Chapter 3

Stability and Performance

Given a model of a system, we can talk about the stability of equilibrium points (or other dynamical features) and discuss methods of defining the performance of an input/output system. The goal of this chapter is to describe the different types of local stability of an equilibrium point and discuss the difference between local stability, global stability, and related concepts. We also describe performance measures for (controlled) systems, including transients and steady state response.

3.1 Qualitative features of nonlinear dynamical systems

We begin by given a description of some of the qualitative features of nonlinear dynamical systems, focusing on ODE representations.

Systems of ODEs

In the last chapter, we saw that one of the methods of modeling dynamical systems is through the use of ordinary differential equations. A state space, input/ouput system has the form

$$\dot{x} = f(x, u)$$
$$y = h(x),$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, and $y \in \mathbb{R}^p$ is the output. The smooth maps $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ represent the dynamics and measurement for the system. We will focus in this text on single input, single output (SISO) sytems, for which $p = q = 1$.

We begin by studying the stability of the closed loop system. That is, we assume that a feedback law, $u = \alpha(x)$ has been defined and hence we are left with the dynamics and our system of ordinary differential equations becomes

$$\dot{x} = f(x, \alpha(x)) = F(x). \tag{3.1}$$

We write $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ for the state vector. Note that we do not bother to write the vector x and differently than a scalar variable. It will generally be clear from context whether a variable is a vector or scalar quantity.

When an equation is written in the form of equation (3.1), we say that it is in *state space form*. Higher order differential equations, such as those given in the last chapter, can always be converted to state space form by defining $x = (y, \dot{y}, \cdots, y^{(n-1)})$.

MATLAB Example 1 *(Simulating ODEs in MATLAB).* MATLAB provides several tools for representing, simulating, and analyzing ordinary differential equations of the form in equation 3.1. To define an ODE in MATLAB, we define a function representing the right hand side of equation (3.1):

```
function dxdt = fode(t, x)
dxdt = [
  F1(x);
  F2(x);
  ...
  Fn(x);
];
```

Each function `Fi(x)` takes a (column) vector x and returns the ith element of the differential equation. The first argument, t, represents the current time and allows for the possibility of time-varying differential equations, in which the right hand side of the ODE in equation (3.1) depends explicitly on time.

ODEs define in this fashion can be simulated by using the MATLAB `ode45` command:

```
ode45('file', [0,T], [x10, x20, ..., xn0])
```

The first argument is the name of the file containing the ODE declaration, the second argument gives the time interval over which the simulation should
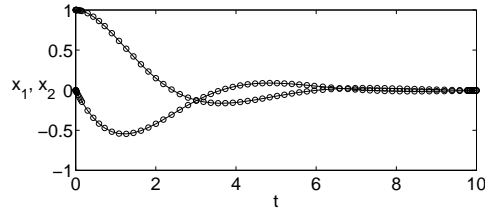
Figure 3.1: Simulation of a damped oscillator, as produced by MATLAB.

be performed and the final argument gives the vector of initial conditions. The default action of the `ode45` command is to plot the time response of each of the states of the system.

Example 5 *(Damped oscillator)*. Consider a damped oscillator (mass, spring, damper systems), as derived in the last chapter. The equations of motion for the system are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2.$$

In vector form, the right hand side can be written as

$$F(x) = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}$$

The output of a MATLAB simulation for this system is shown in Figure **??**.

Phase portraits

A convenient way to understand the qualitative dynamics of dynamical systems with state x $in\mathbb{R}^2$ is to plot the *phase portrait* of the system. Phase portraits can generally only be plotted for two dimensions (or pl"planar") dynamical systems, but they often give insight into the dynamics of much more complicated systems.

