19 Jacobian Linearizations, equilibrium points

In modeling systems, we see that nearly all systems are nonlinear, in that the differential equations governing the evolution of the system's variables are nonlinear. However, most of the theory we have developed has centered on linear systems. So, a question arises: "In what limited sense can a nonlinear system be viewed as a linear system?" In this section we develop what is called a "Jacobian linearization of a nonlinear system," about a specific operating point, called an equilibrium point.

19.1 Equilibrium Points

Consider a nonlinear differential equation

$$\dot{x}(t) = f(x(t), u(t)) \tag{72}$$

where f is a function mapping $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$. A point $\bar{x} \in \mathbf{R}^n$ is called an **equilibrium point** if there is a specific $\bar{u} \in \mathbf{R}^m$ (called the **equilibrium input**) such that

$$f(\bar{x}, \bar{u}) = 0_n$$

Suppose $\bar{x}$ is an equilibrium point (with equilibrium input $\bar{u}$). Consider starting the system (72) from initial condition $x(t_0) = \bar{x}$, and applying the input $u(t) \equiv \bar{u}$ for all $t \geq t_0$. The resulting solution $x(t)$ satisfies

$$x(t) = \bar{x}$$

for all $t \geq t_0$. That is why it is called an **equilibrium point**.

19.2 Deviation Variables

Suppose $(\bar{x}, \bar{u})$ is an equilibrium point and input. We know that if we start the system at $x(t_0) = \bar{x}$, and apply the constant input $u(t) \equiv \bar{u}$, then the state of the system will remain fixed at $x(t) = \bar{x}$ for all t. What happens if we start a little bit away from $\bar{x}$, and we apply a slightly different input from $\bar{u}$? Define deviation variables to measure the difference.

$$\begin{aligned} \delta_x(t) &:= x(t) - \bar{x} \\ \delta_u(t) &:= u(t) - \bar{u} \end{aligned}$$

In this way, we are simply relabling where we call 0. Now, the variables $x(t)$ and $u(t)$ are related by the differential equation

$$\dot{x}(t) = f(x(t), u(t))$$

Substituting in, using the constant and deviation variables, we get

$$\dot{\delta}_x(t) = f\left(\bar{x} + \delta_x(t), \bar{u} + \delta_u(t)\right)$$

This is exact. Now however, let's do a Taylor expansion of the right hand side, and neglect all higher (higher than 1st) order terms

$$\dot{\delta}_x(t) \approx f\left(\bar{x}, \bar{u}\right) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} \delta_x(t) \quad + \quad \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} \delta_u(t)$$

But $f(\bar{x}, \bar{u}) = 0$, leaving

$$\dot{\delta}_x(t) \approx \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} \delta_x(t) \quad + \quad \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} \delta_u(t)$$

This differential equation approximately governs (we are neglecting 2nd order and higher terms) the deviation variables $\delta_x(t)$ and $\delta_u(t)$, **as long as they remain small**. It is a linear, time-invariant, differential equation, since the derivatives of δ_x are linear combinations of the δ_x variables and the deviation inputs, δ_u. The matrices

$$A := \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} \in \mathbf{R}^{n \times n} \quad , \quad B := \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} \in \mathbf{R}^{n \times m} \tag{73}$$

are constant matrices. With the matrices A and B as defined in (73), the linear system

$$\dot{\delta}_x(t) = A\delta_x(t) \quad + \quad B\delta_u(t)$$

is called the **Jacobian Linearization** of the original nonlinear system (72), about the equilibrium point $(\bar{x}, \bar{u})$. For "small" values of δ_x and δ_u, the linear equation **approximately** governs the exact relationship between the deviation variables δ_u and δ_x.

For "small" δ_u (ie., while $u(t)$ remains close to $\bar{u}$), and while δ_x remains "small" (ie., while $x(t)$ remains close to $\bar{x}$), the variables δ_x and δ_u are related by the differential equation

$$\dot{\delta}_x(t) = A\delta_x(t) \quad + \quad B\delta_u(t)$$

In some of the rigid body problems we considered earlier, we treated problems by making a small-angle approximation, taking θ and its derivatives $\dot{\theta}$ and $\ddot{\theta}$ very small, so that certain terms were ignored ($\dot{\theta}^2, \ddot{\theta}\sin\theta$) and other terms simplified ($\sin\theta \approx \theta, \cos\theta \approx 1$). In the context of this discussion, the linear models we obtained were, in fact, the Jacobian linearizations around the equilibrium point $\theta = 0, \dot{\theta} = 0$.

If we design a controller that effectively controls the deviations δ_x, then we have designed a controller that works well when the system is operating **near** the equilibrium point $(\bar{x}, \bar{u})$. We will cover this idea in greater detail later. This is a common, and somewhat effective way to deal with nonlinear systems in a linear manner.

19.3 Tank Example

Consider a mixing tank, with constant supply temperatures T_C and T_H. Let the inputs be the two flow rates $q_C(t)$ and $q_H(t)$. The equations for the tank are

$$\begin{aligned}\dot{h}(t) &= \tfrac{1}{A_T}\left(q_C(t)+q_H(t)-c_D A_o\sqrt{2gh(t)}\right)\\ \dot{T}_T(t) &= \tfrac{1}{h(t)A_T}\left(q_C(t)\left[T_C-T_T(t)\right]+q_H(t)\left[T_H-T_T(t)\right]\right)\end{aligned}$$

Let the state vector x and input vector u be defined as

$$x(t) := \begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} \quad , \quad u(t) := \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix}$$

$$\begin{aligned}f_1(x,u) &= \tfrac{1}{A_T}\left(u_1+u_2-c_D A_o\sqrt{2gx_1}\right)\\ f_2(x,u) &= \tfrac{1}{x_1 A_T}\left(u_1\left[T_C-x_2\right]+u_2\left[T_H-x_2\right]\right)\end{aligned}$$

Intuitively, any height $\bar{h}>0$ and any tank temperature $\bar{T}_T$ satisfying

$$T_C \le \bar{T}_T \le T_H$$

should be a possible equilibrium point (after specifying the correct values of the equilibrium inputs). In fact, with $\bar{h}$ and $\bar{T}_T$ chosen, the equation $f(\bar{x},\bar{u})=0$ can be written as

$$\begin{bmatrix} 1 & 1 \\ T_C-\bar{x}_2 & T_H-\bar{x}_2 \end{bmatrix}\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} c_D A_o\sqrt{2g\bar{x}_1} \\ 0 \end{bmatrix}$$

The 2×2 matrix is invertible if and only if $T_C \neq T_H$. Hence, as long as $T_C \neq T_H$, there is a unique equilibrium input for any choice of $\bar{x}$. It is given by

$$\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \frac{1}{T_H-T_C}\begin{bmatrix} T_H-\bar{x}_2 & -1 \\ \bar{x}_2-T_C & 1 \end{bmatrix}\begin{bmatrix} c_D A_o\sqrt{2g\bar{x}_1} \\ 0 \end{bmatrix}$$

This is simply

$$\bar{u}_1 = \frac{c_D A_o\sqrt{2g\bar{x}_1}\left(T_H-\bar{x}_2\right)}{T_H-T_C} \quad , \quad \bar{u}_2 = \frac{c_D A_o\sqrt{2g\bar{x}_1}\left(\bar{x}_2-T_C\right)}{T_H-T_C}$$

Since the u_i represent flow rates **into** the tank, physical considerations restrict them to be nonegative real numbers. This implies that $\bar{x}_1 \ge 0$ and $T_C \le \bar{T}_T \le T_H$. Looking at the differential equation for T_T, we see that its rate of change is inversely related to h. Hence, the differential equation model is valid while $h(t)>0$, so we further restrict $\bar{x}_1>0$. Under those restrictions, the state $\bar{x}$ is indeed an equilibrium point, and there is a unique equilibrium input given by the equations above.

Next we compute the necessary partial derivatives.

