5 Transfer functions

Associated with the linear system (input u, output y) governed by the ODE

$$\begin{aligned} y^{[n]}(t) + a_1 y^{[n-1]}(t) + \cdots + a_{n-1} y^{[1]}(t) + a_n y(t) \\ = b_0 u^{[n]}(t) + b_1 u^{[n-1]}(t) + \cdots + b_{n-1} u^{[1]}(t) + b_n u(t) \end{aligned} \tag{35}$$

we write "in transfer function form"

$$Y = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} U \tag{36}$$

The expression in (36) is interpreted to be equivalent to the ODE in (35), just a different way of writing the coefficients. The notation in (36) is suggestive of multiplication, and we will see that such an interpretation is indeed useful. The function

$$G(s) := \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

is called *the transfer function from u to y*, and is sometimes denoted $G_{u \to y}(s)$ to indicate this. At this point, the expression in equation (36),

$$Y = G_{u \to y}(s) U$$

is nothing more than a new notation for the differential equation in (35). The differential equation has a well-defined meaning, and we understand what each term represents, and the meaning of the equality sign, =. In the transfer function expression, (36), there is no specific meaning to the individual terms, or the equality symbol. The expression, as a whole, simply means the differential equation it is associated with.

In this section, we will see that, in fact, we can assign proper equality, and make algebraic substitutions and manipulations of transfer function expressions, which will aid our manipulation of linear differential equations. But all of that requires proof, and that is the purpose of this section.

5.1 Linear Differential Operators (LDOs)

Note that in the expression (36), the symbol s plays the role of $\frac{d}{dt}$, and higher powers of s mean higher order derivatives, ie., s^k means $\frac{d^k}{dt^k}$. If z is a function of time, let the notation

$$\left[b_0 \frac{d^n}{dt^n} + b_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + b_{n-1} \frac{d}{dt} + b_n \right](z) := b_0 \frac{d^n z}{dt^n} + b_1 \frac{d^{n-1} z}{dt^{n-1}} + \cdots + b_{n-1} \frac{dz}{dt} + b_n z$$

We will call this type of operation a *linear differential operation*, or LDO. For the purposes of this section, we will denote these by capital letters, say

$$\begin{aligned} L &:= \left[\tfrac{d^n}{dt^n} + a_1 \tfrac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1} \tfrac{d}{dt} + a_n \right] \\ R &:= \left[b_0 \tfrac{d^n}{dt^n} + b_1 \tfrac{d^{n-1}}{dt^{n-1}} + \cdots + b_{n-1} \tfrac{d}{dt} + b_n \right] \end{aligned}$$

Using this shorthand notation, we can write the original ODE in (39) as

$$L(y) = R(u)$$

With each LDO, we naturally associate a polynomial. Specifically, if

$$L := \left[\frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}\frac{d}{dt} + a_n\right]$$

then $p_L(s)$ is defined as

$$p_L(s) := s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

Similarly, with each polynomial, we associate an LDO – if

$$q(s) := s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m$$

then L_q is defined as

$$L_q := \left[\frac{d^m}{dt^m} + b_1 \frac{d^{m-1}}{dt^{m-1}} + \cdots + b_{m-1}\frac{d}{dt} + b_m\right]$$

Therefore, if a linear system is governed by an ODE of the form $L(y) = R(u)$, then the transfer function description is simply

$$Y = \frac{p_R(s)}{p_L(s)} U$$

Similarly, if the transfer function description of a system is

$$V = \frac{n(s)}{d(s)} W$$

then the ODE description is $L_d(v) = L_n(w)$.

5.2 Algebra of Linear differential operations

Note that two successive linear differential operations can be done in either order. For example let

$$L_1 := \left[\frac{d^2}{dt^2} + 5\frac{d}{dt} + 6\right]$$

and

$$L_2 := \left[\frac{d^3}{dt^3} - 2\frac{d^2}{dt^2} + 3\frac{d}{dt} - 4\right]$$

Then, on a differentiable signal z, simple calculations gives

$$\begin{aligned}
L_1\left(L_2(z)\right) &= \left[\frac{d^2}{dt^2} + 5\frac{d}{dt} + 6\right]\left(\left[\frac{d^3}{dt^3} - 2\frac{d^2}{dt^2} + 3\frac{d}{dt} - 4\right](z)\right] \\
&= \left[\frac{d^2}{dt^2} + 5\frac{d}{dt} + 6\right]\left(z^{[3]} - 2\ddot{z} + 3\dot{z} - 4z\right) \\
&= z^{[5]} - 2z^{[4]} + 3z^{[3]} - 4z^{[2]} \\
&\qquad\qquad 5z^{[4]} - 10z^{[3]} + 15z^{[2]} - 20z^{[1]} \\
&\qquad\qquad\qquad 6z^{[3]} - 12z^{[2]} + 18z^{[1]} - 24z \\
&= z^{[5]} + 3z^{[4]} - z^{[3]} - z^{[2]} - 2z^{[1]} - 24z
\end{aligned}$$

