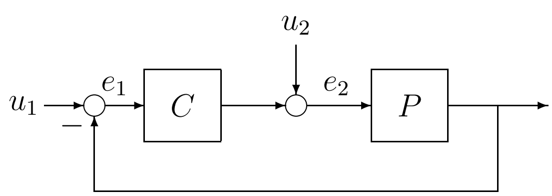
30 Nyquist Criterion: Preliminaries

In order to study the Nyquist Stability Criterion, we first review, and recast the stability results we already know for the standard controller/plant feedback system



By way of notation, we will use the abbreviation RHP to mean "right half-plane," namely all complex numbers λ with $\mathrm{Re}(\lambda) \geq 0$. Similarly, LHP refers to the left half-plane, which is all complex numbers λ with $\mathrm{Re}(\lambda) < 0$.

30.1 Stability

Represent the plant and controller transfer functions as P and C, with

$$P(s) = \frac{n_P(s)}{d_P(s)}, \qquad C(s) = \frac{n_C(s)}{d_C(s)}$$

where n_P, d_P, n_C and d_C are polynomials. The closed-loop characteristic equation is

$$n_P(s)n_C(s) \ + \ d_P(s)d_C(s) = 0$$

If <u>all</u> of the roots of this equation are in the open-left half plane (ie., have negative real parts), then the closed-loop system is stable. If <u>any</u> of the roots of this equation are in the closed-right half plane (ie., have real parts ≥ 0), then the closed-loop system is unstable.

30.2 Pole/Zero Cancellations

If n_P and d_C share any common factors in the right half plane, then this is called a right-half-plane pole/zero cancellation. **It leads to instability, and is not an acceptable control design technique.** To see this, suppose that $\alpha \in \mathbf{C}$ has $\mathrm{Re}(\alpha) \geq 0$, and α is a root of both n_P and d_C. Then $n_P = (s-\alpha)\hat{n}_P(s)$ and $d_C = (s-\alpha)\hat{d}_C(s)$ for some polynomials $\hat{n}_P$ and $\hat{d}_C$. the closed-loop characteristic equation

$$\begin{aligned} n_P(s)n_C(s) \ + \ d_P(s)d_C(s) &= (s-\alpha)\hat{n}_P(s)n_C(s) + d_P(s)(s-\alpha)\hat{d}_C(s) \\ &= (s-\alpha)\left[\hat{n}_P(s)n_C(s) + d_P(s)\hat{d}_C(s)\right] \end{aligned}$$

clearly has a root at $s = \alpha$, and hence the closed loop system is unstable. Similarly, if d_P and n_C share any common factors in the right half plane, then the system is also unstable. For the same reasons, it is easy to see that if n_P and d_P (or n_C and d_C) have a common, right-half plane root, then the closed-loop system will also be unstable.

Hence, for the time being, we make the following assumption:

Assumption: The 4 polynomials, n_P, n_C, d_C and d_P have no RHP roots in common.

In other words, at every point in the right-half plane, at least 3 of the polynomials are nonzero. Otherwise, using the methods above, we can trivially conclude that the closed-loop system is unstable. So, while this assumption does not say anything about the stability of the closed-loop system, if the assumption is violated, then the closed-loop system is definitely unstable.

Suppose that λ is a RHP root of $n_P n_C + d_P d_C = 0$. By our assumptions, it must be that $n_P(\lambda) \neq 0$, $d_P(\lambda) \neq 0$, $n_C(\lambda) \neq 0$ and $d_C(\lambda) \neq 0$. Hence, $n_P(\lambda)n_C(\lambda) = -d_P(\lambda)d_C(\lambda) \neq 0$. Therefore, the quotient $\frac{n_P(\lambda)n_C(\lambda)}{d_P(\lambda)d_C(\lambda)}$ makes sense and is equal to -1. But, this quotient is simply the product $P(s)C(s)|_{s=\lambda}$. Now, this relies completely on the fact that $\text{Re}(\lambda) \geq 0$, and that we have ruled out RHP pole/zero cancellations. We have not ruled out left-half plane cancellations (because they do not cause instability, and are often used in control design). Take, for example

$$P(s) := \frac{1}{s+2}, \qquad C(s) := \frac{10(s+2)}{s+5}$$

so

$$n_P(s) = 1, \qquad d_P(s) = s+2, \qquad n_C(s) = 10(s+2), \qquad d_C(s) = s+5$$

Clearly, -2 is a root of $n_P n_C + d_P d_C = 0$. However, the product $P(s)C(s)$ has a removable singularity at $s = -2$, and evaluating it there gives

$$P(s)C(s)|_{s=-2} = \frac{10}{3} \neq -1$$

So, we can now easily relate the RHP zeros of $n_P n_C + d_P d_C$ to the RHP zeros of $1 + PC$. It is possible to do this for RHP zeros, but not for LHP zeros (since we allow LHP pole/zero cancellations in P and C) .

Take $\lambda \in \mathbf{C}$, with $\text{Re}(\lambda) \geq 0$. Now, if λ is a RHP zero of $n_P n_C + d_P d_C$, then since none of the terms can individually be zero, it follows that (by division), the expression

$$1 + PC = 1 \ + \ \frac{n_P n_C}{d_P d_C}$$

is well defined at λ, and indeed is 0. Conversely, if $1 + PC$ has a RHP zero at λ, then it must be that $PC(\lambda)$ is equal to -1. Hence $n_P(\lambda)n_C(\lambda) = -d_P(\lambda)d_C(\lambda)$.

The RHP poles of $1 + PC$ are the same as the RHP poles of PC (since 1 is a constant). Also, since we rule out RHP pole/zero cancellation, the RHP poles of PC are simply the union of the RHP poles of P and the RHP poles of C.

Summarizing:

1. The RHP zeros of $n_P(s)n_C(s) + d_P(s)d_C(s) = 0$ are the same as the RHP zeros of $1 + P(s)C(s)$.
2. The RHP poles of P along with the RHP poles of C are the same as the RHP poles of $1 + PC$.

30.3 Nyquist Curves

Suppose that $G(s)$ is a rational (fraction of polynomials) function, usually representing a transfer function of a linear system. Let Γ be a simple closed curve in the complex plane that does not pass through any poles or zeros of G. A typical Γ is shown below, along with the locations of the poles and zeros of G.

Traverse Γ in the clockwise (we need to agree on a direction – clockwise is one of two options...) direction. Let γ be the complex number at some point along the curve. As γ takes on values along the curve Γ, plot (in another complex plane) the values of $G(\gamma)$. Call this resulting curve $G(\Gamma)$, note that it will necessarily be a closed curve, but it may cross itself.