We start by introducing the concept of a vector field. For a system of ordinary differential equations

$$\dot{x} = F(x)$$

the right hand side of the differential equation defines at every $x \in \mathbb{R}^n$ a velocity. This velocity tells us how x changes and can be represented as a vector $f(x) \in \mathbb{R}^n$. For planar dynamical systems, we can plot these vectors
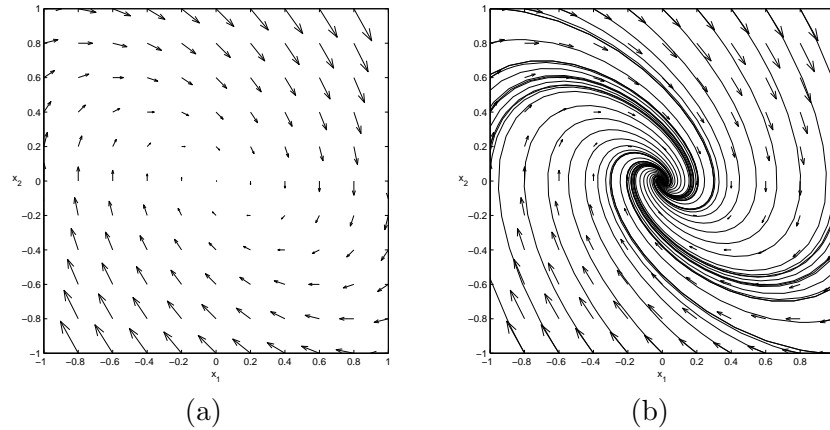
Figure 3.2: Vector field plot (a) and phase portrait (b) for a damped oscillator. This plots were produced using the `phaseplot` command in MATLAB.

at a grid of points in the plane and obtain a visual image of the dynamics of the system, as shown in Figure 3.2a.

A phase portrait is constructed by plotting the flow of the vector field corresponding to the planar dynamical system. That is, for a set of initial conditions $x_0 \in \mathbb{R}^n$, we plot the solution of the differential equation in the plane $\mathbb{R}^2$. This corresponds to following the arrows at each point in the phase plane and drawing the resulting trajectory. By plotting the resulting trajectories for several different initial conditions, we obtain a phase portrait, as show in Figure 3.2b.

Phase portraits give us insight into the dynamics of the system by showing us the trajectories plotted in the (two dimensional) state space of the system. For example, we can see whether all trajectories tend to a single point as time increase or whether there are more complicted behaviors as the system evolves. In the example in Figure 3.2, we see that for all intitial conditions, the system approaches the origin ($x0$). This is consistent with our simulation in Figure 3.1, but it allows us to infer the behavior for all initial conditions rather than a single initial condition. However, the phase portrait does not readily tell us the rate of change of the states (although this can be inferred from the length of the arrows in the vector field plot).

Equilibrium points

An *equilibrium point* of a dynamical system represents a stationary condition for the dynamics. We say that at state x_e is an equilibrium point for a

dynamical system

$$\dot{x} = F(x)$$

if $F(x_e) = 0$. If a dynamical system has an initial condition $x(0) = x_e$ then it will stay at the equilibrium point: $x(t) = x_e$ for all $t > 0$.

Equilibrium points are one of the most important features of a dynamical system since they define the states corresponding to constant operating conditions. A dynamical system can have zero, one or more equilibrium points.

Example 6. Mechanical pendulum One example of a system with multiple equilibrium points is the simple pendulum. The dynamics of this system were derived in the previous chapter and are given by

$$m\ddot{\theta} = mglsin(\theta)$$

where *theta* is the angle that the pendulum makes with respect to the verticle ($\theta = 0$ corresponding to pointing down), m is the mass of the pendulum, l is the length, and g is the gravitational constant.

We can write this system in state space form by defining $x = (\theta, \dot{\theta})$ so that

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -gl \sin(x_1) \end{bmatrix}.$$

The equilibrium points for the system are given by

$$x_e = \begin{bmatrix} 0 \\ \pm n\pi \end{bmatrix}$$

where $n = 0, 1, 2, \ldots$. The equilibrium points for n even correspond to the pendulum is hanging down and those for n odd correspond to the pendulum pointing up. A phase portrait for this system is shown in Figure 3.3.