which is the same as

$$\begin{aligned}
L_2\left(L_1(z)\right) &= \left[\frac{d^3}{dt^3} - 2\frac{d^2}{dt^2} + 3\frac{d}{dt} - 4\right]\left(\left[\frac{d^2}{dt^2} + 5\frac{d}{dt} + 6\right](z)\right] \\
&= \left[\frac{d^3}{dt^3} - 2\frac{d^2}{dt^2} + 3\frac{d}{dt} - 4\right]\left(z^{[2]} + 5\dot{z} + 6z\right) \\
&= z^{[5]} + 5z^{[4]} + 6z^{[3]} \\
&\qquad\qquad -2z^{[4]} - 10z^{[3]} - 12z^{[2]} \\
&\qquad\qquad\qquad z^{[3]} + 15z^{[2]} + 18z^{[1]} \\
&\qquad\qquad\qquad\qquad -4z^{[2]} - 20z^{[1]} - 24z \\
&= z^{[5]} + 3z^{[4]} - z^{[3]} - z^{[2]} - 2z^{[1]} - 24z
\end{aligned}$$

This equality is easily associated with the fact that multiplication of polynomials is a commutative operation, specifically

$$\begin{aligned}
\left(s^2 + 5s + 6\right)\left(s^3 - 2s^2 + 3s - 4\right) &= \left(s^3 - 2s^2 + 3s - 4\right)\left(s^2 + 5s + 6\right) \\
&= s^5 + 3s^4 - s^3 - s^2 - 2s + 24
\end{aligned}$$

Since composition of two linear differential operations behaves like polynomial multiplication, we sometimes notate the "product" L_1L_2 to be composition, in other words, given LDOs L_1 and L_2, the LDO L_1L_2 is defined by composition, and on a differentiable signal z, it is

$$\left[L_1L_2\right](z) := L_1\left(L_2(z)\right)$$

We will often use the notation $[L_1 \circ L_2]$ to also denote this composition of LDOs.

Similarly, if L_1 and L_2 are LDOs, then the sum $L_1 + L_2$ is an LDO defined by its operation on a signal z as $[L_1 + L_2](z) := L_1(z) + L_2(z)$.

It is clear that the following manipulations are always true for every differentiable signal z,

$$L_1\left(L_2(z)\right) + L_3\left(L_4(z)\right) = \left(L_1L_2 + L_3L_4\right)(z)$$

and

$$L\left(z_1 + z_2\right) = L\left(z_1\right) + L\left(z_2\right)$$

and

$$\left[L_1 \circ L_2\right](z) = \left[L_2 \circ L_1\right](z)$$

In terms of LDOs and their associated polynomials, we have the relationships

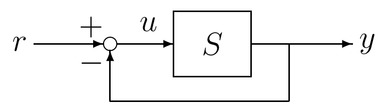
$$\begin{aligned}
p_{[L_1+L_2]}(s) &= p_{L_1}(s) + p_{L_2}(s) \\
p_{[L_1\circ L_2]}(s) &= p_{L_1}(s)p_{L_2}(s)
\end{aligned}$$

In the next several subsections, we derive the LDO representation of an interconnection from the LDO representation of the subsystems.

5.3 Feedback Connection

The most important interconnection we know of is the basic feedback loop. It is also the easiest interconnection for which we derive the differential equation governing the interconnection from the differential equation governing the components.

Consider the simple unity-feedback system shown below



Assume that system S is described by the LDO $L(y) = D(u)$. The feedback interconnection yields $u(t) = r(t) - y(t)$. Eliminate u by substitution, yielding an LDO relationship between r and y

$$L(y) = D(r - y) = D(r) - D(y)$$

This is rearranged to the closed-loop LDO

$$(L + D)(y) = D(r).$$

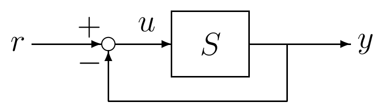
That's a pretty simple derivation. Based on the ODE description of the closed-loop, we can immediately write the closed-loop transfer function,

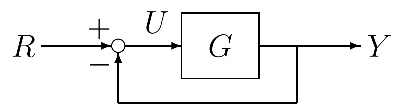
$$\begin{aligned} Y &= \frac{p_D(s)}{p_{[L+D]}(s)} R \\ &= \frac{p_D(s)}{p_L(s) + p_D(s)} R. \end{aligned}$$