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -\frac{g c_D A_o}{A_T\sqrt{2gx_1}} & 0 \\ -\frac{u_1[T_C-x_2]+u_2[T_H-x_2]}{x_1^2 A_T} & \frac{-(u_1+u_2)}{x_1 A_T} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{A_T} & \frac{1}{A_T} \\ \frac{T_C - x_2}{x_1 A_T} & \frac{T_H - x_2}{x_1 A_T} \end{bmatrix}$$

The linearization requires that the matrices of partial derivatives be evaluated **at** the equilibrium points. Let's pick some realistic numbers, and see how things vary with different equilibrium points. Suppose that $T_C = 10°, T_H = 90°, A_T = 3\mathtt{m}^2, A_o = 0.05\mathtt{m}, c_D = 0.7$. Try $\bar{h} = 1\mathtt{m}$ and $\bar{h} = 3\mathtt{m}$, and for $\bar{T}_T$, try $\bar{T}_T = 25°$ and $\bar{T}_T = 75°$. That gives 4 combinations. Plugging into the formulae give the 4 cases

1. $\left(\bar{h}, \bar{T}_T\right) = (1\mathtt{m}, 25°)$. The equilibrium inputs are

   $$\bar{u}_1 = \bar{q}_C = 0.126 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.029$$

   The linearized matrices are

   $$A = \begin{bmatrix} -0.0258 & 0 \\ 0 & -0.517 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -5.00 & 21.67 \end{bmatrix}$$

2. $\left(\bar{h}, \bar{T}_T\right) = (1\mathtt{m}, 75°)$. The equilibrium inputs are

   $$\bar{u}_1 = \bar{q}_C = 0.029 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.126$$

   The linearized matrices are

   $$A = \begin{bmatrix} -0.0258 & 0 \\ 0 & -0.0517 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -21.67 & 5.00 \end{bmatrix}$$

3. $\left(\bar{h}, \bar{T}_T\right) = (3\mathtt{m}, 25°)$. The equilibrium inputs are

   $$\bar{u}_1 = \bar{q}_C = 0.218 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.0503$$

   The linearized matrices are

   $$A = \begin{bmatrix} -0.0149 & 0 \\ 0 & -0.0298 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -1.667 & 7.22 \end{bmatrix}$$

4. $\left(\bar{h}, \bar{T}_T\right) = (3\mathtt{m}, 75°)$. The equilibrium inputs are

   $$\bar{u}_1 = \bar{q}_C = 0.0503 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.2181$$

   The linearized matrices are

   $$A = \begin{bmatrix} -0.0149 & 0 \\ 0 & -0.0298 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -7.22 & 1.667 \end{bmatrix}$$

We can try a simple simulation, both in the exact nonlinear equation, and the linearization, and compare answers.

We will simulate the system

$$\dot{x}(t) = f(x(t), u(t))$$

subject to the following conditions

$$x(0) = \begin{bmatrix} 1.10 \\ 81.5 \end{bmatrix}$$

and

$$u_1(t) = \begin{cases} 0.022 & \text{for} \quad 0 \leq t \leq 25 \\ 0.043 & \text{for} \quad 25 < t \leq 100 \end{cases}$$

$$u_2(t) = \begin{cases} 0.14 & \text{for} \quad 0 \leq t \leq 60 \\ 0.105 & \text{for} \quad 60 < t \leq 100 \end{cases}$$

This is close to equilibrium condition #2. So, in the linearization, we will use linearization #2, and the following conditions

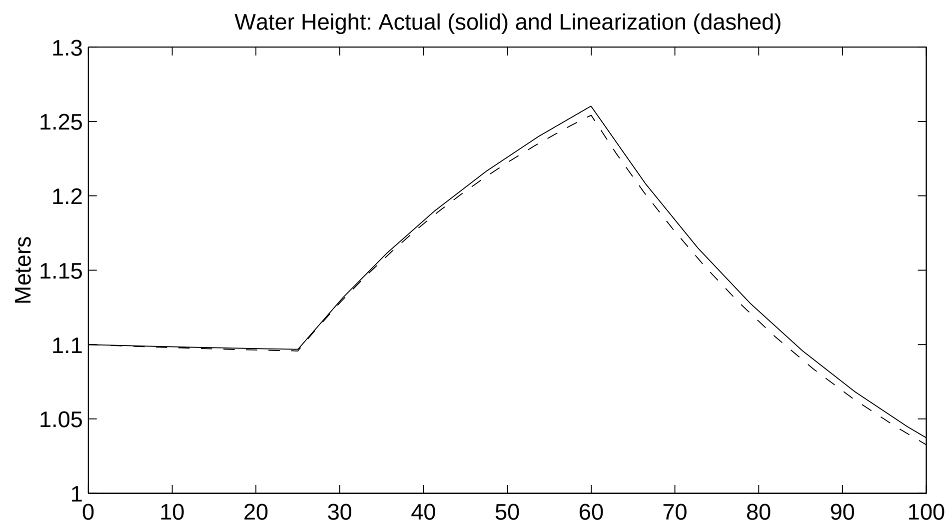
$$\delta_x(0) = x(0) - \begin{bmatrix} 1 \\ 75 \end{bmatrix} = \begin{bmatrix} 0.10 \\ 6.5 \end{bmatrix}$$

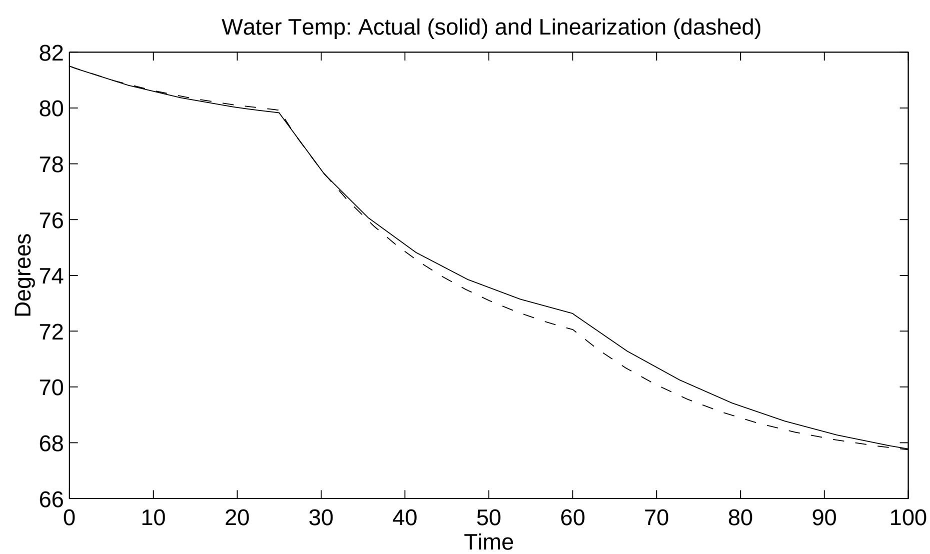
and

$$\delta_{u_1}(t) := u_1(t) - \bar{u}_1 = \begin{cases} -0.007 & \text{for} \quad 0 \leq t \leq 25 \\ 0.014 & \text{for} \quad 25 < t \leq 100 \end{cases}$$

$$\delta_{u_2}(t) := u_2(t) - \bar{u}_2 = \begin{cases} 0.014 & \text{for} \quad 0 \leq t \leq 60 \\ -0.021 & \text{for} \quad 60 < t \leq 100 \end{cases}$$

To compare the simulations, we must first plot $x(t)$ from the nonlinear simulation. This is the "true" answer. For the linearization, we know that δ_x **approximately** governs the deviations from $\bar{x}$. Hence, for that simulation we should plot $\bar{x} + \delta_x(t)$. These are shown below for both h and T_T.





19.4 Linearizing about general solution

In section 19.2, we discussed the linear differential equation which governs small deviations away from an equilibrium point. This resulted in a linear, time-invariant differential equation.

Often times, more complicated situations arise. Consider the task of controlling a rocket trajectory from Earth to the moon. By writing out the equations of motion, from Physics, we obtain state equations of the form

$$\dot{x}(t) = f(x(t), u(t), d(t), t) \tag{74}$$

where u is the control input (thrusts), d are external disturbances.

Through much computer simulation, a preplanned input schedule is developed, which, under ideal circumstances (ie., $d(t) \equiv 0$), would get the rocket from Earth to the moon. Denote this preplanned input by $\bar{u}(t)$, and the ideal disturbance by $\bar{d}(t)$ (which we assume is 0). This results in an ideal trajectory $\bar{x}(t)$, which solves the differential equation,

$$\dot{\bar{x}}(t) = f\left(\bar{x}(t), \bar{u}(t), \bar{d}(t), t\right)$$

Now, small nonzero disturbances are expected, which lead to small deviations in x, which must be corrected by small variations in the pre-planned input u. Hence, engineers need to have a model for how a slightly different input $\bar{u}(t) + \delta_u(t)$ and slightly different disturbance $\bar{d}(t) + \delta_d(t)$ will cause a different trajectory. Write $x(t)$ in terms of a deviation from $\bar{x}(t)$, defining $\delta_x(t) := x(t) - \bar{x}(t)$, giving

$$x(t) = \bar{x}(t) + \delta_x(t)$$

Now x, u and d must satisfy the differential equation, which gives

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), u(t), d(t), t\right) \\ \dot{\bar{x}}(t) + \dot{\delta}_x(t) &= f\left(\bar{x}(t) + \delta_x(t), \bar{u}(t) + \delta_u(t), \bar{d}(t) + \delta_d(t), t\right) \\ &\approx f\left(\bar{x}(t), \bar{u}(t), 0, t\right) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}} \delta_x(t) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}} \delta_u(t) + \left.\frac{\partial f}{\partial d}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}} \delta_d(t) \end{aligned}$$