3.2 Stability

The stability of an equilibrium point determines whether or not solutions nearby the equilibrium point remain nearby, get closer, or get further away.

Definitions

An equilibrium point is *stable* if initial conditions that start near an equilibrium point stay near that equilibrium point. Formally, we say that an equilibrium point x_e is stable if for all $\epsilon > 0$, there exists an $\delta > 0$ such that

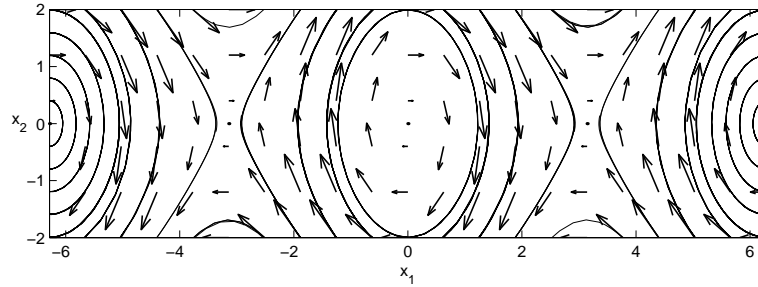$$\|x(0) - x_e\| < \delta \quad \implies \quad \|x(t) - x_e\| < \epsilon \quad \text{for all } t > 0.$$

Figure 3.3: Phase portrait for a simple pendulum. The equilibrium points are marked by solid dots along the $x_2 = 0$ line.
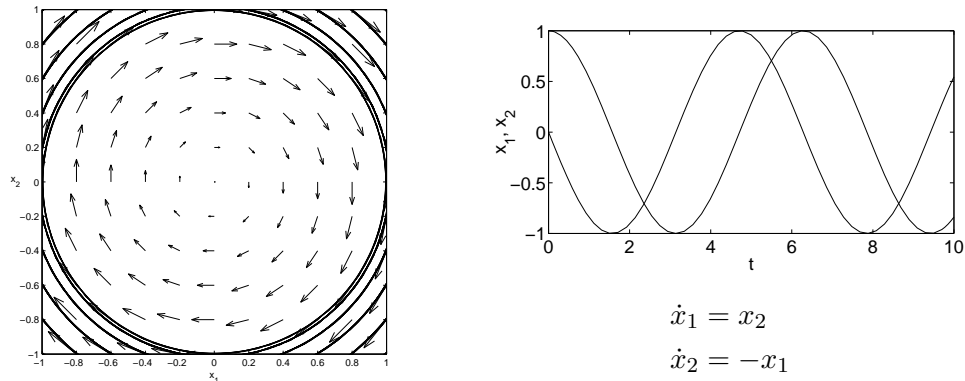

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

Figure 3.4: Phase portrait and time domain simulation for a system with a single stable equilibrium point.

Note that this definition does not imply that $x(t)$ gets closer to x_e as time increases, but rather just that it stays nearby. Furthermore, the value of δ may depend on ϵ, so that if we wish to stay very close to the equilibrium point, we may have to start very, very close ($\delta \ll \epsilon$). This type of stability is sometimes called stability "in the sense of Lyapunov".

An example of a stable equilibrium point is shown in Figure **??**. From the phase portrait, we see that if we start near the equilibrium then we stay near the equilibrium. Indeed, for this example, given any ϵ that defines the range of possible initial conditions, we can simply choose $\delta = \epsilon$ to satisfy the definition of stability.