Since Γ does not pass through any poles of G, it the values $G(\gamma)$ are all well-defined. Since Γ does not pass through any zeros of G, the curve $G(\Gamma)$ does not pass through the point 0. **Question:** How many times does $G(\Gamma)$ encircle the origin? **Answer:** Interestingly, all that matters is the number of poles and zeros of G that are inside the curve Γ. To see this, write G as

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_n)}$$

At any value $\gamma \in \mathbf{C}$, the angle of $G(\gamma)$ is

$$\begin{aligned}\angle G(\gamma) = \angle K + \angle(\gamma - z_1) + \cdots + \angle(\gamma - z_m) \\ -\angle(\gamma - p_1) - \cdots - \angle(\gamma - p_n)\end{aligned}$$

The phasor interpretation is most useful. At any point $\gamma \in \Gamma$, the complex number

$G(\gamma)$ is represented as the product of the complex phasors shown below.

Take r, a complex number inside Γ. Look at the complex phasor $(\gamma - r)$ as γ traverses Γ.

Clearly, the figure shows that during the traversal, the angle of the complex number $(\gamma - r)$ undergoes a net-change of -2π. Similarly, take w a complex number outside

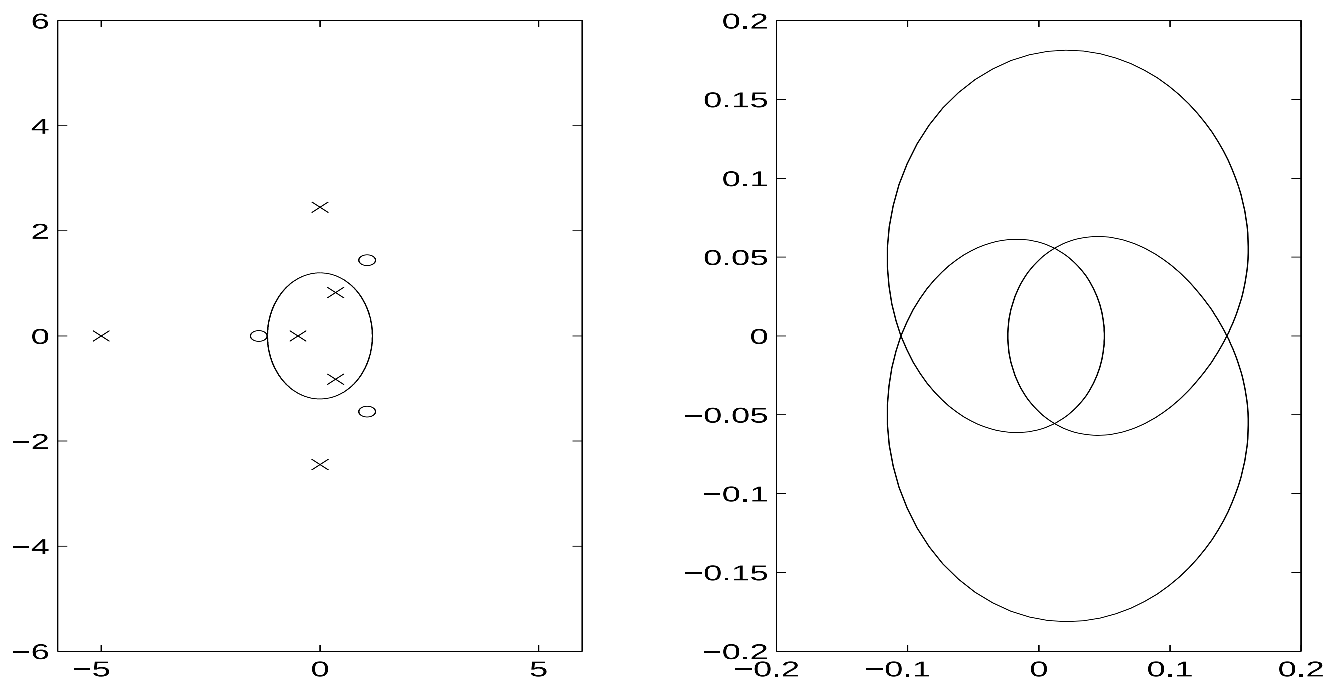
Γ. Look at the complex phasor $(\gamma - w)$ as γ traverses Γ.

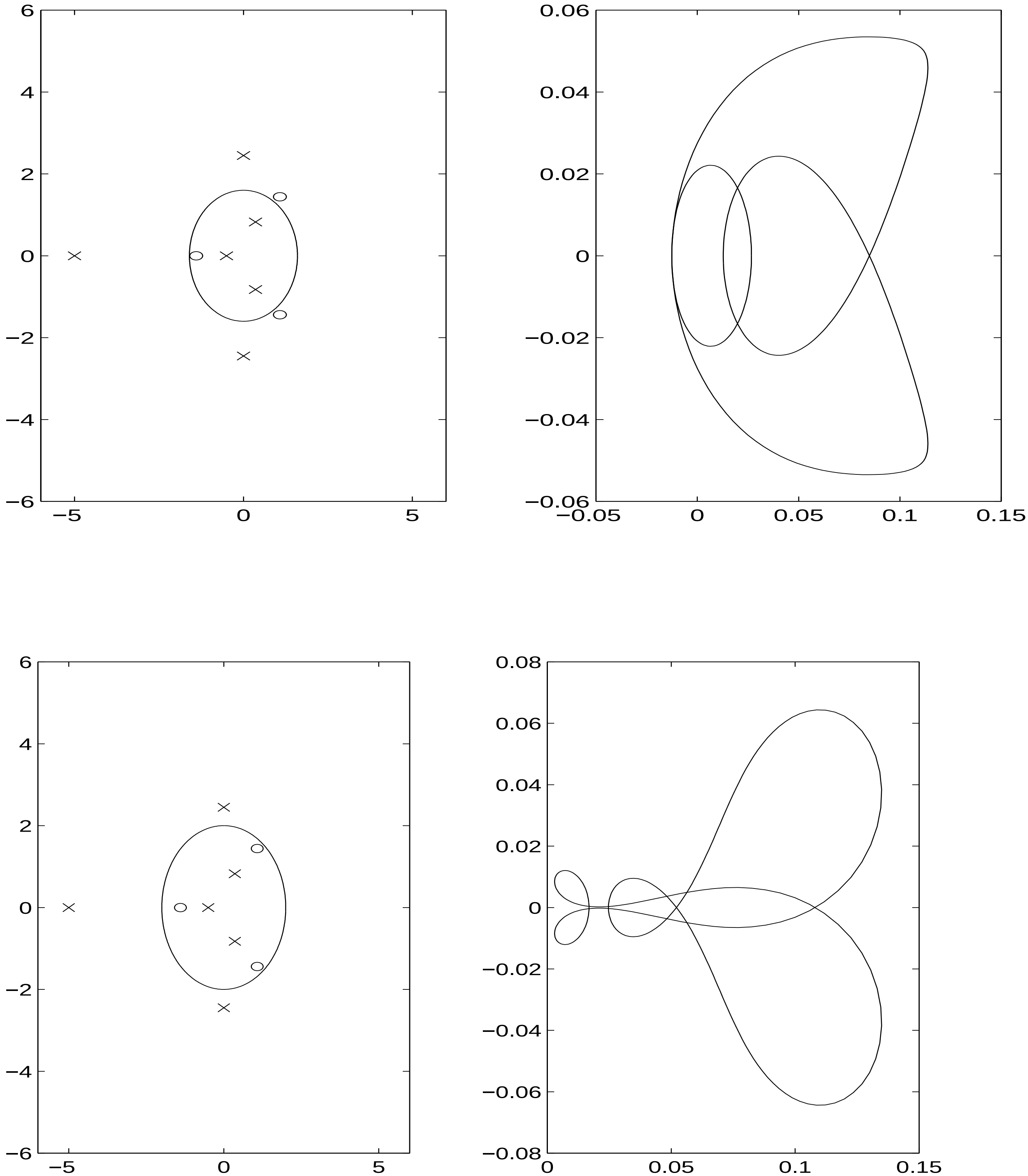
Clearly, the figure shows that during the traversal, the angle of the complex number $(\gamma - w)$ alternates from 0 to negative to 0 to positive and back to 0, with no <u>net</u> change in phase.