Additional manipulation leads to further interpretation. Let $G(s)$ denote the transfer function of S, so $G = \frac{p_D(s)}{p_L(s)}$. Then

$$\begin{aligned} Y &= \frac{p_D(s)}{p_L(s) + p_D(s)} R \\ &= \frac{\frac{p_D(s)}{p_L(s)}}{1 + \frac{p_D(s)}{p_L(s)}} R \\ &= \frac{G(s)}{1 + G(s)} R \end{aligned}$$

This can be interpreted rather easily. Based on the original system interconnection, redraw, replacing signals with their capital letter equivalents, and replacing the system S with its transfer function G. This is shown below.





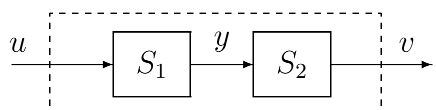
The diagram on the right is interpreted as a diagram of the equations $U = R - Y$, and $Y = GU$. Note that manipulating these as though they are arithmetic expressions gives

$$\begin{aligned}
&Y = G(R - Y) && \text{after substituting for } U \\
&(1 + G)Y = GR && \text{moving } GY \text{ over to left} - \text{hand} - \text{side} \\
&Y = \tfrac{G}{1+G}R && \text{solving for } Y.
\end{aligned}$$

This is is precisely what we want!

5.4 Cascade Connection

Suppose that we have two linear systems, as shown below,



with S_1 governed by

$$y^{[n]}(t) + a_1 y^{[n-1]}(t) + \cdots + a_n y(t) = b_0 u^{[n]}(t) + b_1 u^{[n-1]}(t) + \cdots + b_n u(t)$$

and S_2 governed by

$$v^{[m]}(t) + c_1 v^{[m-1]}(t) + \cdots + c_m v(t) = d_0 y^{[m]}(t) + d_1 y^{[m-1]}(t) + \cdots + d_m y(t)$$

Let $G_1(s)$ denote the transfer function of S_1, and $G_2(s)$ denote the transfer function of S_2. Define the differential operations

$$L_1 := \left[\frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}\frac{d}{dt} + a_n\right]$$

$$R_1 := \left[b_0 \frac{d^n}{dt^n} + b_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + b_{n-1}\frac{d}{dt} + b_n\right]$$

and

$$L_2 := \left[\frac{d^m}{dt^m} + c_1 \frac{d^{m-1}}{dt^{m-1}} + \cdots + c_{m-1}\frac{d}{dt} + c_m\right]$$

$$R_2 := \left[d_0 \frac{d^m}{dt^m} + d_1 \frac{d^{m-1}}{dt^{m-1}} + \cdots + d_{m-1}\frac{d}{dt} + d_m\right]$$

Hence, the governing equation for system S_1 is $L_1(y) = R_1(u)$, while the governing equation for system S_2 is $L_2(v) = R_2(y)$. Moreover, in terms of transfer functions, we have

$$G_1(s) = \frac{p_{R_1}(s)}{p_{L_1}(s)}, \qquad G_2(s) = \frac{p_{R_2}(s)}{p_{L_2}(s)}$$

Now, apply the differential operation R_2 to the first system, leaving

$$R_2\left(L_1(y)\right) = R_2\left(R_1(u)\right)$$

Apply the differential operation L_1 to system 2, leaving

$$L_1\left(L_2(v)\right) = L_1\left(R_2(y)\right)$$

But, in the last section, we saw that two linear differential operations can be applied in any order, hence $L_1\left(R_2(y)\right) = R_2\left(L_1(y)\right)$. This means that the governing differential equation for the cascaded system is

$$L_1\left(L_2(v)\right) = R_2\left(R_1(u)\right)$$

which can be rearranged into

$$L_2\left(L_1(v)\right) = R_2\left(R_1(u)\right)$$

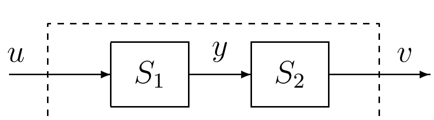
or, in different notation

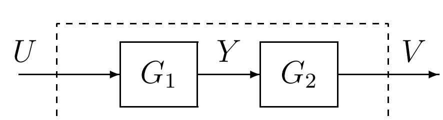
$$\left[L_2 \circ L_1\right](v) = \left[R_2 \circ R_1\right](u)$$

In transfer function form, this means

$$\begin{aligned} V &= \frac{p_{[R_2 \circ R_1]}(s)}{p_{[L_2 \circ L_1]}(s)} U \\ &= \frac{p_{R_2}(s) p_{R_1}(s)}{p_{L_2}(s) p_{L_1}(s)} U \\ &= G_2(s) G_1(s) U \end{aligned}$$

Again, this has a nice interpretation. Redraw the interconnection, replacing the signals with the capital letter equivalents, and the systems by their transfer functions.