But the functions $\bar{x}$ and $\bar{u}$ satisfy the governing differential equation, so $\dot{\bar{x}}(t) = f\left(\bar{x}(t), \bar{u}(t), 0, t\right)$, leaving the (approximate) governing equation for δ_x

$$\dot{\bar{x}}(t) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}} \delta_x(t) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}} \delta_u(t) + \left.\frac{\partial f}{\partial d}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}} \delta_d(t)$$

Define *time-varying* matrices A, B_1 and B_2 by

$$A(t) := \left.\frac{\partial f}{\partial x}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}} \qquad B_1(t) := \left.\frac{\partial f}{\partial u}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}} \qquad B_2(t) := \left.\frac{\partial f}{\partial d}\right|_{\substack{x(t)=\bar{x}(t)\\u(t)=\bar{u}(t)\\d(t)=\bar{d}(t)}}$$

The deviation variables are approximately governed by

$$\dot{\delta}_x(t) = A(t)\delta_x(t) + B_1(t)\delta_u(t) + B_2(t)\delta_d(t) = A(t)\delta_x(t) + \begin{bmatrix} B_1(t) & B_2(t) \end{bmatrix} \begin{bmatrix} \delta_u(t) \\ \delta_d(t) \end{bmatrix}$$

This is called the linearization of system (74) *about the trajectory* $\left(\bar{x}, \bar{u}, \bar{d}\right)$. This type of linearization is a generalization of linearizations about equilibrium points. Linearizing about an equilibrium point yields a LTI system, while linearizing about a trajectory yields an LTV system.

In general, these types of calculations are carried out numerically,

- `simulink` to get solution $\bar{x}(t)$ given particular inputs $\bar{u}(t)$ and $\bar{d}(t)$

- Numerical evaluation of $\frac{\partial f}{\partial x}$, etc, evaluated at the solution points
- Storing the time-varying matrices $A(t)$, $B_1(t)$, $B_2(t)$ for later use

In a simple case it is possible to analytically determine the time-varying matrices.

19.5 Problems

1. The temperature in a particular 3-dimensional solid is a function of position, and is known to be

$$T(x, y, z) = 42 + (x-2)^2 + 3(y-4)^2 - 5(z-6)^2 + 2yz$$

   (a) Find the first order approximation (linearization) of the temperature near the location $(4, 6, 0)$. Use δ_x, δ_y and δ_z as your deviation variables.

   (b) What is the maximum error between the actual temperature and the first order approximation formula for $|\delta_x| \le 0.3$, $|\delta_y| \le 0.2$, $|\delta_z| \le 0.1$. You may use Matlab to solve this numerically, or analytically if you wish.

   (c) More generally, suppose that $\bar{x} \in \mathbf{R}$, $\bar{y} \in \mathbf{R}$, $\bar{z} \in \mathbf{R}$. Find the first order approximation of the temperature near the location $(\bar{x}, \bar{y}, \bar{z})$.

2. The pitching-axis of a tail-fin controlled missile is governed by the nonlinear state equations

$$\begin{aligned} \dot{\alpha}(t) &= K_1 M f_n(\alpha(t), M) \cos\alpha(t) \; + \; q(t) \\ \dot{q}(t) &= K_2 M^2 \left[f_m(\alpha(t), M) + Eu(t) \right] \end{aligned}$$

   Here, the states are $x_1 := \alpha$, the angle-of-attack, and $x_2 := q$, the angular velocity of the pitching axis. The input variable, u, is the deflection of the fin which is mounted at the tail of the missile. K_1, K_2, and E are physical constants, with $E > 0$. M is the speed (Mach) of the missile, and f_n and f_m are known, differentiable functions (from wind-tunnel data) of α and M. Assume that M is a constant, and $M > 0$.

   (a) Show that for any specific value of $\bar{\alpha}$, with $|\bar{\alpha}| < \frac{\pi}{2}$, there is a pair $(\bar{q}, \bar{u})$ such that

$$\begin{bmatrix} \bar{\alpha} \\ \bar{q} \end{bmatrix}, \; \bar{u}$$

   is an equilibrium point of the system (this represents a turn at a constant rate). Your answer should clearly show how $\bar{q}$ and $\bar{u}$ are functions of $\bar{\alpha}$, and will most likely involve the functions f_n and f_m.

   (b) Calculate the Jacobian Linearization of the missile system about the equilibrium point. Your answer is fairly symbolic, and may depend on partial derivatives of the functions f_n and f_m. Be sure to indicate where the various terms are evaluated.

3. (Taken from "Introduction to Dynamic Systems Analysis," T.D. Burton, 1994) Constants α and β are given. Let $f_1(x_1, x_2) := x_1 - x_1^3 + \alpha x_1 x_2$, and $f_2(x_1, x_2) := -x_2 + \beta x_1 x_2$. Consider the 2-state system $\dot{x}(t) = f(x(t))$.

   (a) Note that there are no inputs. Find all equilibrium points of this system

   (b) Derive the Jacobian linearization which describes the solutions near <u>each</u> equilibrium point.

4. Car Engine Model (the reference for this problem is Cho and Hedrick, "Automotive Powertrain Modelling for Control," *ASME Journal of Dynamic Systems, Measurement and Control*, vol. 111, No. 4, December 1989): In this problem we consider a 1-state model for an automotive engine, with the transmission engaged in 4th gear. The engine state is m_a, the mass of air (in kilograms) in the intake manifold. The state of the drivetrain is the angular velocity, ω_e, of the engine. The input is the throttle angle, α, (in radians). The equations for the engine is

$$\begin{aligned} \dot{m}_a(t) &= c_1 T(\alpha(t)) - c_2 \omega_e(t) m_a(t) \\ T_e &= c_3 m_a(t) \end{aligned}$$

   where we treat ω_e and α as inputs, and T_e as an output. The drivetrain is modeled

$$\dot{\omega}_e(t) = \frac{1}{J_e}\left[T_e(t) - T_f(t) - T_d(t) - T_r(t)\right]$$

   The meanings of the terms are as follows:

   - $T(\alpha)$ is a throttle flow characteristic depending on throttle angle,

$$T(\alpha) = \begin{cases} T(-\alpha) & \text{for} \quad \alpha < 0 \\ 1 - \cos(1.14\alpha - 0.0252) & \text{for} \quad 0 \le \alpha \le 1.4 \\ 1 & \text{for} \quad \alpha > 1.4 \end{cases}$$

   - T_e is the torque from engine on driveshaft, $c_3 = 47500$ `Nm/kg`.
   - T_f is engine friction torque (`Nm`), $T_f = 0.106\omega_e + 15.1$
   - T_d is torque due to wind drag (`Nm`), $T_d = c_4\omega_e^2$, with $c_4 = 0.0026$ $\texttt{Nms}^2$
   - T_r is torque due to rolling resistance at wheels (`Nm`), $T_r = 21.5$.
   - J_e is the effective moment of inertia of the engine, transmission, wheels, car, $J_e = 36.4\texttt{kgm}^2$
   - $c_1 = 0.6$`kg/s`, $c_2 = 0.095$
   - In 4th gear, the speed of car, v (`m/s`), is related to ω_e as $v = 0.129\omega_e$.

   (a) Combine these equations into state variable form.

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned}$$

   where $x_1 = m_a$, $x_2 = \omega_e$, $u = \alpha$ and $y = v$.

(b) For the purpose of calculating the Jacobian linearization, explicitly write out the function $f(x, u)$ (without any t dependence). Note that f maps 3 real numbers (x_1, x_2, u) into 2 real numbers. There is **no** explicit time dependence in this particular f.

(c) Compute the equilibrium values of $\bar{m}_a$, $\bar{\omega}_e$ and $\bar{\alpha}$ so that the car travels at a constant speed of 22 `m/s` . Repeat the calculation for a constant speed of 32 `m/s` .

(d) Consider deviations from these equilibrium values,

$$\begin{aligned}
\alpha(t) &= \bar{\alpha} + \delta_\alpha(t) \\
\omega_e(t) &= \bar{\omega}_e + \delta_{\omega_e}(t) \\
m_a(t) &= \bar{m}_a + \delta_{m_a}(t) \\
y(t) &= \bar{y} + \delta_y(t)
\end{aligned}$$

Find the (linear) differential equations that approximately govern these deviation variables (Jacobian Linearzation discussed in class), in both the 22 `m/s` , and 32 `m/s` case.