An equilibrium point is *asymptotically stable* if it is stable and also $x(t) \to 0$ and $t \to \infty$. This corresponds to the case where all nearby trajectories converge to the equilibrum point for large time. Figure **??** shows an
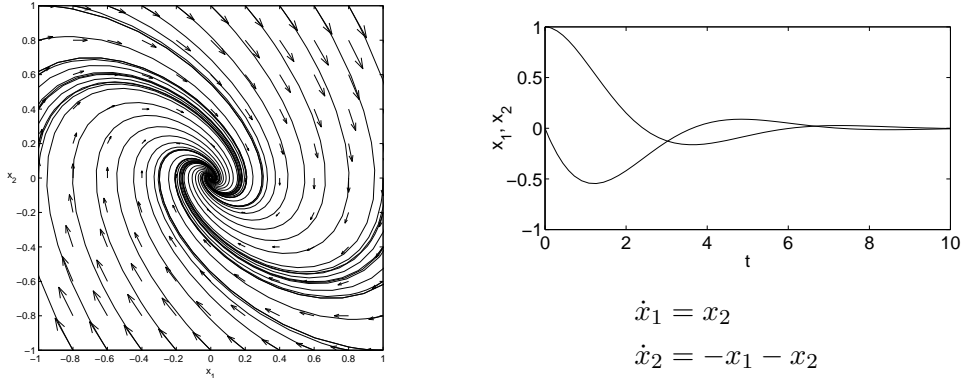
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$

Figure 3.5: Phase portrait and time domain simulation for a system with a single asymptotically stable equilibrium point.

example of an asymptotically stable equilibrium point. Note from the phase portraints that all trajectories not only stay near the equilibrium point at the origin, but they all approach the origin as t gets large (the directions of the arrows on the phase plot show the direction in which the trajectories move).

An equilibrium point is *unstable* if it is not stable. More specifically, we say that an equilibrium point is unstable if given any $\epsilon > 0$, there always exists an initial condition $x(0)$ with $\|x(0) - x_e\| < \epsilon$ such that $x(t)$ is arbitrarily large as time increases. An example of an unstable equilibrium point is shown in Figure 3.6.

For planar dynamical systems, equilibrium points have been assigned names based on their stability type. An asymptotically stable equilibrium point is called a *sink* or sometimes an *attractor*. An unstable equilibrium point can either be a *source*, if all trajectories lead away from the equilibrium point, or a *saddle*, if some trajectories lead to the equilibrium point and others move away (this is the situation pictures in Figure 3.5. Finally, an equilibrium point which is stable but not asymptotically stable (such as the one in Figure 3.4 is called a *center*.

Lyapunov functions

Advanced

A powerful tool for determining stability is the use of Lyapunov functions. A *Lyapunov function* $V(x)$ is an energy-like function that can be used to determine stability of a system. Roughly speaking, if we can find a non-negative function that always decreases along trajectories of the system, we

$$\dot{x}_1 = 2x_1 - x_2$$
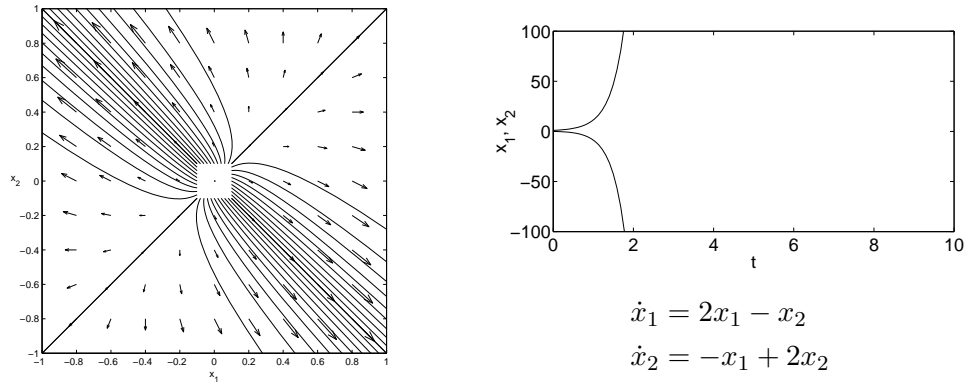$$\dot{x}_2 = -x_1 + 2x_2$$

Figure 3.6: Phase portrait and time domain simulation for a system with a single unstable equilibrium point.

can conclude that the minimum of the function is a stable equilibrium point (locally).