Take, for example,

$$G(s) = \frac{(s+1.4)(s^2 - 2(0.6)(1.8)s + 1.8^2)}{(s+0.5)(s+5)(s^2 - 2(0.4)(0.9) + 0.9^2)} s^2 + 6)$$

In the next 9 figures, we show circular paths Γ, of different radii, and centers, and the corresponding curves $G(\Gamma)$. In each case, the number of encirclements of the origin by $G(\Gamma)$ is correct.

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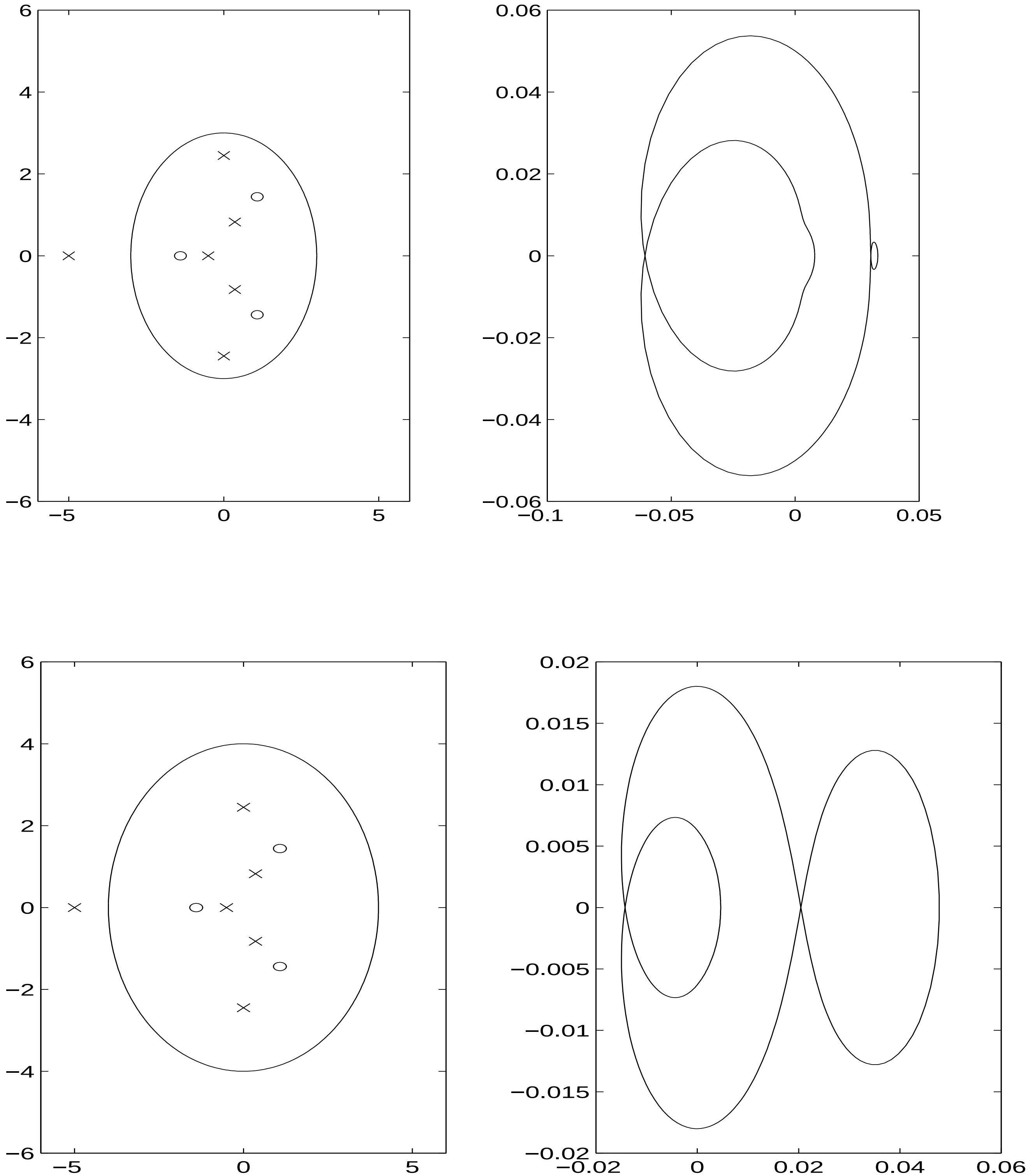
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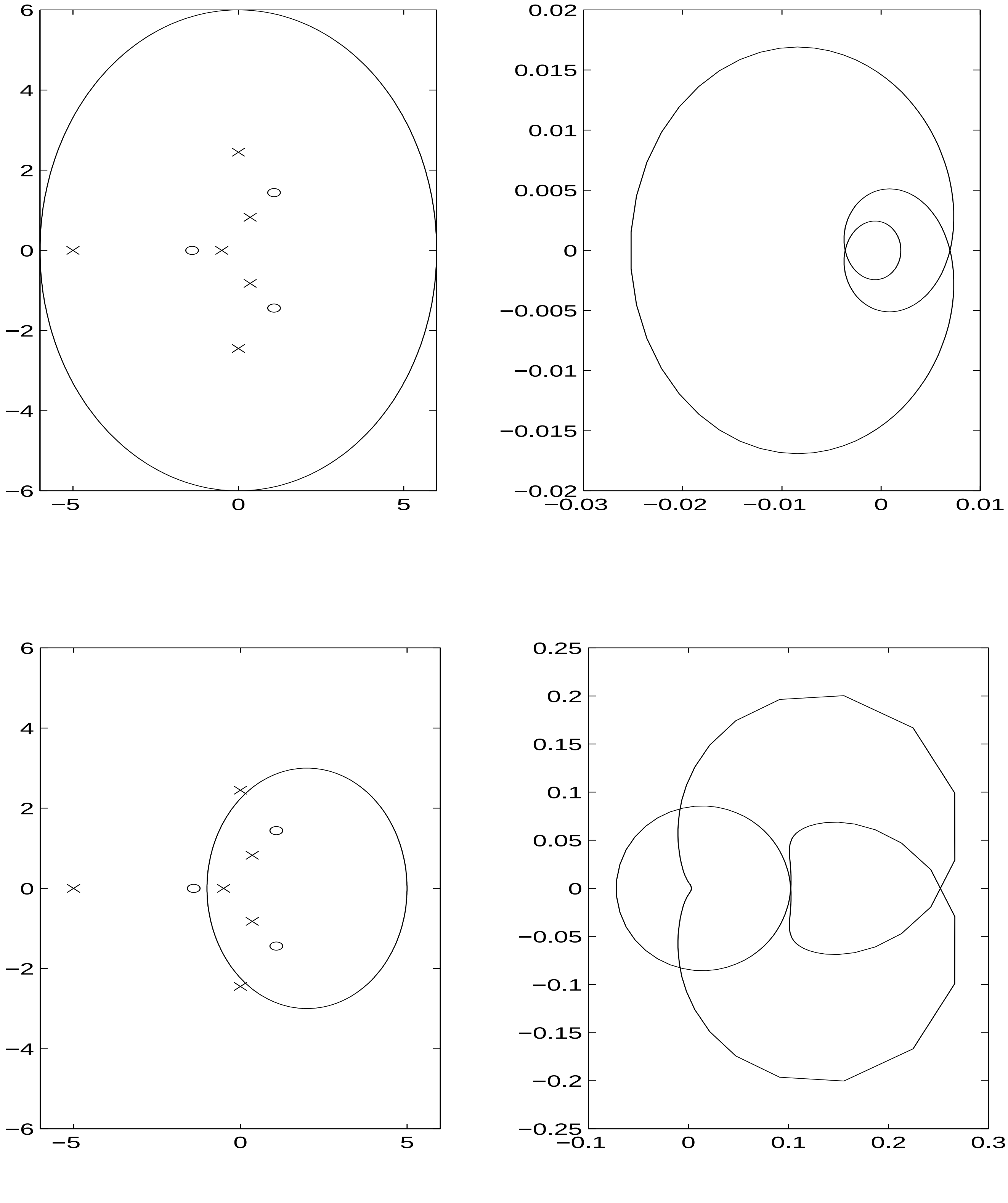
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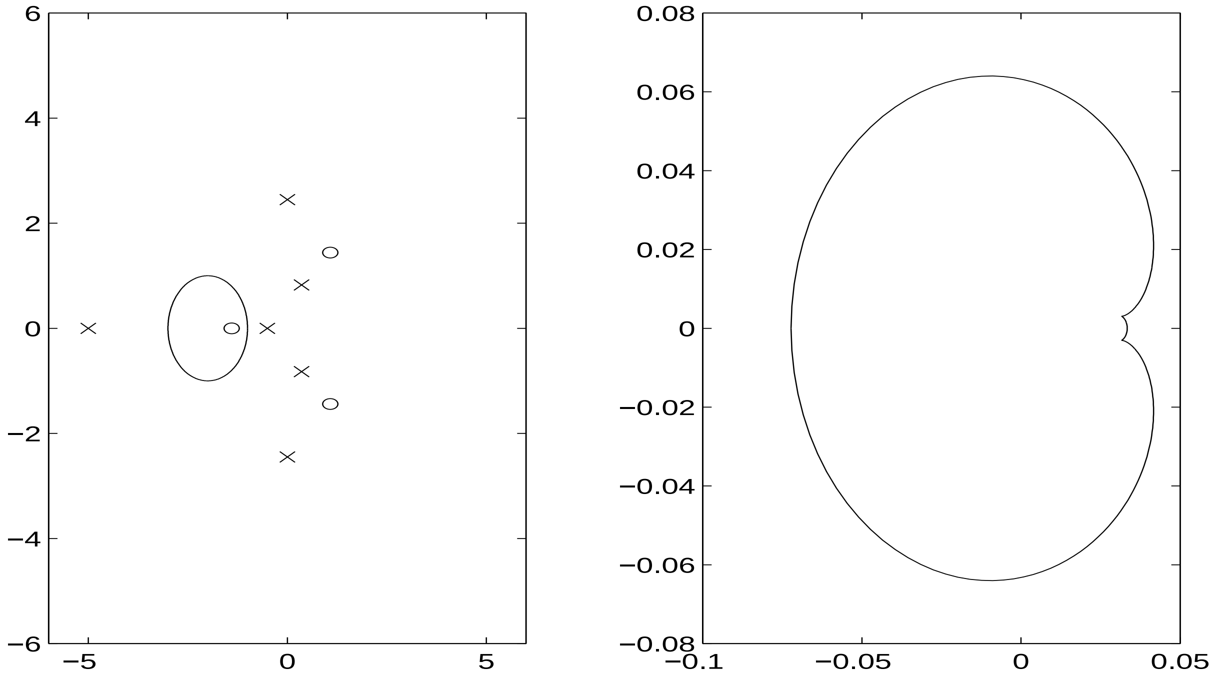
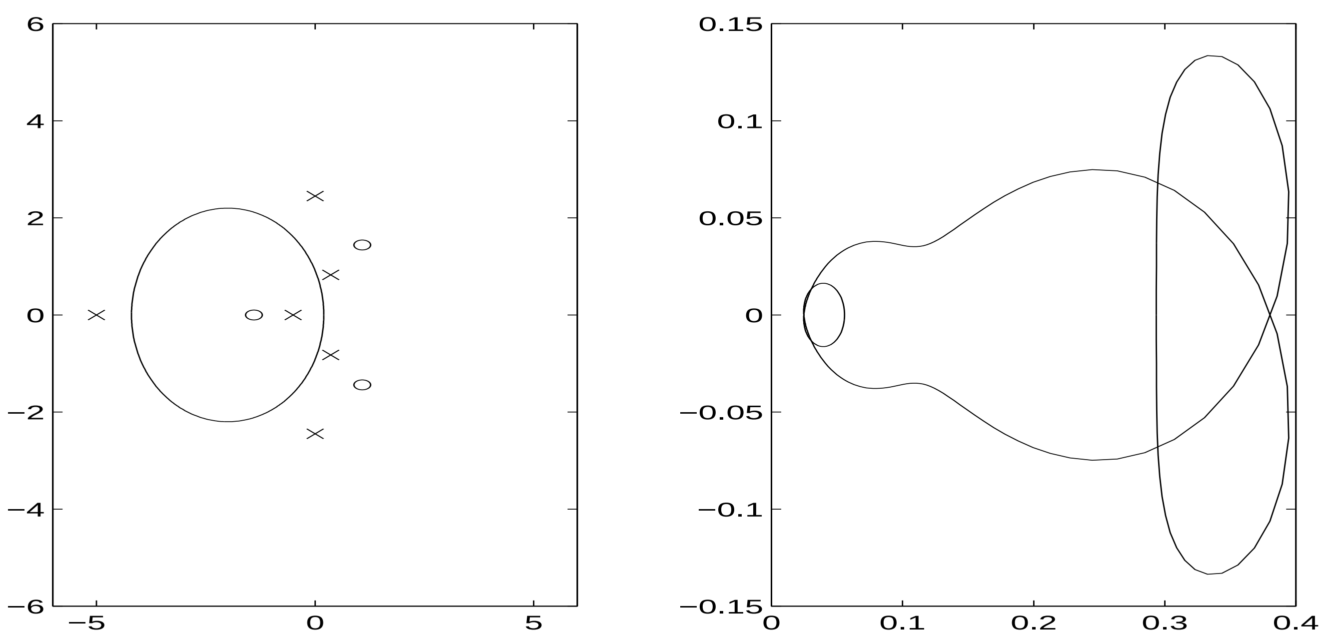




