25 Stabilization by State-Feedback

25.1 Theory

Consider the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

As usual, let $x(t) \in \mathbf{R}^n$, and input $u(t) \in \mathbf{R}^m$. Suppose that the states $x(t)$ are available for measurement, so that a control law of $u(t) = Kx(t)$ is possible. Dimensions dictate that $K \in \mathbf{R}^{m \times n}$. How can the values that make up the gain matrix K be chosen to ensure closed-loop stability? An obvious approach is to

1. Pick n **desired closed-loop eigenvalues**, $\lambda_1, \lambda_2, \ldots, \lambda_n$

2. Calculate the coefficients of the desired closed-loop characteristic polynomial,

   $$p_{\text{des}}(s) := (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n) = s^n + c_1 s^{n_1} + \cdots + c_n$$

   Here the c_i are complicated functions of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

3. Explicitly calculate the closed-loop characteristic polynomial symbolically in the entries of K,

   $$p_{A+BK}(s) = s^n + f_1(K)s^{n-1} + f_2(K)s^{n-2} + \cdots + f_{n-1}(K)s^1 + f_n(K)$$

4. Choose K so that for each $1 \le i \le n$, the equation

   $$f_i(K) = c_i \tag{94}$$

   is satisfied.

Suppose that $u(t) \in \mathbf{R}$ is a single input ($m = 1$). Then the gain matrix $K \in \mathbf{R}^{1 \times n}$. In this case, we can actually show that the coefficients of the closed-loop characteristic equation are affine (linear plus constant) functions of the entries of the K matrix. This means that solving the n equations in (94) will be relatively "easy," involving a matrix inversion problem.

$$\begin{aligned}
p_{A+BK}(s) &:= \det\left[sI - (A + BK)\right] \\
&= \det\left[sI - (A + BK)\right] \\
&= \det\left[(sI - A) - BK\right] \\
&= \det(sI - A)\left[I - (sI - A)^{-1} BK\right] \\
&= \det(sI - A)\det\left[I - (sI - A)^{-1} BK\right] \\
&= \det(sI - A)\left[1 - K(sI - A)^{-1} B\right] \\
&= \det(sI - A) - K\,adj\,(sI - A)\,B
\end{aligned}$$

25.2 Example

As an example, we consider the inverted pendulum problem, described in section 24. The linearized equations of motion about the unstable equilibrium point $\left(\bar{\theta} = 0, \dot{\bar{\theta}} = 0, \bar{w} = 0, \dot{\bar{w}} = 0\right)$ is

$$\dot{\delta}_x(t) = A\delta_x(t) + B\delta_u(t)$$

where the structure of A and B are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \beta & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \gamma \\ 0 \\ \Omega \end{bmatrix}$$

Simple calculations give

$$\det(sI - A) = \det \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -\alpha & 0 \\ 0 & 0 & s & -1 \\ 0 & 0 & -\beta & s \end{bmatrix} = s^2\left(s^2 - \beta\right)$$

and

$$\begin{aligned} adj\,(sI - A) &= \begin{bmatrix} s(s^2-\beta) & (-1)0 & 0 & (-1)0 \\ (-1)(\beta - s^2) & s(s^2-\beta) & (-1)0 & 0 \\ \alpha s & (-1)(-\alpha s^2) & s^3 & (-1)(-\beta s^2) \\ (-1)(-\alpha) & \alpha s & (-1)(-s^2) & s^3 \end{bmatrix}^T \\ &= \begin{bmatrix} s(s^2-\beta) & s^2-\beta & \alpha s & \alpha \\ 0 & s(s^2-\beta) & \alpha s^2 & \alpha s \\ 0 & 0 & s^3 & s^2 \\ 0 & 0 & \beta s^2 & s^3 \end{bmatrix} \end{aligned}$$