To define this more formally, we make a few definitions. We say that a function $V(x)$ is *positive definite* if there exists a strictly increasing, scalar function α with $\alpha(0) = 0$ such that $V(0) = 0$ and $V(x) \geq \alpha(\|x\|)$. We will often write this as "$V(x) > 0$" (even though $V(0) = 0$). Similarly, a function is *negative definite* if $V(0) = 0$ and $V(x) \leq -\alpha(\|x\|)$. We say tthat a function $V(x)$ is *positive semidefinite* if $V(x)$ can be zero at points other than $x = 0$ but otherwise $V(x)$ is strictly positive. We write this as "$V(x) \geq 0$" and define negative semi-definite functions analogously.

We can now characterize the stability of a system

$$\dot{x} = F(x) \qquad x \in \mathbb{R}^n$$

Let $V(x)$ be a non-negative function on $\mathbb{R}^n$ and let $\dot{V}$ represent the time derivative of V along trajectories of the system dynamics:

$$\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} F(x).$$

The following table characterizes the stability of the origin, $x = 0$:

$$V(x) > 0, \quad \dot{V}(x) \leq 0 \quad \implies \quad x = 0 \text{ is stable}$$
$$V(x) > 0, \quad \dot{V}(x) < 0 \quad \implies \quad x = 0 \text{ is asymptotically stable}$$

If V satisfies one of the conditions above, we say that V is a *Lyapunov function* for the system.

Lyapunov functions are not unique and hence we can use many different methods to find one. Indeed, one of the main difficulties in using Lyapunov functions is finding them.[1] It turns out that Lyapunov functions can always be found for any stable system (under certain conditions) and hence one knows that if a system is stable, a Lyapunov function exists (and vice versa).

Example 7. Consider a planar dynamical system

$$\dot{x}_1 = -x_1 - x_2$$
$$\dot{x}_2 = -x_2$$

We choose as a Lyapunov function candidate the function

$$V(x) = x_1^2 + x_2^2$$

This function is clearly positive definite and has time derivative

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$
$$= -2x_1^2 - 2x_1x_2 - 2x_2^2$$
$$= -(x_1 + x_2)^2 - x_1^2 - x_2^2.$$

Using the table above, we conclude that the origin is asymptotically stable.

3.3 Local Versus Global Behavior

As we have already seen through some of our examples, we can have more than one equilibrium point in a given system and these equilibrium points can have differing stability types. When there is more than one equilibrium point in a system, none of the equilibrium points can be globally stable (since starting at the other equilibrum point, we will necessary to move). In this section we explore more careful the relationship between local and global stability and give some conditions for characterizing stability regions.

Local stability and regions of attraction

A system may contain many equilibrium points and each of these equilibrium points could be *locally* stable. By this we mean that if we perturb the initial condition slighty, then the system stays in the neighborhood of that equilibrium point (or, for asymptotic stability, returns to the equilibrium

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[1]Fortunately, there are systematic tools available for searching for special classes of Lyapunov functions, such as sums of squares [**?**].

point). The definitions of stability that we gave in Section 3.2 reflect this local nature.

The Lyapunov tests that we derived for checking stability were global in nature. That is, we asked that a Lyapunov function satisfy $V > 0$ and $\dot{V} < 0$ for *all* $x \in \mathbb{R}^n$. To check for local stability, it is sufficient to ask that V be *locally* positive definite and $\dot{V}$ locally negative definite. More formally, we say that a $V(x)$ is *locally positive definite* (lpd) if there exists a strictly increasing, scalar function α with $\alpha(0) = 0$ and scuh that $V(0) = 0$ and $V(x) \geq \alpha(\|x\|)$ for all x in some open neighborhood $\mathcal{N} \subset \mathbb{R}^n$ containing the equilibrium point $x_e = 0$.