(e) Using `Simulink`, **starting from the equilibrium point computed in part 4c** (not `[0;0]`), apply a sinusoidal throttle angle of

$$\alpha(t) = \bar{\alpha} + \beta \sin(0.5t)$$

for 3 values of β, $\beta = 0.01, 0.04, 0.1$ and obtain the response for v and m_a. Compare (and comment) on the difference between the results of the "true" nonlinear simulation and the results from the linearized analysis. Do this for both cases.

20 Linear Systems and Time-Invariance

We have studied the governing equations for many types of systems. Usually, these are nonlinear differential equations which govern the evolution of the variables as time progresses. In section 19.2, we saw that we could "linearize" a nonlinear system about an equilibrium point, to obtain a linear differential equation which giverns the approximate behavior of the system near the equilibrium point. Also, in some systems, the governing equations were already linear. Linear differential equations are important class of systems to study, and these are the topic of the next several section.

20.1 Linearity of solution

Consider a vector differential equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)d(t) \\ x(t_0) &= x_0 \end{aligned} \tag{75}$$

where for each t, $A(t) \in \mathbf{R}^{n \times n}$, $B(t) \in \mathbf{R}^{n \times m}$, and for each time t, $x(t) \in \mathbf{R}^n$ and $d(t) \in \mathbf{R}^m$.

Claim: The solution function $x(\cdot)$, on any interval $[t_0, t_1]$ is a **linear function of the pair** $\left(x_0, d(t)_{[t_0 \ t_1]}\right)$.

The precise meaning of this statement is as follows: Pick any constants $\alpha, \beta \in \mathbf{R}$,

- if $x_1(t)$ is the solution to (75) starting from initial condition $x_{1,0}$ (at t_0) and forced with input $d_1(t)$, and
- if $x_2(t)$ is the solution to (75) starting from initial condition $x_{2,0}$ (at t_0) and forced with input $d_2(t)$,

then $\alpha x_1(t) + \beta x_2(t)$ is <u>the</u> solution to (75) starting from initial condition $\alpha x_{1,0} + \beta x_{2,0}$ (at t_0) and forced with input $\alpha d_1(t) + \beta d_2(t)$.

This can be easily checked, note that for every time t, we have

$$\begin{aligned} \dot{x}_1(t) &= A(t)x_1(t) + B(t)d_1(t) \\ \dot{x}_2(t) &= A(t)x_2(t) + B(t)d_2(t) \end{aligned}$$

Multiply the first equation by the constant α and the second equation by β, and add them together, giving

$$\begin{aligned} \frac{d}{dt}\left[\alpha x_1(t) + \beta x_2(t)\right] &= \alpha \dot{x}_1(t) + \beta \dot{x}_2(t) \\ &= \alpha\left[Ax_1(t) + Bd_1(t)\right] + \beta\left[Ax_2(t) + Bd_2(t)\right] \\ &= A\left[\alpha x_1(t) + \beta x_2(t)\right] + B\left[\alpha d_1(t) + \beta d_2(t)\right] \end{aligned}$$

which shows that the linear combination $\alpha x_1 + \beta x_2$ does indeed solve the differential equation. The initial condition is easily checked. Finally, the existence and uniqueness theorem for differential equations tells us that this $(\alpha x_1 + \beta x_2)$ is the only solution which satisfies both the differential equation **and** the initial conditions.

Linearity of the solution in the pair (initial condition, forcing function) is often called **The Principal of Superposition** and is an extremely useful property of linear systems.

20.2 Time-Invariance

A separate issue, unrelated to linearity, is time-invariance. A system described by $\dot{x}(t) = f(x(t), d(t), t)$ is called *time-invariant* if (roughly) the behavior of the system does depend *explicitly* on the absolute time. In other words, shifting the time axis does not affect solutions.

Precisely, suppose that x_1 is a solution for the system, starting from x_{10} at $t = t_0$ subject to the forcing $d_1(t)$, defined for $t \geq t_0$.

Now, let $\tilde{x}$ be the solution to the equations starting from x_0 at $t = t_0 + \Delta$, subject to the forcing $\tilde{d}(t) := d(t-\Delta)$, defined for $t \geq t_0+\Delta$. Suppose that for all choices of t_0, Δ, x_0 and $d(\cdot)$, the two responses are related by $\tilde{x}(t) = x(t-\Delta)$ for all $t \geq t_0+\Delta$. Then the system described by $\dot{x}(t) = f(x(t), d(t), t)$ is called *time-invariant*.

In practice, the easiest manner to recognize time-invariance is that the right-hand side of the state equations (the first-order differential equations governing the process) do not explicitly depend on time. For instance, the system

$$\begin{aligned} \dot{x}_1(t) &= 2 * x_2(t) - \sin\left[x_1(t)x_2(t)d_2(t)\right] \\ \dot{x}_2(t) &= -\left|x_2(t)\right| - x_2(t)d_1(t) \end{aligned}$$

is nonlinear, yet time-invariant.

21 Matrix Exponential

21.1 Definition

Recall that for the scalar differential equation

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) \\ x(t_0) &= x_0 \end{aligned}$$

the solution for $t \geq t_0$ is given by the formula

$$x(t) = e^{a(t-t_0)} + \int_{t_0}^{t} e^{a(t-\tau)} bu(\tau) d\tau$$

What makes this work? The main thing is the special structure of the exponential function. Namely that

$$\frac{d}{dt} e^{at} = ae^{at}$$

Now consider a vector differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(t_0) &= x_0 \end{aligned} \qquad (76)$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$ are constant matrices, and for each time t, $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$. The solution can be derived by proceeding analogously to the scalar case.

For a matrix $A \in \mathbf{C}^{n \times n}$ define a matrix function of time, $e^{At} \in \mathbf{C}^{n \times n}$ as

$$\begin{aligned} e^{At} &:= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \\ &= I + tA + \tfrac{t^2}{2!} A^2 + \tfrac{t^3}{3!} A^3 + \cdots \end{aligned}$$

This is exactly the same as the definition in the scalar case. Now, for any $T > 0$, every element of this matrix power series converges absolutely and uniformly on the interval $[0, T]$. Hence, it can be differentiated term-by-term to correctly get the derivative. Therefore

$$\begin{aligned} \tfrac{d}{dt} e^{At} &:= \sum_{k=0}^{\infty} \frac{k\ t^{k-1}}{k!} A^k \\ &= A + tA^2 + + \tfrac{t^2}{2!} A^3 + \cdots \\ &= A\left(I + tA + \tfrac{t^2}{2!} A^2 + \cdots\right) \\ &= Ae^{At} \end{aligned}$$

This is the most important property of the function e^{At}. Also, in deriving this result, A could have been pulled out of the summation on either side. Summarizing these important identities,

$$e^{A0} = I_n, \qquad \frac{d}{dt} e^{At} = Ae^{At} = e^{At} A$$

So, the matrix exponential has properties similar to the scalar exponential function. However, there are two important facts to watch out for:

- **WARNING:** Let a_{ij} denote the (i, j)th entry of A. The (i, j)th entry of e^{At} **IS NOT EQUAL TO** $e^{a_{ij}t}$. This is most convincingly seen with a nontrivial example. Consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

  A few calculations show that

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad \cdots \qquad A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

  The definition for e^{At} is

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

  Plugging in, and evaluating on a term-by-term basis gives

$$e^{At} = \begin{bmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots & 0 + t + 2\frac{t^2}{2!} + 3\frac{t^3}{3!} + \cdots \\ 0 + 0 + 0 + 0 + \cdots & 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \end{bmatrix}$$

  The $(1, 1)$ and $(2, 2)$ entries are easily seen to be the power series for e^t, and the $(2, 1)$ entry is clearly 0. After a few manipulations, the $(1, 2)$ entry is te^t. Hence, in this case

$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

  which is very different than the element-by-element exponentiation of At,

$$\begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{bmatrix} = \begin{bmatrix} e^t & e^t \\ 1 & e^t \end{bmatrix}$$

- **WARNING:** In general,

$$e^{(A_1+A_2)t} \neq e^{A_1 t} e^{A_2 t}$$

  unless $t = 0$ (trivial) or $A_1 A_2 = A_2 A_1$. However, the identity

$$e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$$

  is always true. Hence $e^{At}e^{-At} = e^{-At}e^{At} = I$ for all matrices A and all t, and therefore for all A and t, e^{At} is invertible.