Back to the theory: we can combine the effect of all of the poles and zeros of G in this manner, and determine that

- For every zero of G inside Γ, the quantity $G(s)$ undergoes a phase-change of -2π as s traverses Γ once, clockwise.

- For every pole of G inside Γ, the quantity $G(s)$ undergoes a phase-change of 2π as s traverses Γ once, clockwise.
- For every pole or zero of G outside Γ, the quantity $G(s)$ undergoes no net phase-change as s traverses Γ once, clockwise.

Since a phase change of 2π represents a counterclockwise (CCW) encirclement of the origin, we can combine all of these ideas into one statement:

Theorem: Let $\#p_{G,\Gamma}$ denote the number of poles of G inside Γ. Let $\#z_{G,\Gamma}$ denote the number of zeros of G inside Γ. If Γ is traversed clockwise, then the curve $G(\Gamma)$ encircles the origin $(\#p_{G,\Gamma} - \#z_{G,\Gamma})$ times, counterclockwise.

Application: As we have already seen, the RHP zeros of $1 + P(s)C(s)$ are the closed-loop RHP poles. The RHP poles of $1 + P(s)C(s)$ are the open-loop (ie., individually, of P and C) RHP poles. A Nyquist plot of $1 + PC$, with Γ enclosing the whole right half-plane should relate encirclements, RHP open-loop poles, and RHP closed-loop poles.

31 Nyquist Analysis: Examples

In this section, we perform a simple Nyquist Analysis of a closed-loop system with open-loop poles on the Imaginary Axis. This requires us to use the indentation method, and we see that regardless of which way we indent, the final answers obtained are consistent.

The plant is usnstable, and also has a pole at the origin,

$$P(s) = \frac{s+1}{s(s-1)}$$

The controller is taken to be constant gain, $C = K$, where K is a constant.

Since the controller is just a constant, we will do the Nyquist plots for P alone, and easily determine how a positive or negative value of K would influence the plots.

31.1 Contours and Nyquist Diagrams

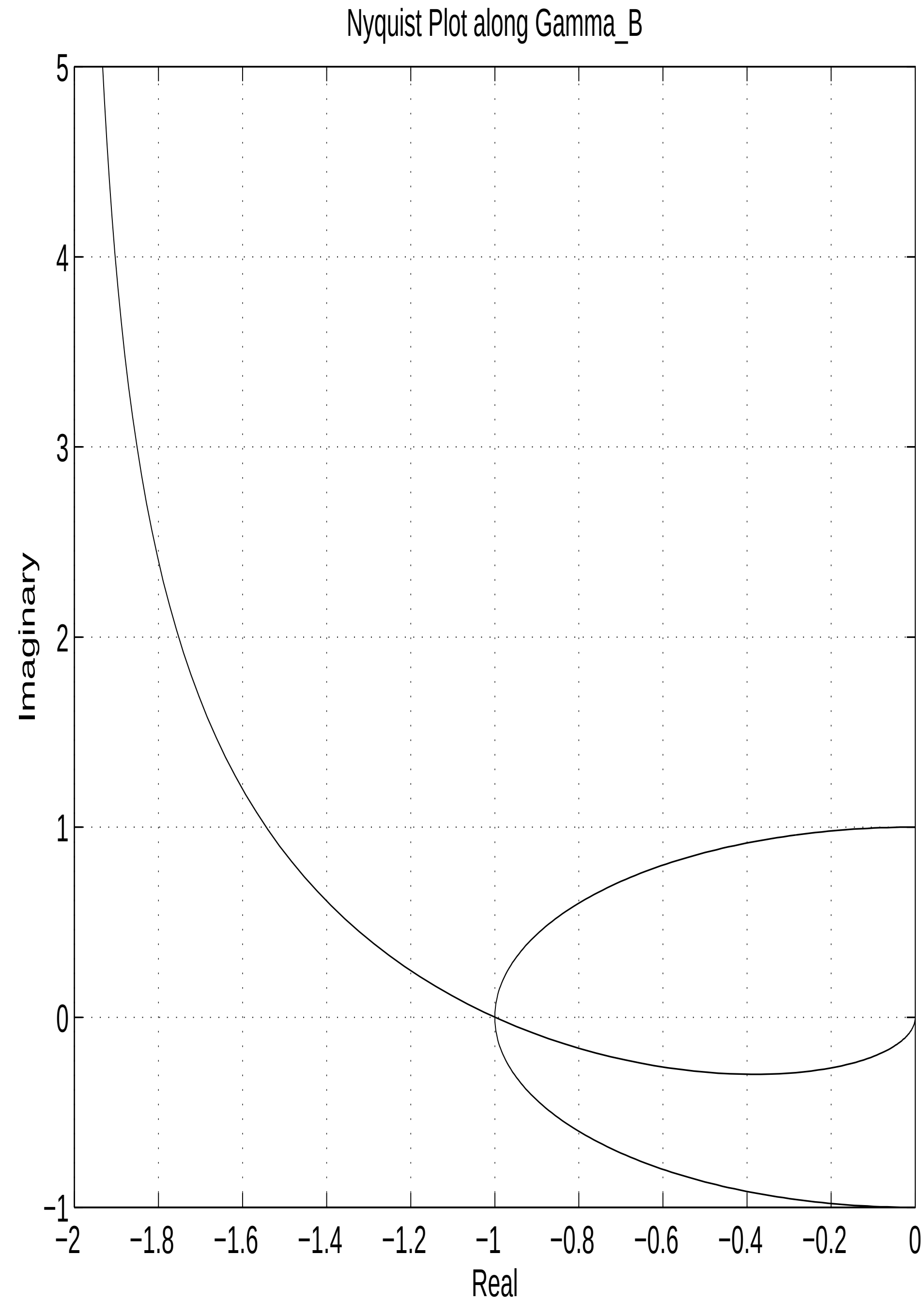
We will consider two different contours, Γ and Γ'. They only differ by the direction of indentation around the plant pole at $s = 0$.