The diagram on the right depicts the equations $Y = G_1 U$, and $V = G_2 Y$. Treating these as arithmetic equalities allows substitution for Y, which yields $V = G_2 G_1 U$, as desired.

Example: Suppose S_1 is governed by

$$\ddot{y}(t) + 3\dot{y}(t) + y(t) = 3\dot{u}(t) - u(t)$$

and S_2 is governed by

$$\ddot{v}(t) - 6\dot{v}(t) + 2v(t) = \dot{y}(t) + 4y(t)$$

Then for S_1 we have

$$L_1 = \left[\frac{d^2}{dt^2} + 3\frac{d}{dt} + 1\right], \qquad R_1 = \left[3\frac{d}{dt} - 1\right], \qquad G_1(s) = \frac{3s-1}{s^2+3s+1}$$

while for S_2 we have

$$L_2 = \left[\frac{d^2}{dt^2} - 6\frac{d}{dt} + 2\right], \qquad R_2 = \left[\frac{d}{dt} + 4\right], \qquad G_2(s) = \frac{s+4}{s^2-6s+2}$$

The product of the transfer functions is easily calculated as

$$G(s) := G_2(s)G_1(s) = \frac{3s^2+11s-4}{s^4-3s^3-15s^2+2}$$

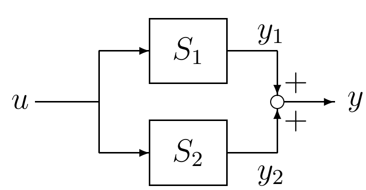
so that the differential equation governing u and v is

$$v^{[4]}(t) - 3v^{[3]}(t) - 15v^{[2]}(t) + 2v(t) = 3u^{[2]}(t) + 11u^{[1]}(t) - 4u(t)$$

which can also be verified again, by direct manipulation of the ODEs.

5.5 Parallel Connection

Suppose that we have two linear systems, as shown below,



System S_1 is governed by

$$y_1^{[n]}(t) + a_1 y_1^{[n-1]}(t) + \cdots + a_n y_1(t) = b_0 u^{[n]}(t) + b_1 u^{[n-1]}(t) + \cdots + b_n u(t)$$

and denoted as $L_1(y_1) = R_1(u)$. Likewise, system S_2 is governed by

$$y_2^{[m]}(t) + c_1 y_2^{[m-1]}(t) + \cdots + c_m y_2(t) = d_0 u^{[m]}(t) + d_1 u^{[m-1]}(t) + \cdots + d_m u(t)$$

and denoted $L_2(y_2) = R_2(u)$.

Apply the differential operation L_2 to the governing equation for S_1, yielding

$$L_2\left(L_1(y_1)\right) = L_2\left(R_1(u)\right) \tag{37}$$

Similarly, apply the differential operation L_1 to the governing equation for S_2, yielding and

$$L_1\left(L_2(y_2)\right) = L_1\left(R_2(u)\right)$$

But the linear differential operations can be carried out is either order, hence we also have

$$L_2\left(L_1(y_2)\right) = L_1\left(R_2(u)\right) \tag{38}$$

Add the expressions in (37) and (38), to get

$$\begin{aligned}
L_2\left(L_1(y)\right) &= L_2\left(L_1(y_1 + y_2)\right) \\
&= L_2\left(L_1(y_1)\right) + L_2\left(L_1(y_2)\right) \\
&= L_2\left(R_1(u)\right) + L_1\left(R_2(u)\right) \\
&= \left[L_2 \circ R_1\right](u) + \left[L_1 \circ R_2\right](u) \\
&= \left[L_2 \circ R_1 + L_1 \circ R_2\right](u)
\end{aligned}$$

In transfer function form this is

$$\begin{aligned}
Y &= \frac{p_{[L_2 \circ R_1 + L_1 \circ R_2]}(s)}{p_{[L_2 \circ L_1]}(s)} U \\
&= \frac{p_{[L_2 \circ R_1]}(s) + p_{[L_1 \circ R_2]}(s)}{p_{L_2}(s) p_{L_1}(s)} U \\
&= \frac{p_{L_2}(s) p_{R_1}(s) + p_{L_1}(s) p_{R_2}(s)}{p_{L_2}(s) p_{L_1}(s)} U \\
&= \left[\frac{p_{R_1}(s)}{p_{L_1}(s)} + \frac{p_{R_2}(s)}{p_{L_2}(s)}\right] U \\
&= \left[G_1(s) + G_2(s)\right] U
\end{aligned}$$

So, *the transfer function of the parallel connection is the sum of the individual transfer functions.*

This is **extremely important!** The transfer function of an interconnection of systems is simply the algebraic gain of the closed-loop systems, treating individual subsystems as complex gains, with their "gain" taking on the value of the transfer function.