21.2 Diagonal A

If $A \in \mathbf{C}^{n \times n}$ is diagonal, then e^{At} is easy to compute. Specifically, if A is diagonal, it is easy to see that for any k,

$$A = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n \end{bmatrix} \qquad A^k = \begin{bmatrix} \beta_1^k & 0 & \cdots & 0 \\ 0 & \beta_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n^k \end{bmatrix}$$

In the power series definition for e^{At}, any off-diagonal terms are identically zero, and the i'th diagonal term is simply

$$\left[e^{At}\right]_{ii} = 1 + t\beta_i + \frac{t^2}{2!}\beta_i^2 + \frac{t^3}{3!}\beta_i^3 + \cdots = e^{\beta_i t}$$

21.3 Block Diagonal A

If $A_1 \in \mathbf{C}^{n_1 \times n_1}$ and $A_2 \in \mathbf{C}^{n_2 \times n_2}$, define

$$A := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

Question: How is e^{At} related to $e^{A_1 t}$ and $e^{A_2 t}$? Very simply – note that for any $k \geq 0$,

$$A^k = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}^k = \begin{bmatrix} A_1^k & 0 \\ 0 & A_2^k \end{bmatrix}$$

Hence, the power series definition for e^{At} gives

$$e^{At} = \begin{bmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_2 t} \end{bmatrix}$$

21.4 Effect of Similarity Transformations

For any invertible $T \in \mathbf{C}^{n \times n}$,

$$e^{T^{-1}ATt} = T^{-1}e^{At}T$$

Equivalently,

$$Te^{T^{-1}ATt}T^{-1} = e^{At}$$

This can easily be shown from the power series definition.

$$e^{T^{-1}ATt} = I + t\left(T^{-1}AT\right) + \frac{t^2}{2!}\left(T^{-1}AT\right)^2 + \frac{t^3}{3!}\left(T^{-1}AT\right)^3 + \cdots$$

It is easy to verify that for every integer k

$$\left(T^{-1}AT\right)^k = T^{-1}A^kT$$

Hence, we have (with I written as $T^{-1}IT$)

$$e^{T^{-1}ATt} = T^{-1}IT + tT^{-1}AT + \frac{t^2}{2!}T^{-1}A^2T + \frac{t^3}{3!}T^{-1}A^3T + \cdots$$

Pull T^{-1} out on the left side, and T on the right to give

$$e^{T^{-1}ATt} = T^{-1}\left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)T$$

which is simply

$$e^{T^{-1}ATt} = T^{-1}e^{At}T$$

as desired.

21.5 Candidate Solution To State Equations

The matrix exponential is extremely important in the solution of the vector differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{77}$$

starting from the initial condition $x(t_0) = x_0$. First, consider the case when $B \equiv 0_{n\times m}$, so that the differential equation is

$$\dot{x}(t) = Ax(t) \tag{78}$$

with $x(t_0) = x_0$. Proceeding as in the scalar case, a candidate for the solution for $t \geq t_0$ is

$$x_c(t) := e^{A(t-t_0)}x_0.$$

We need to check two things

- x_c should satisfy the initial condition – plugging in gives

  $$x_c(t_0) = e^{A(t_0-t_0)}x_0 = Ix_0 = x_0$$

  which verifies the first condition.

- x_c should satisfy the differential equation – differentiating the definition of x_c gives

  $$\begin{aligned}\dot{x}_c(t) &= \tfrac{d}{dt}\left[e^{A(t-t_0)}x_0\right]\\ &= \tfrac{d}{dt}\left[e^{A(t-t_0)}\right]x_0\\ &= Ae^{A(t-t_0)}x_0\\ &= Ax_c(t)\end{aligned}$$

  which verifies the second condition

Hence $x_c(t) := e^{A(t-t_0)}x_0$ is **the** solution to the undriven system in equation (78) for all $t \geq t_0$.

Now, consider the original equation in (77). We can derive the solution to the forced equation using the "integrating factor" methid, proceeding in the same manner as in the scalar case, with extra care for the matrix-vector operations. Suppose a function x satisfies (77). Multiply both sides by e^{-At} to give

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t)$$

Move one term to the left, leaving,

$$\begin{aligned} e^{-At}Bu(t) &= e^{-At}\dot{x}(t) - e^{-At}Ax(t) \\ &= e^{-At}\dot{x}(t) - Ae^{-At}x(t) \\ &= \tfrac{d}{dt}\left[e^{-At}x(t)\right] \end{aligned} \tag{79}$$

Since these two functions are equal at every time, we can integrate them over the interval $[t_0 \ t_1]$. Note that the right-hand side of (79) is an exact derivative, giving

$$\begin{aligned} \int_{t_0}^{t_1} e^{-At}Bu(t)dt &= \int_{t_0}^{t_1} \frac{d}{dt}\left[e^{-At}x(t)\right]dt \\ &= e^{-At}x(t)\Big|_{t_0}^{t_1} \\ &= e^{-At_1}x(t_1) - e^{-At_0}x(t_0) \end{aligned}$$

Note that $x(t_0) = x_0$. Also, multiply both sides by e^{At_1}, to yield

$$e^{At_1}\int_{t_0}^{t_1} e^{-At}Bu(t)dt = x(t_1) - e^{A(t_1-t_0)}x_0$$

This is rearranged into

$$x(t_1) = e^{A(t_1-t_0)}x_0 + \int_{t_0}^{t_1} e^{A(t_1-t)}Bu(t)dt$$

Finally, switch variable names, letting τ be the variable of integration, and letting t be the right end point (as opposed to t_1). In these new letters, the expression for the solution of the (77) for $t \geq t_0$, subject to initial condition $x(t_0) = x_0$ is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

consisting of a *free* and *forced* response.

21.6 Examples

Given $\beta \in \mathbf{R}$, define

$$A := \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$$

Calculate

$$A^2 = \begin{bmatrix} -\beta^2 & 0 \\ 0 & -\beta^2 \end{bmatrix}$$

and hence for any k,

$$A^{2k} = \begin{bmatrix} (-1)^k\beta^{2k} & 0 \\ 0 & (-1)^k\beta^{2k} \end{bmatrix}, \quad A^{2k+1} = \begin{bmatrix} 0 & (-1)^k\beta^{2k+1} \\ (-1)^{k+1}\beta^{2k+1} & 0 \end{bmatrix}$$

Therefore, we can write out the first few terms of the power series for e^{At} as

$$e^{At} = \begin{bmatrix} 1 - \frac{1}{2}\beta^2t^2 + \frac{1}{4!}\beta^4t^4 - \cdots & \beta t - \frac{1}{3!}\beta^3t^3 + \frac{1}{5!}\beta^5t^5 - \cdots \\ -\beta t + \frac{1}{3!}\beta^3t^3 - \frac{1}{5!}\beta^5t^5 + \cdots & 1 - \frac{1}{2}\beta^2t^2 + \frac{1}{4!}\beta^4t^4 - \cdots \end{bmatrix}$$

which is recognized as

$$e^{At} = \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix}$$

Similarly, suppose $\alpha, \beta \in \mathbf{R}$, and

$$A := \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

Then, A can be decomposed as $A = A_1 + A_2$, where

$$A_1 := \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$$

Note: **in this special case**, $A_1A_2 = A_2A_1$, hence

$$e^{(A_1+A_2)t} = e^{A_1t}e^{A_2t}$$

Since A_1 is diagonal, we know e^{A_1t}, and e^{A_2t} follows from our previous example. Hence

$$e^{At} = \begin{bmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ -e^{\alpha t}\sin\beta t & e^{\alpha t}\cos\beta t \end{bmatrix}$$

This is an important case to remember. Finally, suppose $\lambda \in \mathbf{F}$, and

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

A few calculations show that

$$A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$$

Hence, the power series gives

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

22 Eigenvalues, eigenvectors, stability

22.1 Diagonalization: Motivation

Recall two facts from Section 21: For diagonal matrices $\Lambda \in \mathbf{F}^{n\times n}$,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \Rightarrow \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

and: If $A \in \mathbf{F}^{n\times n}$, and $T \in \mathbf{F}^{n\times n}$ is invertible, and $\tilde{A} := T^{-1}AT$, then

$$e^{At} = Te^{\tilde{A}t}T^{-1}$$

Clearly, for a general $A \in \mathbf{F}^{n\times n}$, we need to study the invertible transformations $T \in \mathbf{F}^{n\times n}$ such that $T^{-1}AT$ is a diagonal matrix.

Suppose that T is invertible, and Λ is a diagonal matrix, and $T^{-1}AT = \Lambda$. Moving the T^{-1} to the other side of the equality gives $AT = T\Lambda$. Let t_i denote the i'th column of the matrix T. SInce T is assumed to be invertible, none of the columns of T can be identically zero, hence $t_i \neq \Theta_n$. Also, let λ_i denote the (i,i)'th entry of Λ. The i'th column of the matrix equation $AT = T\Lambda$ is just

$$At_i = t_i\lambda_i = \lambda_i t_i$$

This observation leads to the next section.