Note that the contours only differ in the A region.

In both contours case, Γ_B is simply the frequency response function of the plant for $\omega > 0$. We can determine that fairly accurately from a straight-line Bode plot.

This gives the Nyquist plot for region approximately as

The actual plot is



Region Γ_C gets mapped to the single point, 0. Region Γ_D is the complex conjugate of region B, hence we are left with regions Γ_A and $\Gamma_{A'}$.

31.1.1 Indent to Right

Region Γ_A will represent indentation to the right. Along Γ_A, we have

$$s = \epsilon e^{j\theta}, \quad \theta = -\frac{\pi}{2} \to \frac{\pi}{2}, \text{CCW}$$

Evaluating P along Γ_A gives

$$\begin{aligned} P(s) &= \tfrac{\epsilon e^{j\theta}+1}{\epsilon e^{j\theta}(\epsilon e^{j\theta}-1)} \\ &\approx \tfrac{1}{\epsilon e^{j\theta}(-1)} \\ &= -\tfrac{1}{\epsilon}e^{-j\theta} \end{aligned}$$

Hence, as s transverse Γ_A, $P(s)$ is a large complex number, traversing CW (due to the negative sign in the complex exponential), starting from $-e^{j\frac{\pi}{2}}$ and ending at $-e^{-j\frac{\pi}{2}}$.

31.1.2 Indent to Left

Region $\Gamma_{A'}$ will represent indentation to the left. Along $\Gamma_{A'}$, we have

$$s = \epsilon e^{j\theta}, \quad \theta = \frac{3\pi}{2} \to \frac{\pi}{2}, \text{CW}$$

Evaluating P along $\Gamma_{A'}$ gives (this is the same...)

$$\begin{aligned} P(s) &= \tfrac{\epsilon e^{j\theta}+1}{\epsilon e^{j\theta}(\epsilon e^{j\theta}-1)} \\ &\approx \tfrac{1}{\epsilon e^{j\theta}(-1)} \\ &= -\tfrac{1}{\epsilon}e^{-j\theta} \end{aligned}$$

Hence, as s transverse $\Gamma_{A'}$, $P(s)$ is a large complex number, traversing CCW (due to the negative sign in the complex exponential), starting from $-e^{-j\frac{3\pi}{2}}$ and ending at $-e^{-j\frac{\pi}{2}}$.

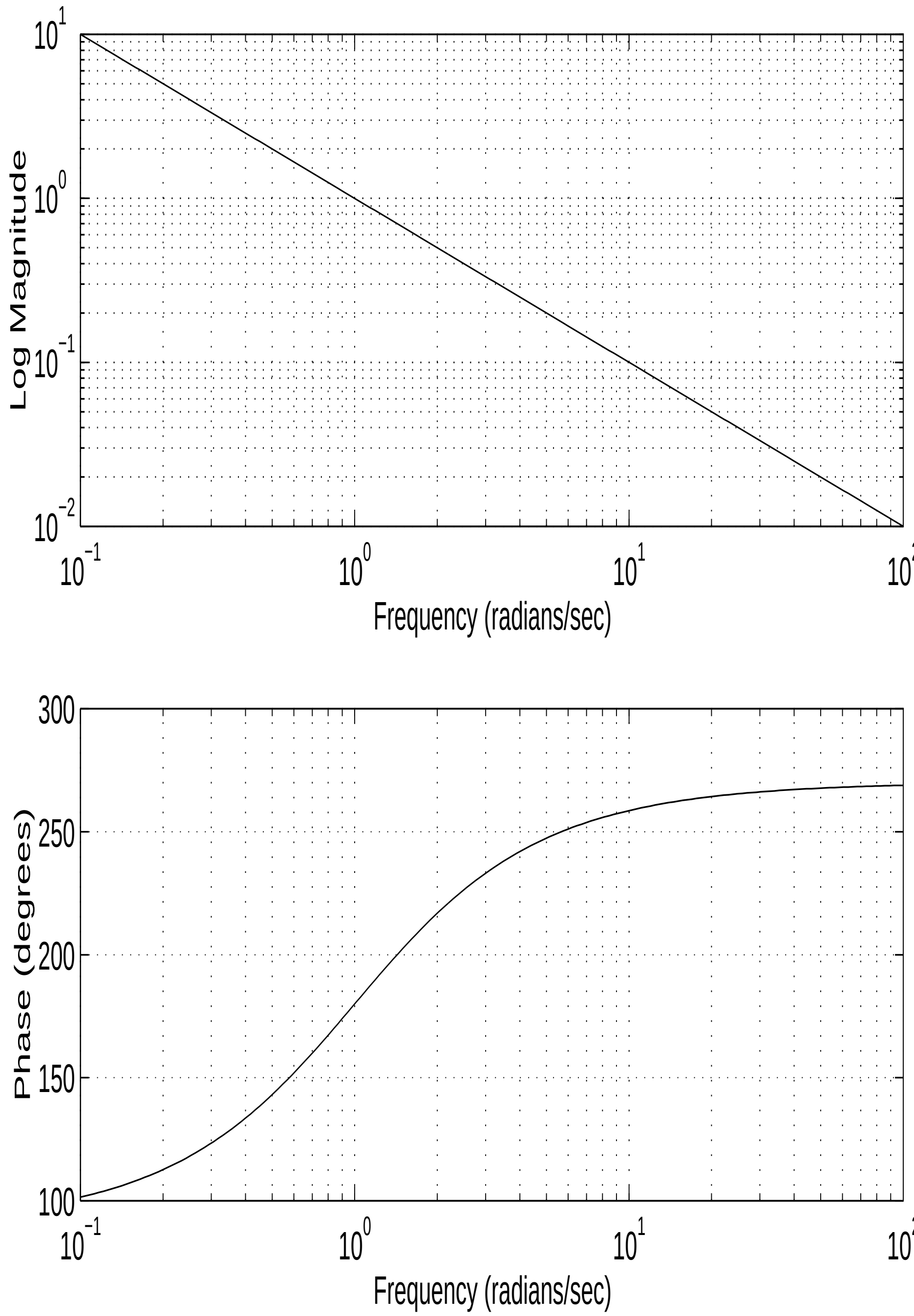
31.2 Stability Analysis

This allows us to conclude the analysis. The table below summarizes our findings.