5.6 General Connection

The following steps are used for a general interconnection of systems, wach governed by a linear differential equation relating the inputs and outputs.

- Redraw the block diagram of the interconnection. Change signals (lowercase) to upper case, and replace each system with its transfer function.
- Write down the equations, in transfer function form, that are implied by the diagram.
- Manipulate the equations as though they are arithmetic expressions. Addition and multiplication commute, and the distributive laws hold.

5.7 Systems with multiple inputs

Associated with the multi-input, single-output linear ODE

$$L(y) = R_1(u) + R_2(w) + R_3(v) \tag{39}$$

we write

$$Y = \frac{p_{R_1}(s)}{p_L(s)}U + \frac{p_{R_2}(s)}{p_L(s)}W + \frac{p_{R_3}(s)}{p_L(s)}V \tag{40}$$

5.8 Problems

1. Find the transfer function from u to y for the systems governed by the differential equations

   (a) $\dot{y}(t) = \frac{1}{\tau}\left[u(t) - y(t)\right]$

   (b) $\dot{y}(t) + a_1 y(t) = b_0\dot{u}(t) + b_1 u(t)$

   (c) $\dot{y}(t) = u(t)$ (explain connection to Simulink icon for integrator...)

   (d) $\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t)$

2. (a) Suppose that the transfer function of a controller, relating reference signal r and measurement y to control signal u is

   $$U = C(s)\left[R - Y\right]$$

   Suppose that the plant has transfer function relating control signal u and disturbance d to output y as

   $$Y = G_3(s)\left[G_1(s)U + G_2(s)D\right]$$

   Draw a simple diagram, and determine the closed-loop transfer functions relating r to y and d to y.

(b) Carry out the calculations for

$$C(s) = K_P + \frac{K_I}{s}, \quad G_1(s) = \frac{E}{\tau s + 1}, \quad G_2(s) = G, \quad G_3(s) = \frac{1}{ms + \alpha}$$

Directly from this closed-loop transfer function calculation, determine the differential equation for the closed-loop system, relating r and d to y.

(c) Given the transfer functions for the plant and controller in (2b),

i. Determine the differential equation for the controller, which relates r and y to u.

ii. Determine the differential equation for the plant, which relates d and u to y.

iii. Combining these differential equations, eliminate u and determine the closed-loop differential equation relating r and d to y.

3. Find the transfer function from e to u for the PI controller equations

$$\begin{aligned} \dot{z}(t) &= e(t) \\ u(t) &= K_P e(t) + K_I z(t) \end{aligned}$$

4. Suppose that the transfer function of a controller, relating reference signal r and measurement y_m to control signal u is

$$U = C(s)\left[R - Y_M\right]$$

Suppose that the plant has transfer function relating control signal u and disturbance d to output y as

$$Y = \left[G_1(s)U + G_2(s)D\right]$$

Suppose the measurement y_m is related to the actual y with additional noise (n), and a filter (with transfer function F)

$$Y_M = F(s)\left[Y + N\right]$$

(a) Draw a block diagram

(b) In one calculation, determine the 3 closed-loop transfer functions relating inputs r, d and n to the output y.

(c) In one calculation, determine the 3 closed-loop transfer functions relating inputs r, d and n to the control signal u. and d to y.

6 Frequency Responses of Linear Systems

In this section, we consider the steady-state response of a linear system due to a sinusoidal input. The linear system is the standard one,

$$\begin{aligned} y^{[n]}(t) + a_1 y^{[n-1]}(t) + \cdots + a_{n-1} y^{[1]}(t) + a_n y(t) \\ = b_0 u^{[n]}(t) + b_1 u^{[n-1]}(t) + \cdots + b_{n-1} u^{[1]}(t) + b_n u(t) \end{aligned} \tag{41}$$

with y the dependent variable (output), and u the independent variable (input).

Assume that the system is stable, so that the roots of the characteristic equation are in the open left-half of the complex plane. This guarantees that **all** homogeneous solutions decay exponentially to zero as $t \to \infty$.