22.2 Eigenvalues

Definition: Given a matrix $A \in \mathbf{F}^{n\times n}$. A complex number λ is an **eigenvalue** of A if there is a nonzero vector $v \in \mathbf{C}^n$ such that

$$Av = v\lambda = \lambda v$$

The nonzero vector v is called an **eigenvector** of A associated with the eigenvalue λ.

Remark: Consider the differential equation $\dot{x}(t) = Ax(t)$, with initial condition $x(0) = v$. Then $x(t) = ve^{\lambda t}$ is the solution (check that it satisfies initial condition and differential equation). So, an eigenvector is "direction" in the state-space such that if you start in the direction of the eigenvector, you stay in the direction of the eigenvector.

Note that if λ is an eigenvalue of A, then there is a vector $v \in \mathbf{C}^n$, $v \neq \Theta_n$ such that

$$Av = v\lambda = (\lambda I)\, v$$

Hence

$$(\lambda I - A) v = \Theta_n$$

Since $v \neq \Theta_n$, it must be that $\det(\lambda I - A) = 0$.

For an $n \times n$ matrix A, define a polynomial, $p_A(\cdot)$, called **the characteristic polynomial of A** by

$$p_A(s) := \det(\lambda I - A)$$

Here, the symbol s is simply the indeterminate variable of the polynomial.

For example, take

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & -1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

Straightforward manipulation gives $p_A(s) = s^3 - s^2 + 3s - 2$. Hence, we have shown that the eigenvalues of A are **necessarily** roots of the equation

$$p_A(s) = 0.$$

For a general $n \times n$ matrix A, we will write

$$p_A(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

where the $a_1, a_2, \ldots, a_n$ are complicated products and sums involving the entries of A. Since the characteristic polynomial of an $n \times n$ matrix is a n'th order polynomial, the equation $p_A(s) = 0$ has **at most** n distinct roots (some roots could be repeated). **Therefore, a $n \times n$ matrix A has at most n distinct eigenvalues**.

Conversely, suppose that $\lambda \in \mathbf{C}$ is a root of the polynomial equation

$$p_A(s)|_{s=\lambda} = 0$$

Question: Is λ an eigenvalue of A?

Answer: Yes. Since $p_A(\lambda) = 0$, it means that

$$\det(\lambda I - A) = 0$$

Hence, the matrix $\lambda I - A$ is singular (not invertible). Therefore, by the matrix facts, the equation

$$(\lambda I - A) v = \Theta_n$$

has a **nonzero** solution vector v (which you can find by Gaussian elimination). This means that

$$\lambda v = Av$$

for a nonzero vector v, which means that λ is an eigenvalue of A, and v is an associated eigenvector.

We summarize these facts as:

- A is a $n \times n$ matrix
- The characteristic polynomial of A is

$$\begin{aligned} p_A(s) &:= \det(sI - A) \\ &= s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \end{aligned}$$

- A complex number λ is an eigenvalue of A if and only if λ is a root of the "characteristic equation" $p_a(\lambda) = 0$.

Next, we have a useful fact from linear algebra: Suppose A is a given $n \times n$ matrix, and $(\lambda_1, v_1), (\lambda_2, v_2), \ldots, (\lambda_n, v_n)$ are eigenvalue/eigenvector pairs. So, for each i, $v_i \neq \Theta_n$ and $Av_i = v_i\lambda_i$. **Fact:** If all of the $\{\lambda_i\}_{i=1}^n$ are distinct, then the set of vectors

$$\{v_1, v_2, \ldots, v_n\}$$

are a linearly independent set. In other words, the matrix

$$V := [v_1 \ v_2 \ \cdots \ v_n] \in \mathbf{C}^{n\times n}$$

is invertible.

Proof: We'll prove this for 3×3 matrices – check your linear algebra book for the generalization, which is basically the same proof.

Suppose that there are scalars $\alpha_1, \alpha_2, \alpha_3$, such that

$$\sum_{i=1}^{3} \alpha_i v_i = \Theta_3$$

This means that

$$\begin{aligned} \Theta_3 &= (A - \lambda_3 I)\,\Theta_3 \\ &= (A - \lambda_3 I) \textstyle\sum_{i=1}^3 \alpha_i v_i \\ &= \alpha_1 (A - \lambda_3 I)\, v_1 + \alpha_2 (A - \lambda_3 I)\, v_2 + \alpha_3 (A - \lambda_3 I)\, v_3 \\ &= \alpha_1 (\lambda_1 - \lambda_3)\, v_1 + \alpha_2 (\lambda_2 - \lambda_3)\, v_2 + \Theta_3 \end{aligned} \tag{80}$$

Now multiply by $(A - \lambda_2 I)$, giving

$$\begin{aligned} \Theta_3 = (A - \lambda_2 I)\,\Theta_3 &= (A - \lambda_2 I)\,[\alpha_1(\lambda_1 - \lambda_3)v_1 + \alpha_2(\lambda_2 - \lambda_3)v_2] \\ &= \alpha_1(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)v_1 \end{aligned}$$

Since $\lambda_1 \neq \lambda_3, \lambda_1 \neq \lambda_2, v_1 \neq \Theta_3$, it must be that $\alpha_1 = 0$. Using equation (80), and the fact that $\lambda_2 \neq \lambda_3, v_2 \neq \Theta_3$ we get that $\alpha_2 = 0$. Finally, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \Theta_3$ (by assumption), and $v_3 \neq \Theta_3$, so it must be that $\alpha_3 = 0$. $\sharp$

22.3 Diagonalization Procedure

In this section, we summarize all of the previous ideas into a step-by-step diagonalization procedure for $n \times n$ matrices.

1. Calculate the characteristic polynomial of A, $p_A(s) := \det(sI - A)$.
2. Find the n roots of the equation $p_A(s) = 0$, and call the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$.
3. For each i, find a nonzero vector $t_i \in \mathbf{C}_n$ such that
$$(A - \lambda_i I)\, t_i = \Theta_n$$
4. Form the matrix
$$T := [t_1 \; t_2 \; \cdots \; t_n] \in \mathbf{C}^{n \times n}$$
(note that if all of the $\{\lambda_i\}_{i=1}^{n}$ are distinct from one another, then T is guaranteed to be invertible).
5. Note that $AT = T\Lambda$, where
$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$
6. If T is invertible, then $T^{-1}AT = \Lambda$. Hence
$$e^{At} = Te^{\Lambda t}T^{-1} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} T^{-1}$$

We will talk about the case of nondistinct eigenvalues later.

22.4 e^{At} as $t \to \infty$

For the remainder of the section, assume A has distinct eigenvalues.

- if **all** of the eigenvalues (which may be complex) of A satisfy
$$\operatorname{Re}(\lambda_i) < 0$$
then $e^{\lambda_i t} \to 0$ as $t \to \infty$, so **all** entries of $\left[e^{At}\right]$ decay to zero

- If there is one (or more) eigenvalues of A with

$$\mathrm{Re}\,(\lambda_i) \geq 0$$

then

$$e^{\lambda_i t} \rightarrow \begin{array}{l} \text{bounded} \neq 0 \\ \infty \end{array} \quad \text{as} \rightarrow \infty$$

Hence, **some** of the entries of e^{At} either do not decay to zero, or, in fact, diverge to ∞.

So, the eigenvalues are an indicator (the key indicator) of stability of the differential equation

$$\dot{x}(t) = Ax(t)$$

- if **all** of the eigenvalues of A have **negative real parts**, then from any initial condition x_0, the solution

$$x(t) = e^{At}x_0$$

decays to Θ_n as $t \rightarrow \infty$ (all coordinates of $x(t)$ decay to 0 as $t \rightarrow \infty$). In this case, A is said to be a *Hurwitz* matrix.

- if **any** of the eigenvalues of A have **nonnegative real parts**, then from some initial conditions x_0, the solution to

$$\dot{x}(t) = Ax(t)$$

does **not** decay to zero.

22.5 Complex Eigenvalues

In many systems, the eigenvalues are complex, rather than real. This seems unusual, since the system itself (ie., the physical meaning of the state and input variables, the coefficients in the state equations, etc.) is very much Real. The procedure outlined in section 22.3 for matrix diagonalization is applicable to both real and complex eigenvalues, and if A is real, all intermediate complex terms will cancel, resulting in e^{At} being purely real, as expected. However, in the case of complex eigenvalues it may be more advantageous to use a different similarity transformation, which does not lead to a diagonalized matrix. Instead, it leads to a real block-diagonal matrix, whose structure is easily interpreted.