	A	A'
# of poles of P in RHP$_\epsilon$	1	2
# of poles of C in RHP$_\epsilon$	0	0
# of CCW enc. of -1 ($K > 1$)	1	2
# of unstable closed-loop poles	0	0
# of CCW enc. of -1 ($0 < K < 1$)	-1	0
# of unstable closed-loop poles	2	2
# of CCW enc. of -1 ($K < 0$)	-1	0
# of unstable closed-loop poles	2	2

31.3 Gain and Phase Margins

An accurate Bode plot of the plant is given below



Note that if the controller gain is equal to 10, then the combined plant/controller gain is equal to 1 at a frequency of ______. This is called the *gain crossover frequency*. In any system, there may be multiple gain crossover frequencies. At that frequency, the combined phase of the plant/controller is ______. Hence, if additional negative phase of ______ were somehow added to the plant, without changing the gain, then the closed-loop system would actually be unstable (encirclements would change from correct number, to incorrect number). This is called the *Phase Margin* of the closed-loop system. You should always report the Gain-Crossover frequency and Phase Margin together, as a piece of information about how much the plant could change, and stability would still be maintained.

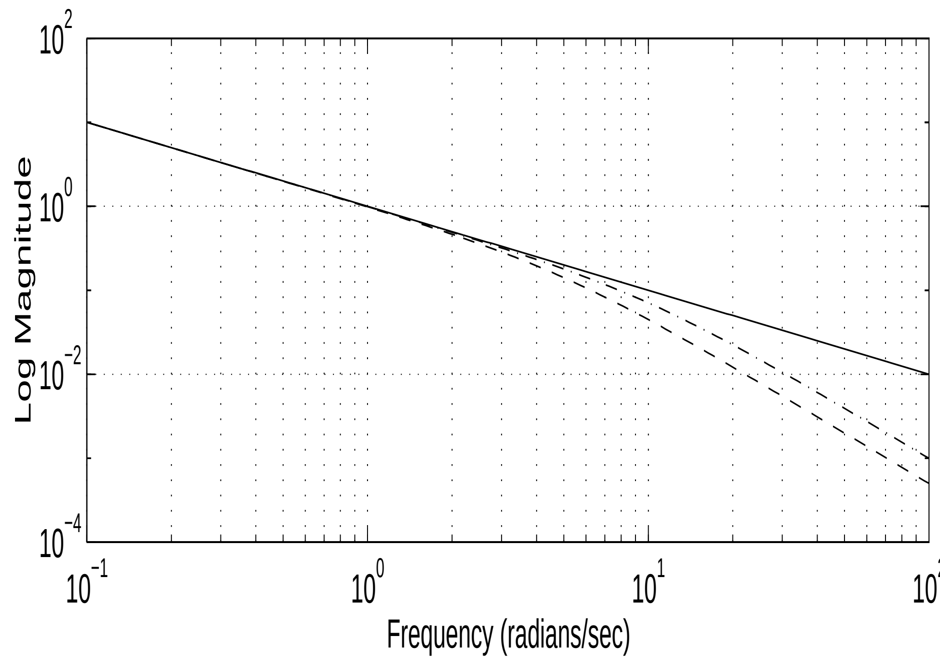
Similarly, we notice that if the controller gain is equal to 10, then the phase of the combined plant/controller gain is equal to π at a frequency of ______. This is called the *phase crossover frequency.* In any system, there may be multiple phase crossover frequencies. At that frequency, the gain of the combined plant/controller is ______. Hence, if the gain of the plant were to change by a factor of ______ without changing the phase, then the closed-loop system would actually be unstable. This is called a *Gain Margin* of the closed-loop system. You should always report the Phase-Crossover frequency and Gain Margin together, as a piece of information about how much the plant could change, and stability would still be maintained.