Suppose that the forcing function $u(t)$ is chosen as a complex exponential, namely ω is a fixed real number, and $u(t) = e^{j\omega t}$. Note that the derivatives are particularly easy to compute, namely

$$u^{[k]}(t) = (j\omega)^k e^{j\omega t}$$

It is easy to show that for some complex number H, **one** particular solution is of the form

$$y_P(t) = He^{j\omega t}$$

How? Simply plug it in to the ODE, leaving

$$\begin{aligned} H\left[(j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_{n-1}(j\omega) + a_n\right] e^{j\omega t} \\ = \left[b_0(j\omega)^n + b_1(j\omega)^{n-1} + \cdots + b_{n-1}(j\omega) + b_n\right] e^{j\omega t} \end{aligned}$$

For all t, the quantity $e^{j\omega t}$ is never zero, so we can divide out leaving

$$\begin{aligned} H\left[(j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_{n-1}(j\omega) + a_n\right] \\ = \left[b_0(j\omega)^n + b_1(j\omega)^{n-1} + \cdots + b_{n-1}(j\omega) + b_n\right] \end{aligned}$$

Now, since the system is stable, the roots of the polynomial

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

all have negative real part. Hence, $\lambda = j\omega$, which has 0 real part, is <u>not</u> a root. Therefore, we can explicitly solve for H as

$$H = \frac{b_0(j\omega)^n + b_1(j\omega)^{n-1} + \cdots + b_{n-1}(j\omega) + b_n}{(j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_{n-1}(j\omega) + a_n} \tag{42}$$

Moreover, since actual solution differs from this particular solution by some homogeneous solution.

$$y(t) = y_P(t) + y_H(t)$$

In the limit, the homogeneous solution decays, regardless of the initial conditions, and we have

$$\lim_{t\to\infty} y(t) = y_P(t) = He^{j\omega t}$$

The explanation we have given was valid at an arbitrary value of the forcing frequency, ω. The expression for H in (42) is still valid. Hence, we often write $H(\omega)$ to indicate the dependence of H on the forcing frequency.

$$H(\omega) := \frac{b_0(j\omega)^n + b_1(j\omega)^{n-1} + \cdots + b_{n-1}(j\omega) + b_n}{(j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_{n-1}(j\omega) + a_n} \tag{43}$$

This function is called the "frequency response" of the linear system in (41). Sometimes it is referred to as the "frequency response from u to y," written as $H_{u\to y}(\omega)$. For stable systems, we have proven for fixed value $\bar{u}$ and fixed ω

$$u(t) := \bar{u}e^{j\omega t} \quad \Rightarrow \quad y_{ss}(t) = H(\omega)\bar{u}e^{j\omega t}$$

Recall that the transfer function from u to y is the rational function $G(s)$ given by

$$G(s) := \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Note that $H(\omega) = G(s)\big|_{s=j\omega}$. Hence, we can immediately write down the frequency response function once we have derived the transfer function. Hence, we often do not use different letters to distinguish the transfer function and frequency response, typically writing $G(s)$ to denote the transfer function and $G(j\omega)$ to denote the frequency response function.

6.1 Complex and Real Particular Solutions

What is the meaning of a complex solution to the differential equation (41)? Suppose that functions u and y are complex, and solve the ODE. Denote the real part of the function u as u_R, and the imaginary part as u_I (similar for y). Then u_R and u_I are real-valued functions, and for all t $u(t) = u_R(t) + ju_I(t)$. Differentiating this k times gives

$$u[k](t) = u_R^{[k]}(t) + ju_I^{[k]}(t)$$

Hence, if y and u satisfy the ODE, we have

$$\begin{aligned}\left[y_R^{[n]}(t) + jy_I^{[n]}(t)\right] + a_1\left[y_R^{[n-1]}(t) + jy_I^{[n-1]}(t)\right] + \cdots + a_n\left[y_R(t) + jy_I(t)\right] = \\ = b_0\left[u_R^{[n]}(t) + ju_I^{[n]}(t)\right] + b_1\left[u_R^{[n-1]}(t) + ju_I^{[n-1]}(t)\right] + \cdots + b_n\left[u_R(t) + ju_I(t)\right]\end{aligned}$$

But the real and imaginary parts must be equal individually, so exploiting the fact that the coeffcients a_i and b_j are real numbers, we get

$$\begin{aligned}y_R^{[n]}(t) + a_1 y_R^{[n-1]}(t) + \cdots + a_{n-1}y_R^{[1]}(t) + a_n y_R(t) \\ = b_0 u_R^{[n]}(t) + b_1 u_R^{[n-1]}(t) + \cdots + b_{n-1}u_R^{[1]}(t) + b_n u_R(t)\end{aligned}$$

and

$$\begin{aligned}y_I^{[n]}(t) + a_1 y_I^{[n-1]}(t) + \cdots + a_{n-1}y_I^{[1]}(t) + a_n y_I(t) \\ = b_0 u_I^{[n]}(t) + b_1 u_I^{[n-1]}(t) + \cdots + b_{n-1}u_I^{[1]}(t) + b_n u_I(t)\end{aligned}$$

Hence, if (u, y) are functions which satisfy the ODE, then both (u_R, y_R) and (u_I, y_I) also satisfy the ODE.