Let us consider a second order system, with complex eigenvalues

$$\dot{x}(t) = Ax(t) \tag{81}$$

where $A \in \mathbf{R}^{n \times n}$, $x \in \mathcal{R}^2$ and

$$p_A(\lambda) = \det(\lambda I_2 - A) = (\lambda - \sigma)^2 + \omega^2\,. \tag{82}$$

The two eigenvalues of A are $\lambda_1 = \sigma + j\omega$ and $\lambda_2 = \sigma - j\omega$, while the two eigenvectors of A are given by

$$(\lambda_1 I_2 - A)v_1 = \Theta_2 \qquad (\lambda_2 I_2 - A)v_2 = \Theta_2 \, . \tag{83}$$

Notice that, since λ_2 is the complex conjugate of λ_1, v_2 is the complex conjugate vector of v_1, i.e. if $v_1 = v_r + jv_i$, then $v_2 = v_r - jv_i$. This last fact can be verified as follows. Assume that $\lambda_1 = \sigma + j\omega$, $v_1 = v_r + jv_i$ and insert these expressions in the first of Eqs. (83).

$$[(\sigma + j\omega)I_2 - A)] \, [v_r + jv_i] = \Theta_2 \, ,$$

Separating into its real and imaginary parts we obtain

$$[\sigma v_r - \omega v_i] + j \, [\sigma v_i + \omega v_r] = Av_r + jAv_i$$

$$\begin{aligned} [\sigma v_r - \omega v_i] &= Av_r \\ [\sigma v_i + \omega v_r] &= Av_i \, . \end{aligned} \tag{84}$$

Notice that Eqs. (84) hold if we replace ω by $-\omega$ and v_i by $-v_i$. Thus, if $\lambda_1 = \sigma + j\omega$ and $v_1 = v_r + jv_i$ are respectively an eigenvalue and eigenvector of A, then $\lambda_2 = \sigma - j\omega$ and $v_2 = v_r - jv_i$ are also respectively an eigenvalue and eigenvector.

Eqs. (84) can be rewritten in matrix form as follows

$$A \begin{bmatrix} v_r & v_i \end{bmatrix} = \begin{bmatrix} v_r & v_i \end{bmatrix} \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} . \tag{85}$$

Thus, we can define the similarity transformation matrix

$$T = \begin{bmatrix} v_r & v_i \end{bmatrix} \in \mathcal{R}^{2\times 2} \tag{86}$$

and the matrix

$$J_c = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \tag{87}$$

such that

$$A = T \, J_c \, T^{-1}, \qquad e^{At} = T \, e^{J_c t} \, T^{-1} \, . \tag{88}$$

The matrix exponential $e^{J_c t}$ is easy to calculate. Notice that

$$J_c = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} = \sigma I_2 + S_2$$

where

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } S_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

is *skew-symmetric*, i.e. $S_2^T = -S_2$. Thus,

$$e^{S_2 t} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

This last result can be verified by differentiating with respect to time both sides of the equation:

$$\frac{d}{dt} e^{S_2 t} = S_2\, e^{S_2 t}$$

and

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} &= \begin{bmatrix} -\omega\sin(\omega t) & \omega\cos(\omega t) \\ -\omega\cos(\omega t) & -\omega\sin(\omega t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} = S_2\, e^{S_2 t}. \end{aligned}$$

Since $\sigma I_2\, S_2 = S_2\, \sigma I_2$, then

$$e^{J_c t} = e^{\sigma t}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}. \tag{89}$$

22.5.1 Examples

Consider the system

$$\dot{x}(t) = Ax(t) \tag{90}$$

where $x \in \mathcal{R}^2$ and

$$A = \begin{bmatrix} -1.\bar{3} & 1.\bar{6} \\ -1.\bar{6} & 1.\bar{3} \end{bmatrix}.$$

The two eigenvalues of A are $\lambda_1 = j$ and $\lambda_2 = -j$. Their corresponding eigenvectors are respectively

$$v_1 = \begin{bmatrix} 0.4243 + 0.5657j \\ 0 + 0.7071j \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0.4243 - 0.5657j \\ 0 - 0.7071j \end{bmatrix}.$$

Defining

$$T = \begin{bmatrix} 0.4243 & 0.5657 \\ 0 & 0.7071 \end{bmatrix} \tag{91}$$

we obtain $A\, T_2 = T_2\, J_c$, where

$$J_c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus,

$$e^{J_c t} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

and

$$e^{At} = T\, e^{J_c t} T^{-1} = \begin{bmatrix} 0.4243 & 0.5657 \\ 0 & 0.7071 \end{bmatrix} \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 2.3570 & -1.8856 \\ 0 & 1.4142 \end{bmatrix} .$$

Utilizing the coordinate transformation

$$x^* = T^{-1}x \qquad\qquad x = Tx^* , \tag{92}$$

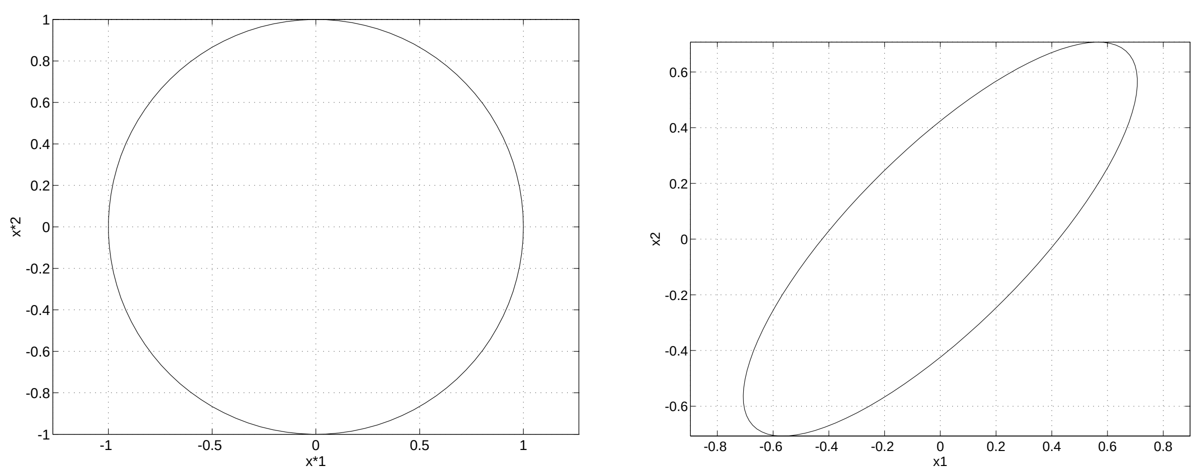
where T is given by Eq. (91), we can obtain from Eqs. (92) and(90)

$$\begin{aligned} \dot{x}^*(t) &= \left\{T^{-1}AT\right\} x^*(t) \\ &= J_c\, x^*(t) . \end{aligned} \tag{93}$$

Consider now the initial condition

$$x(0) = \begin{bmatrix} 0.4243 \\ 0 \end{bmatrix} = Tx^*(0) , \qquad\qquad x^*(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

The phase plot for $x^*(t)$ is given by Fig. 8-(a), while that for $x(t)$ is given by Fig. 8-(b).



(a) $\dot{x}^* = J_c x^*$ (b) $\dot{x} = Ax$

Figure 8: Phase plots

22.6 Problems

1. Find (by hand calculation) the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -3 & -2 & 2 \\ 5 & 7 & -5 \\ 5 & 8 & -6 \end{bmatrix}$$

Does e^{At} have all of its terms decaying to zero?

2. Read about the `MatLab` command `eig` (use the MatLab manual, the Matlab primer, and/or the comnand `>> help eig`). Repeat problem 1 using `eig`, and explain any differences that you get. If a matrix has distinct eigenvalues, in what sense are the eigenvectors not unique?

3. Consider the differential equation $\dot{x}(t) = Ax(t)$, where $x(t)$ is (3×1) and A is from problem (1) above. Suppose that the initial condition is

$$x(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Write the initial condition as a linear combination of the eigenvectors, and find the solution $x(t)$ to the differential equation, written as a time-dependent, linear combination of the eigenvectors.

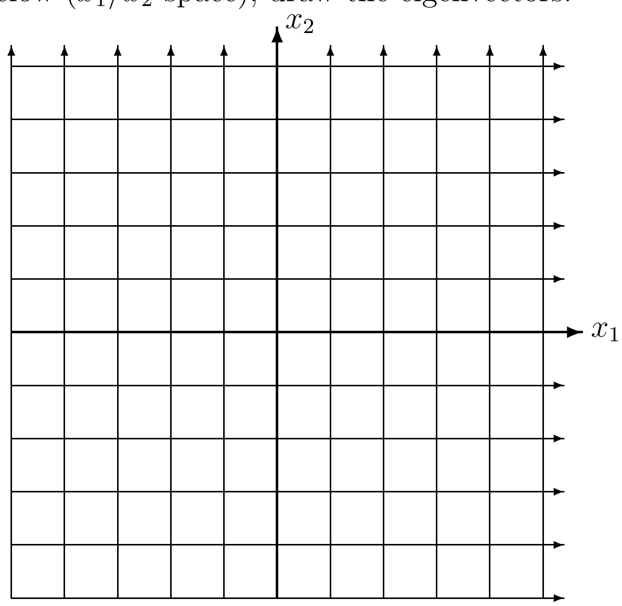
4. Suppose that we have a 2nd order system ($x(t) \in \mathbf{R}^2$) governed by the differential equation

$$\dot{x}(t) = \begin{bmatrix} 0 & -2 \\ 2 & -5 \end{bmatrix} x(t)$$

Let A denote the 2×2 matrix above.

