Appendix A

Linear Algebra

In this appendix we review the notation and concepts from linear algebra that are used in the text.

A.1 Linear spaces and operators

A.2 Jordan Form

Appendix B

Laplace Transforms

The Laplace transform is an essential part of the language of control. Only a few elementary properties are needed for basic control applications. There is a beautiful theory for Laplace transforms which makes it possible to use many powerful tools of the theory of functions of a complex variable to get deep insights into the behavior of systems.

B.1 Basic Concepts

The Laplace transform maps a time function $f : R^+ \to R$ to a function $F = \mathcal{L}f : C \to C$ of a complex variable. It is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)dt. \tag{B.1}$$

The transform has some properties which makes it very well suited to deal with linear systems.

First we observe that the transform is linear because

$$\begin{aligned}\mathcal{L}(af + bg) = aF(s) + bF(s) &= a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt \\ &= \int_0^\infty e^{-st}(af(t) + bg(t))dt = a\mathcal{L}f + b\mathcal{L}g\end{aligned} \tag{B.2}$$

Next we show that the Laplace transform of the derivative of a function is related to the Laplace transform of the function in a very simple way. If $F(s)$ is the Laplace transform of a function $f(t)$ then the transform of the derivative df/dt is given by Next we will calculate the transform of the derivative of a function, i.e. $f'(t) = \frac{df(t)}{dt}$. We have

$$\mathcal{L}\frac{df}{dt} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt = -f(0) + s\mathcal{L}f$$

where the second equality is obtained by integration by parts. We thus obtain the following important formula for the transform of a derivative

$$\mathcal{L}\frac{df}{dt} = s\mathcal{L}f - f(o) = sF(s) - f(0) \tag{B.3}$$

This formula is particularly simple if the initial conditions are zero because *it follows that differentiation of a function corresponds to multiplication of the transform with s.*

Since differentiation corresponds to multiplication with s we can expect that integration corresponds to division by s. This is true as can be seen by calculating the Laplace transform of an integral. We have

$$\begin{aligned}\mathcal{L}\int_0^t f(\tau)d\tau &= \int_0^\infty \Big(e^{-st}\int_0^t f(\tau)d\tau\Big)dt\\ &= -\frac{e^{-st}}{s}\int_0^t e^{-s\tau} f'(\tau)d\tau\Big|_0^\infty + \int_0^\infty \frac{e^{-s\tau}}{s} f(\tau)d\tau = \frac{1}{s}\int_0^\infty e^{-s\tau} f(\tau)d\tau\end{aligned}$$

hence

$$\mathcal{L}\int_0^t f(\tau)d\tau = \frac{1}{s}\mathcal{L}f = \frac{1}{s}F(s) \tag{B.4}$$

Integration of a time function thus corresponds to dividing the Laplace transform by s. This is consistent with the fact that differentiation of a time function corresponds to multiplication of the transform with s.

Consider a linear time-invariant system where the initial state is zero. The relation between the input u and the output y of is given by the convolution integral

$$y(t) = \int_0^\infty g(t-\tau)u(\tau)d\tau$$

see (2.20). We will now consider the Laplace transform of such an expression. We have

$$\begin{aligned}Y(s) &= \int_0^\infty e^{-st} y(t)dt = \int_0^\infty e^{-st}\int_0^\infty g(t-\tau)u(\tau)d\tau dt\\ &= \int_0^\infty\int_0^t e^{-s(t-\tau)}e^{-s\tau} g(t-\tau)u(\tau)d\tau dt\\ &= \int_0^\infty e^{-s\tau}u(\tau)d\tau\int_0^\infty e^{-st} g(t)dt = G(s)U(s)\end{aligned}$$

The result can be written as $Y(s) = G(s)U(s)$ where G, U and Y are the Laplace transforms of g, u and y. The system theoretic interpretation is that the Laplace transform of the output of a linear system is a product of two terms, the Laplace transform of the input $U(s)$ and the Laplace transform of the impulse response of the system $G(s)$. A mathematical interpretation is that the Laplace transform of a convolution is the product of the transforms of the functions that are convoluted. the fact that the formula $Y(s) = G(s)U(s)$ is much simpler than a convolution is one reason why Laplace transforms have become popular in control.

B.2 Additional Properties

For the sake of completeness we will give a few Laplace transforms and some of their properties.

The transform of $f_1(t) = e^{-at}$ is given by

$$F_1(s) = \int_0^\infty e^{-(s+a)t} dt = -\frac{1}{s+a} e^{-st}\Big|_0^\infty = \frac{1}{s+a}$$

By setting $a = 0$ we obtain the Laplace transform for a step. Differentiating the above equation we find that the transform of the function $f_2(t) = te^{-at}$ is

$$F_2(s) = \frac{1}{(s+a)^2}$$

Repeated differentiation shows that the transform of the function $f_3(t) = t^n e^{-at} 0n!$ is

$$F_3(s) = \frac{1}{(s+a)^{n+1}}$$

Setting $a = 0$ in f_1 we find that the transform of the unit step function $f_4(t) = 1$ is

$$F_4(s) = \frac{1}{s}$$

Similarly we find by setting $a = 0$ in f_3 that the transform of $f_5 = t^n/n!$ is

$$F_5(s) = \frac{1}{s^{n+1}}$$

Setting $a = ib$ in f_1 we find that the transform of $f(t) = e^{-ibt} = \cos bt - i \sin bt$ is

$$F(s) = \frac{1}{s+ib} = \frac{s-ib}{s^2+b^2} = \frac{s}{s^2+b^2} - i\frac{b}{s^2+b^2}$$

Separating real and imaginary parts we find that the transform of $f_6(t) = \sin bt$ and $f_7(t) = \cos bt$ are

$$F_6(t) = \frac{b}{s^2 + b^2}, \quad F_7(t) = \frac{s}{s^2 + b^2}$$

Proceeding in this way it is possible to build up tables of transforms that are useful for hand calculations.

The behavior of the time function for small arguments is governed by the behavior of the Laplace transform for large arguments. Or more precisely that the value of $f(t)$ for small t is thus equal to $sF(s)$ for large s. This is shown as follows.

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \int_0^\infty se^{-st} f(t)dt = \lim_{s\to\infty} \int_0^\infty e^{-v} f(\frac{v}{s})dv = f(0)$$

This result, which requires that the limit exists, is called is the initial value theorem. The converse is also true, we have

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \int_0^\infty se^{-st} f(t)dt = \lim_{s\to 0} \int_0^\infty e^{-v} f(\frac{v}{s})dv = f(\infty)$$

The value of $f(t)$ for large t is thus equal to $sF(s)$ for small s, the result is called the final value theorem. These properties are very useful for qualitative assessment of a time functions and Laplace transforms.