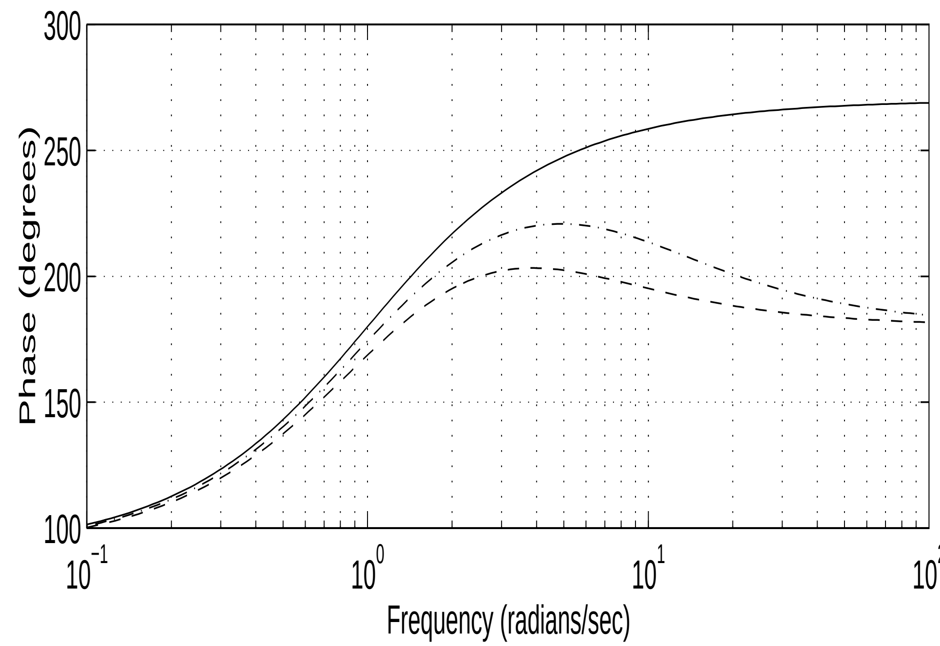
31.4 Other Plants

Accurate Bode plots of the plant, and two others,

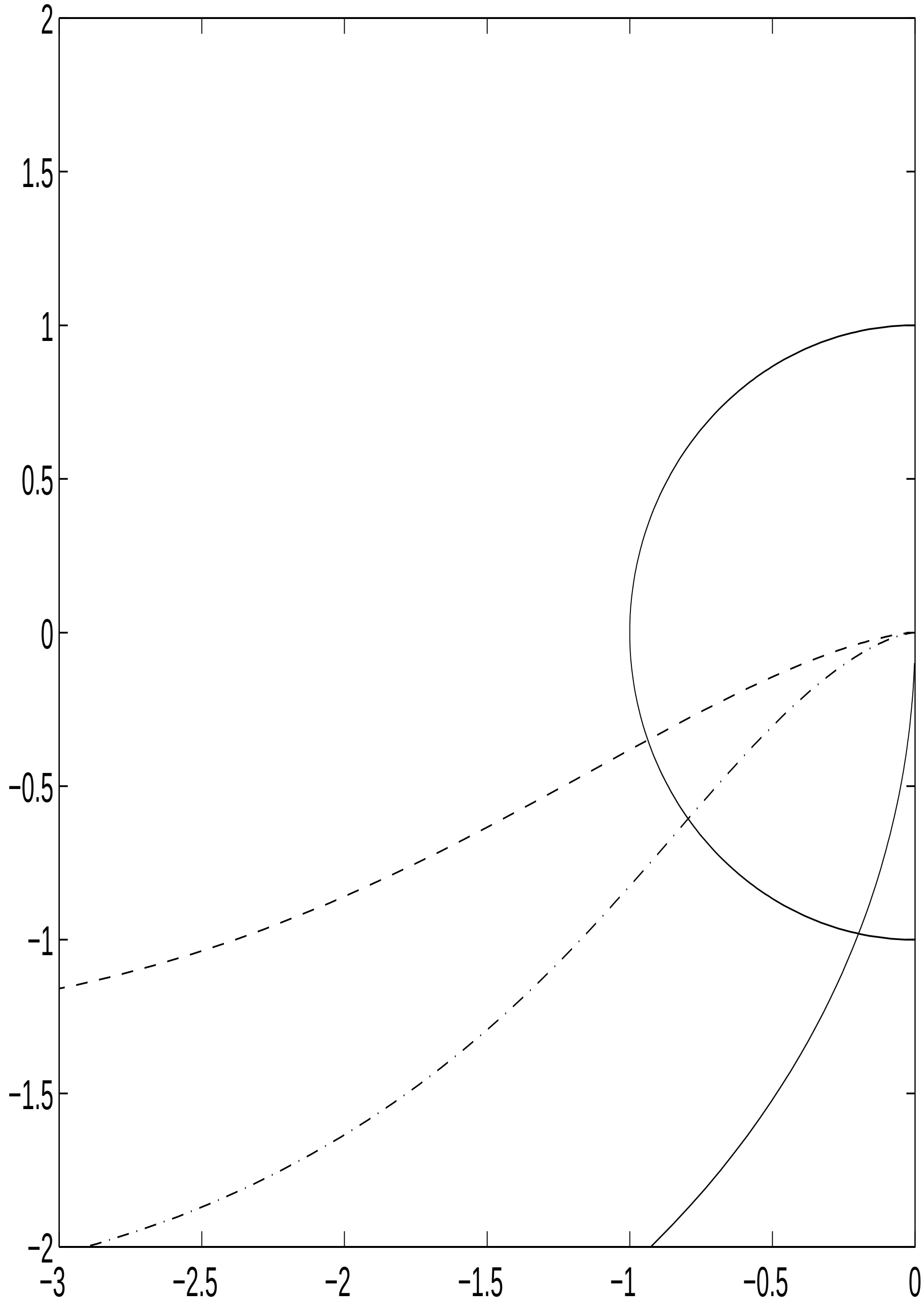
$$P_1(s) := \frac{s+1}{s(s-1)(0.1s+1)}, \qquad P_2(s) := \frac{s+1}{s(s-1)(0.2s+1)}$$

are shown below





These allow simple sketches of the Nyquist plot, and allow for extremely easy evaluation of the crossover frequencies and margins. Accurate Nyquist plots of PC are given (with $C = 10$).



2
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−0.5
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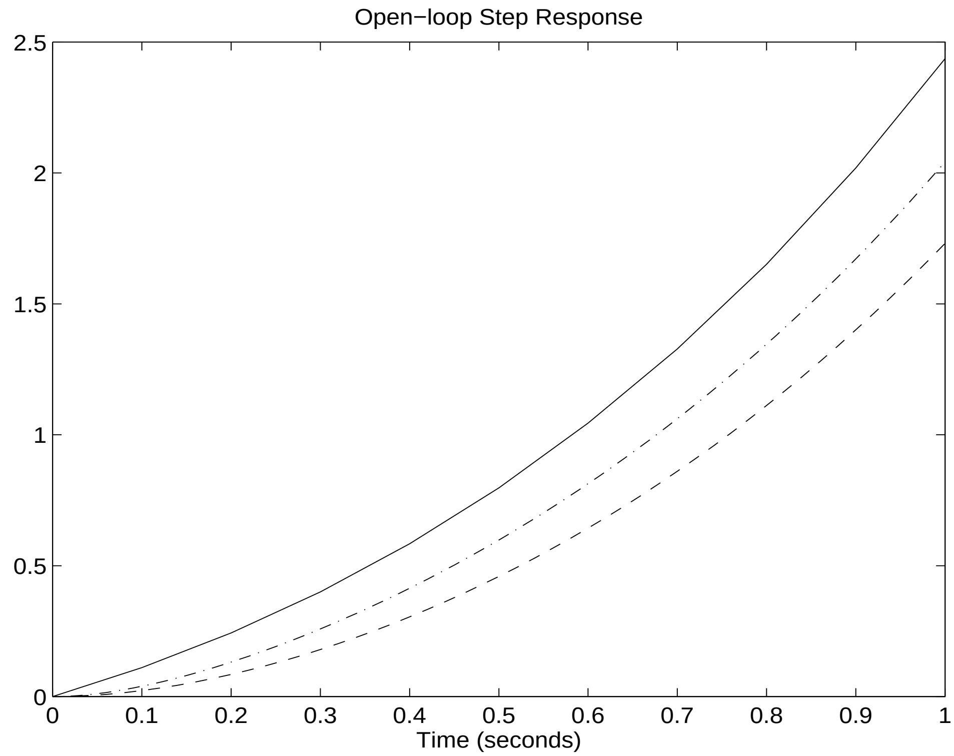
−2
−3
−2.5
−2
−1.5
−1
−0.5
0

The time responses of all of the plants are shown below.



Closed-loop time responses are shown below, and will be discussed.