(a) Find the eigenvalues an eigenvectors of A. In this problem, the eigenvectors **can** be chosen to have all integer entries (recall that eigenvectors can be scaled)

(b) On the grid below (x_1/x_2 space), draw the eigenvectors.



(c) A plot of $x_2(t)$ vs. $x_1(t)$ is called a *phase-plane* plot. The variable t is not explicitly plotted on an axis, rather it is the parameter of the tick marks along the plot. On the grid above, using hand calculations, draw the solution to the equation $\dot{x}(t) = Ax(t)$ for the initial conditions

$$x(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

HINT: Remember that if v_i are eigenvectors, and λ_i are eigenvalues of A, then the solution to $\dot{x}(t) = Ax(t)$ from the initial condition $x(0) = \sum_{i=1}^{2} \alpha_i v_i$ is simply

$$x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i$$

(d) Use Matlab to create a similar picture with many (say 20) different initial conditions spread out in the x_1, x_2 plane

5. Suppose A is a <u>real</u>, $n \times n$ matrix, and λ is an eigenvalue of A, and λ is not real, but λ is complex. Suppose $v \in \mathbf{C}^n$ is an eigenvector of A associated with this eigenvalue λ. Use the notation $\lambda = \lambda_R + j\lambda_I$ and $v = v_r + jv_I$ for the real and imaginary parts of λ, and v (j means $\sqrt{-1}$).

(a) By equating the real and imaginary parts of the equation $Av = \lambda v$, find two equations that relate the various real and imaginary parts of λ and v.

(b) Show that $\bar{\lambda}$ (complex conjugate of λ) is also an eigenvalue of A. What is the associated eigenvector?

(c) Consider the differential equation $\dot{x} = Ax$ for the A described above, with the eigenvalue λ and eigenvector v. Show that the function

$$x(t) = e^{\lambda_R t}\left[\cos\left(\lambda_I t\right) v_R - \sin\left(\lambda_I t\right) v_I\right]$$

satisfies the differential equation. What is $x(0)$ in this case?

(d) Fill in this sentence: If A has complex eigenvalues, then if $x(t)$ starts on the ____________ part of the eigenvector, the solution $x(t)$ oscillates between the ____________ and ____________ parts of the eigenvector, with frequency associated with the ____________ part of the eigenvalue. During the motion, the solution also increases/decreases exponentially, based on the ____________ part of the eigenvalue.

(e) Consider the matrix A

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

It is possible to show that

$$A\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ j\frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ j\frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1+2j & 0 \\ 0 & 1-2j \end{bmatrix}$$

Sketch the trajectory of the solution $x(t)$ in $\mathbf{R}^2$ to the differential equation $\dot{x} = Ax$ for the initial condition $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(f) Find e^{At} for the A given above. NOTE: e^{At} is real whenever A is real. See the notes for tricks in making this easy.

6. A hoop (of radius R) is mounted vertically, and rotates at a constant angular velocity Ω. A bead of mass m slides along the hoop, and θ is the angle that locates the bead location. $\theta = 0$ corresponds to the bead at the bottom of the hoop, while $\theta = \pi$ corresponds to the top of the hoop.

The nonlinear, 2nd order equation (from Newton's law) governing the bead's motion is

$$mR\ddot{\theta} + mg\sin\theta + \alpha\dot{\theta} - m\Omega^2 R\sin\theta\cos\theta = 0$$

All of the parameters m, R, g, α are positive.

(a) Let $x_1(t) := \theta(t)$ and $x_2(t) := \dot{\theta}(t)$. Write the 2nd order nonlinear differential equation in the state-space form

$$\begin{aligned} \dot{x}_1(t) &= f_1\left(x_1(t), x_2(t)\right) \\ \dot{x}_2(t) &= f_2\left(x_1(t), x_2(t)\right) \end{aligned}$$

(b) Show that $\bar{x}_1 = 0, \bar{x}_2 = 0$ is an equilibrium point of the system.

(c) Find the linearized system

$$\dot{\delta}_x(t) = A\delta_x(t)$$

which governs small deviations away from the equilibrium point $(0, 0)$.

(d) Under what conditions (on m, R, Ω, g) is the linearized system stable? Under what conditions is the linearized system unstable?

(e) Show that $\bar{x}_1 = \pi, \bar{x}_2 = 0$ is an equilibrium point of the system.

(f) Find the linearized system $\dot{\delta}_x(t) = A\delta_x(t)$ which governs small deviations away from the equilibrium point $(\pi, 0)$.

(g) Under what conditions is the linearized system stable? Under what conditions is the linearized system unstable?

(h) It would seem that if the hoop is indeed rotating (with angular velocity Ω) then there would other equilibrium point (with $0 < \theta < \pi/2$). Do such equilibrium points exist in the system? Be very careful, and please explain your answer.

(i) Find the linearized system $\dot{\delta}_x(t) = A\delta_x(t)$ which governs small deviations away from this equilibrium point.

(j) Under what conditions is the linearized system stable? Under what conditions is the linearized system unstable?

23 Jordan Form

23.1 Motivation

In the last section, we saw that if $A \in \mathbf{F}^{n\times n}$ has **distinct** eigenvalues, then there exists an invertible matrix $T \in \mathbf{C}^{n\times n}$ such that $T^{-1}AT$ is diagonal. In this sense, we say that A **is diagonalizable by similarity transformation** (the matrix $T^{-1}AT$ is called a similarity transformation of A).

Even some matrices that have repeated eigenvalues can be diagonalized. For instance, take $A = I_2$. Both of the eigenvalues of A are at $\lambda = 1$, yet A is diagonal (in fact, any invertible T makes $T^{-1}AT$ diagonal).

On the other hand, there are other matrices that **cannot** be diagonalized. Take, for instance,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The characteristic equation of A is $p_A(\lambda) = \lambda^2$. Clearly, both of the eigenvalues of A are equal to 0. Hence, **if** A can be diagonalized, the resulting diagonal matrix would be the 0 matrix (recall that if A can be diagonalized, then the diagonal elements are necessarily roots of the equation $p_A(\lambda) = 0$). Hence, there would be an invertible matrix T such that $T^{-1}AT = 0_{2\times 2}$. Rearranging gives

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiplying out both sides gives

$$\begin{bmatrix} t_{21} & t_{22} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

But if $t_{21} = t_{22} = 0$, then T is not invertible. Hence, this 2×2 matrix A cannot be diagonalized with a similarity transformation.

23.2 Details

It turns out that every matrix can be "almost" diagonalized using similarity transformations. Although this result is not difficult to prove, it will take us too far from the main flow of the course. The proof can be found in any decent linear algebra book. Here we simply outline the facts.

Definition: If $J \in \mathbf{C}^{m\times m}$ appears as

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

then J is called a **Jordan** block of dimension m, with associated eigenvalue λ.

Theorem: (Jordan Canonical Form) For every $A \in \mathbf{F}^{n\times n}$, there exists an invertible matrix $T \in \mathbf{C}^{n\times n}$ such that

$$T^{-1}AT = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{bmatrix}$$

where each J_i is a Jordan block.

If $J \in \mathbf{F}^{m\times m}$ is a Jordan block, then using the power series definition, it is straight-forward to show that

$$e^{Jt} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & \ddots & \frac{t^{m-2}}{(m-2)!}e^{\lambda t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda t} \end{bmatrix}$$

Note that if $\mathrm{Re}\,(\lambda) < 0$, then every element of e^{Jt} decays to zero, and if if $\mathrm{Re}\,(\lambda) \geq 0$, then some elements of e^{Jt} diverge to ∞.

23.3 Significance

In general, there is no numerically reliable way to compute the Jordan form of a matrix. It is a conceptual tool, which, among other things, shows us the extent to which e^{At} contains terms other than exponentials.

Notice that if the real part of λ is negative, then for any finite integer m,

$$\lim_{t\to\infty} \left\{ t^m e^{\lambda t} \right\} = 0\,.$$

From this result, we see that even in the case of Jordan blocks, the signs of the real parts of the eigenvalues of A determine the stability of the linear differential equation

$$\dot{x}(t) = Ax(t)$$