} \end{aligned} \tag{4.10}$$

The system (4.9) approximates the original system (4.8) when we are near the equilibrium point that the system was linearized about.

It is important to note that we can only define the linearization of a system about an equilibrium point. To see this, consider a polynomial system

$$\dot{x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + u,$$

where $a_1 \neq 0$. There are a familty of equiblrium points for this system given by $(x_e, u_e) = (-(a_0 + u_0)/a_1, u_0)$ and we can linearize around any of these. Suppose instead that we try to linearize around the origin of the system, $x = 0$, $u = 0$. If we drop the higher order terms in x, then we get

$$\dot{x} = a_0 + a_1 x + u,$$

which is *not* the Jacobian linearization if $a_0 \neq 0$. The constant term must be kept and this is not present in (4.9). Furthermore, even if we kept the constant term in the approximate model, the system would quickly move away from this point (since it is "driven" by the constant term a_0) and hence the approximation could soon fail to hold.

Local stability of nonlinear systems

Having constructed an approximate model around an equilibrium point, we can now ask to what extent this model predicts the behavior of the original nonlinear system. The following theorem gives a partial answer for the case of stability.

Theorem 3. *Consider the system* (4.8) *and let* $A(\cdot)$ *be defined as in equation* (**??**). *If the real part of the eigenvalues of* A *are strictly less than zero, then* x_e *is an asymptotically stable equilibrium point of* (4.8).

This theorem proves that *global* uniform asymptotic stability of the linearization implies *local* uniform asymptotic stability of the original nonlinear system. The estimates provided by the proof of the theorem can be used to give a (conservative) bound on the domain of attraction of the origin. Systematic techniques for estimating the bounds on the regions of attraction of equilibrium points of nonlinear systems is an important area of research and involves searching for the "best" Lyapunov functions.

Feedback linearization

4.4 Discrete time linear systems

4.5 Further Reading