Hence,

$$adj\,(sI - A)\,B = \begin{bmatrix} \gamma(s^2-\beta) + \Omega\alpha \\ \gamma s(s^2-\beta) + \Omega\alpha s \\ \Omega s^2 \\ \Omega s^3 \end{bmatrix}$$

Denote K as $\begin{bmatrix} K_1 & K_2 & K_3 & K_4 \end{bmatrix}$, then

$$p_{A+BK}(s) = s^2(s^2-\beta) - K_1\left[\gamma(s^2-\beta) + \Omega\alpha\right] - K_2\left[\gamma s(s^2-\beta) + \Omega\alpha s\right] - K_3\Omega s^2 - K_4\Omega s^3$$

Rearranging gives that the closed-loop characteristic polynomial $p_{A+BK}(s)$ is

$$s^4 + [-K_2\gamma - K_4\Omega]\,s^3 + [-\beta - K_1\gamma - K_3\Omega]\,s^2 + [K_2(\gamma\beta - \Omega\alpha)]\,s + [K_1(\gamma\beta - \Omega\alpha)]$$

Denote the closed-loop characteristic polynomial as

$$s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4$$

The relationship between c and K is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -\beta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\gamma & 0 & -\Omega \\ -\gamma & 0 & -\Omega & 0 \\ 0 & \gamma\beta - \Omega\alpha & 0 & 0 \\ \gamma\beta - \Omega\alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix}$$

Now, suppose that $\Omega \neq 0$ and $\gamma\beta - \Omega\alpha \neq 0$. Then, the 4×4 matrix which multiplies K is invertible, and so by proper choice of the K_i, we can make the coefficients c take on any desired values. Equivalently, by proper choice of the K_i, we can make $p_{A+BK}(s)$ **any 4th order polynomial** that we want it to be. Hence, we have complete freedom to place the eigenvalues of $A + BK$.

25.3 Problems

1. The model for the tightrope walker derived in Section 16.3, problem 1 is

$$\begin{aligned} \left(I_O^P + m_B L^2\right) \ddot{\theta}(t) &= g\left(m_B L + m_p \bar{L}\right) \sin\theta(t) - u(t) \\ I_B^G \left[\ddot{\theta}(t) + \ddot{\psi}(t)\right] &= u(t) \end{aligned}$$

   (a) Choose states $x_1 := \theta, x_2 := \dot{\theta}, x_3 := \psi, x_4 := \dot{\psi}$, and write the nonlinear equations of motion in state-space form

$$\dot{x}(t) = f\left(x(t), u(t)\right)$$

   (b) Show that the upright position $\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0, \bar{x}_4 = 0, \bar{u} = 0$ is an equilibrium point of the system.

   (c) Find the Jacobian linearization of the system about the equilibrium point. Determine the stability/instability of the linearized system (by determining the eigenvalues). Does this conclusion seem in line with your intuition?

   (d) Take the parameters (all SI units) to be $\bar{L} = 1.3, L = 1.2, g = 9.8, m_p = 60, m_B = 15, I_B^G = 80, I_O^P = 90$. Are these reasonable?

   (e) Calculate state-feedback laws

$$\delta_u(t) = K_\alpha \delta_x(t)$$

   for the linearization such that the closed-loop eigenvalues of the linearization are at the locations

$$\alpha(-0.4 \pm j1.2) \quad , \quad \alpha(-0.6 \pm j0.38)$$

   for several values of α, namely $\alpha = 0.5, 1, 2, 4, 8$. Comment on the approximate dependence of the gains K_α on α.

(f) For the 5 different feedback laws, simulate the linearized system and the actual nonlinear system with this feedback law, starting from the initial condition

$$x(0) = \begin{bmatrix} 2\frac{\pi}{180} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Use `subplot` to plot the responses of the angles θ and ψ, as well as the control moment $u(t)$ on each page. Comment on the suitability of the designs. Note, before printing, use

```
>> set(gcf,'paperposition',[.3 7.9 .8 9.4])
```

to enlarge the area of the printed page that MatLab will use