With this definition, the characterizations of stability and asymptototic stability carry through to the local case.

We can also define the set of all initial conditions that converge to a given asymptoticallly stable equilibrium point. This set is called the *region of attraction* for the equilibrium point. An example is shown in Figure **??**. In general, computing regions of attraction is extremely difficult.

Limit cycles and other attractors

3.4   System Performance Measures

So far, this chapter has only described the stability characteristics of a systems. While stability is often a desirably feature, stability alone may not be sufficient in many applications. We will want to create feedback systems that quickly react to changes and give high performance in measureable ways. In this section, we consider two measures of performance that were introduced already in the last chapter: step response and frequency response.

Transient response versus steady state response

Step response

We return now to the case of an input/output state space system

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
y &= h(x)
\end{aligned}
\tag{3.2}
$$

where $x \in \mathbb{R}^n$ is the state and $u, y \in \mathbb{R}$ are the input and output. The *step response* of the system 3.2 is defined as the output $y(t)$ starting from zero initial condition (or the appropriate equilibrium point) and given a *step*

Figure 3.7: Sample step response

input:

$$u = \begin{cases} 0 & t = 0 \\ 1 & t > 0 \end{cases}$$

We note that the step input is discontinuous and hence is not physically implementable. However, it is a convenient abstract that is widely used in studying input/output systems.

A sample step response is shown in Figure 3.7. Several terms are used when referring to a step response:

Steady state value The steady state value of a step response is the final level of the output, assuming it converges.

Rise time The rise time is the amount of time required for the signal to go from 5% of its final value to 95% of its final value. It is possible to define other limits as well, but in this book we shall use these percentages unless otherwise indicated.

Overshoot The overshoot is the percentage of the final value by which the signal initially rises above the final value. This usually assumes that future values of the signal do not overshoot the final value by more than this initial transient, otherwise the term can be ambiguous.

Settling time The settling time is the amount of time required for the signal to stay within 5% of its final value for all future times.

Frequency response

The frequency response of an input/output system measures the way in which the system responds to a sinusoidal excitation on one of its inputs. As we have already seen (and will see in more detail later), for linear systems the particular solution associated with a sinusoidal excitation is itself a sinusoid at the same frequency. Hence we can compare the magnitude and phase of the output sinusoid as compared to the input. More generally, if a system has a sinuoidal output response at the same frequency as the input forcing, we can speak of the frequency response.

Frequency response is typically measured in terms of *gain* and *phase* at a given forcing frequency, as illustrated in Figurefig:freqresp. The gain the system at a given frequency is given by the ratio of the amplitude of the

output to that of the input. The phase is given by the the fraction of a period by which the output differs from the input. Thus, if we have an input $u = A_u \sin(\omega t + \psi)$ and output $y = A_y \sin(\omega t + \phi)$, we write

$$\text{gain}(\omega) = \frac{A_y}{A_u} \qquad \text{phase}(\omega) = \phi - \psi.$$

If the phase is positive, we say that the output "leads" the input, otherwise we say it "lags" the input. For linear systems, we will see that the size and phase of the input can be set to unity and zero, respectively, simplifying this formula.

Relating stability to performance

Other performance measures

3.5   Second Order Systems

One class of systems that occurs frequency in the analysis and design of feedback systems are second order, linear differential equations. Because of their ubiquitous nature, it is useful to apply the concepts of this chapter to that specific class of systems and build more intution about the relationship between stability and performance.

3.6   Further Reading

The field of dynamical systems has a rich literature that characterizes the possible features of dynamical systems and describes how parametric changes in the dynamics can lead to topological changes in behavior (these are called *bifurcations*). A very readable introduction to dynamical systems is given by Strogatz [**?**].

3.7   Exercises

1. (Exponential stability)