6.2 Response due to real sinusoidal inputs

Suppose that $H \in \mathbf{C}$ is not equal to zero. Recall that $\angle H$ is the real number (unique to within an additive factor of 2π) which has the properties

$$\cos \angle H = \frac{\mathrm{Re} H}{|H|}, \quad \sin \angle H = \frac{\mathrm{Im} H}{|H|}$$

Then,

$$\begin{aligned}
\mathrm{Re}\left(He^{j\theta}\right) &= \mathrm{Re}\left[(H_R + jH_I)(\cos\theta + j\sin\theta)\right] \\
&= H_R\cos\theta - H_I\sin\theta \\
&= |H|\left[\tfrac{H_R}{|H|}\cos\theta - \tfrac{H_I}{|H|}\sin\theta\right] \\
&= |H|\left[\cos\angle H\cos\theta - \sin\angle H\sin\theta\right] \\
&= |H|\cos(\theta + \angle H) \\
\mathrm{Im}\left(He^{j\theta}\right) &= \mathrm{Im}\left[(H_R + jH_I)(\cos\theta + j\sin\theta)\right] \\
&= H_R\sin\theta + H_I\cos\theta \\
&= |H|\left[\tfrac{H_R}{|H|}\sin\theta + \tfrac{H_I}{|H|}\cos\theta\right] \\
&= |H|\left[\cos\angle H\sin\theta + \sin\angle H\cos\theta\right] \\
&= |H|\sin(\theta + \angle H)
\end{aligned}$$

Now consider the differential equation/frequency response case. Let $H(\omega)$ denote the frequency response function. If the input $u(t) = \cos\omega t = \mathrm{Re}\,(e^{j\omega t})$, then the steady-state output y will satisfy

$$y(t) = |H(\omega)|\cos(\omega t + \angle H(\omega))$$

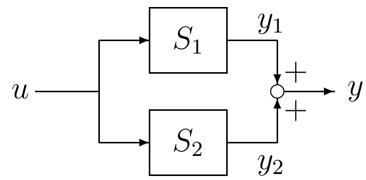
A similar calculation holds for sin, and these are summarized below.

Input	Steady-State Output
1	$H(0) = \frac{b_n}{a_n}$
$\cos\omega t$	$\lvert H(\omega)\rvert\cos(\omega t + \angle H(\omega))$
$\sin\omega t$	$\lvert H(\omega)\rvert\sin(\omega t + \angle H(\omega))$

6.3 Interconnections

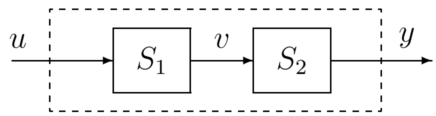
Frequency Responses are a useful concept when working with interconnections of linear systems. Since the frequency response function turned out to be the transfer function evaluated at $s = j\omega$, frequency response functions of interconnections follow the same rules as transfer functions of interconnections. This is extremely important. THE FREQUENCY RESPONSE OF A STABLE INTERCONNECTION OF SYSTEMS (WHICH ARE INDIVIDUALLY POSSIBLY UNSTABLE) IS SIMPLY THE ALGEBRAIC GAIN OF THE CLOSED-LOOP SYSTEMS, TREATING INDIVIDUAL SUBSYSTEMS AS COMPLEX GAINS, WITH THEIR "GAIN" TAKING ON THE VALUE OF THE FREQUENCY RESPONSE FUNCTION. THIS IS TRUE, EVEN IF SOME OF THE SUBSYSTEMS ARE NOT THEMSELVES STABLE.

The frequency response of the parallel connection, shown below



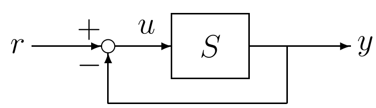
is simply $G(j\omega) = G_1(j\omega) + G_2(j\omega)$, where $G_1(s)$ and $G_2(s)$ are the transfer functions of the dynamic systems S_1 and S_2 respectively.

For the cascade of two stable systems,



the frequency response is $G(j\omega) = G_2(j\omega)G_1(j\omega)$.

The other important interconnection we know of is the basic feedback loop. Consider the simple unity-feedback system shown below



Again, using the transfer function derived earlier, we see that

$$G_{r\to y}(j\omega) = \frac{G_{u\to y}(j\omega)}{1 + G_{u\to y}(j\omega)}$$

6.4 Problems

1. (a) Write a Matlab program to compute the frequency response $G(\omega)$ for the standard system

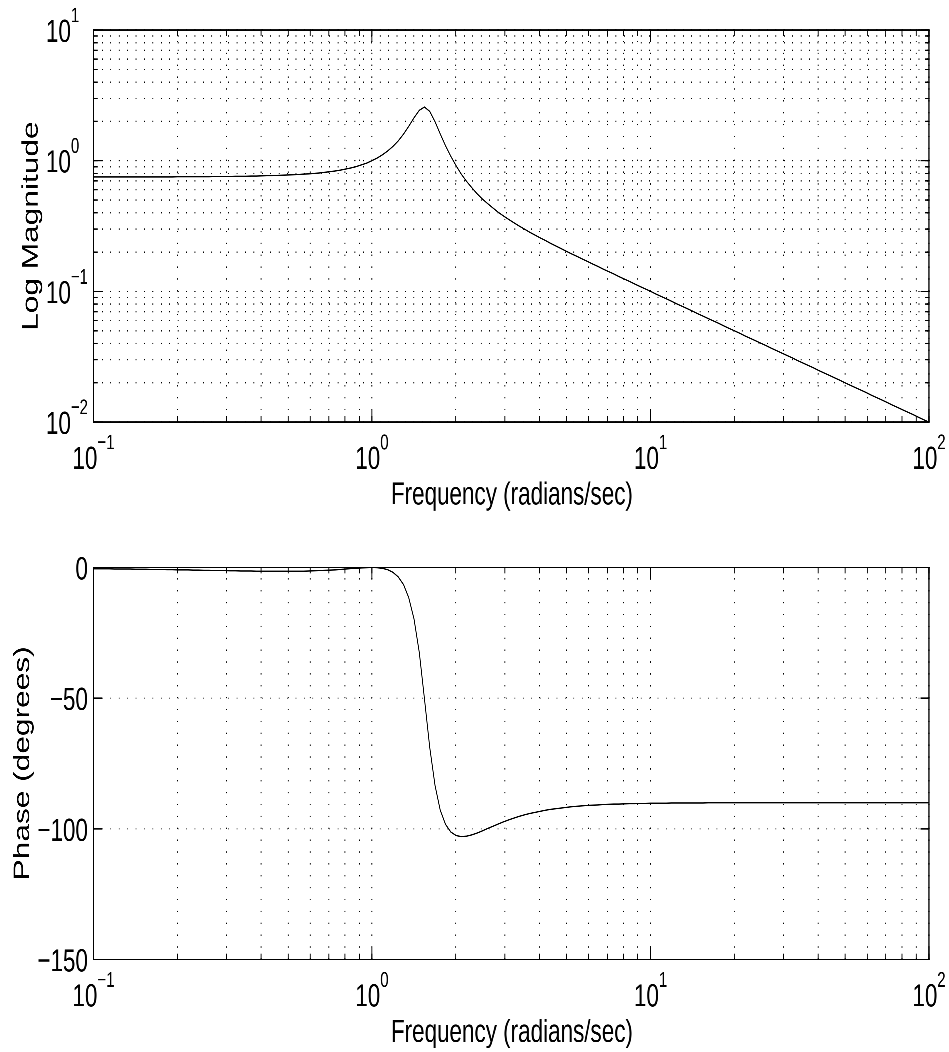
$$y^{[n]}(t) + a_1 y^{[n-1]} + \cdots + a_n y(t) = b_0 u^{[n]}(t) + b_1 u^{[n-1]} + \cdots + b_n u(t)$$

The program syntax should be `[g,gmag,gangle] = frsp132(A,B,omega)`. The `omega` vector would be a row vector of frequencies that the user wants the frequency response computed at. Typically, this will be about 100-200 values, logarithmically spaced between a lower bound and upper bound. All returned quantities should have the same length as `omega`, containing the values of $G(\omega)$, $|G(\omega)|$ and $\angle G(\omega)$ respectively. Hint: read about the command `phase` in Matlab. You can use it to calculate $\angle G$.

Verify that your program can compute the correct response for

$$y^{[3]}(t) + 2y^{[2]}(t) + 3y^{[1]}(t) + 4y(t) = u^{[2]}(t) + 2u^{[1]}(t) + 3u(t)$$

which is shown below



(b) Use the program to calculate and plot the magnitude and angle of the frequency response of the system

$$\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t)$$

for $\omega_n = 2$, $\xi = 0.1, 0.3, 0.7, 1.0, 2.0$. Compare your results to the theoretical results we obtained in class.

(c) Read up on the Matlab command `freqresp`. Use it on the above examples, and verify that it works as advertised.