13 Control of Second-Order System

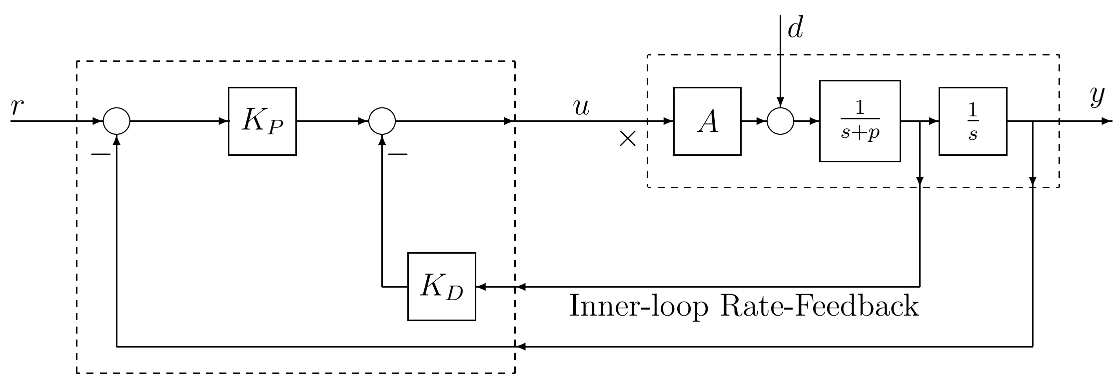
In this section, we analyze PD and PID control of a plant typical in mechanical positioning systems. We also propose a possible design method. The nominal model for the plant is

$$P(s) = \frac{A}{s(s+p)}$$

where A and p are fixed parameters.

13.1 PD control

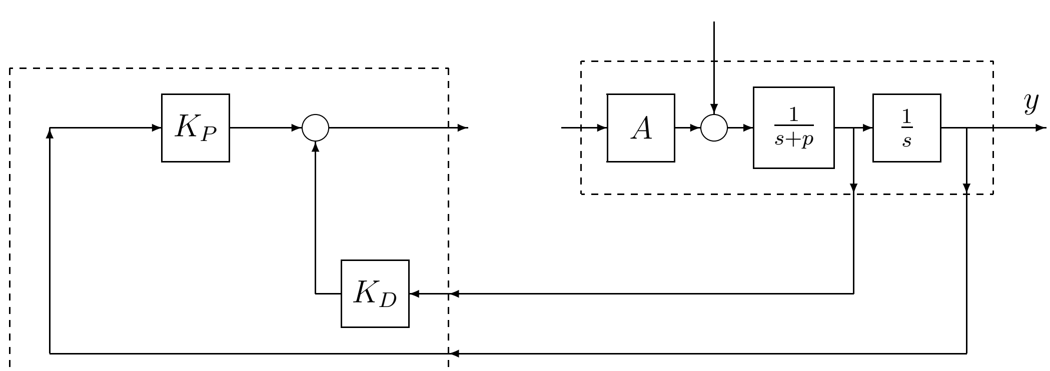
First, consider PD control, specifically proportional control, with inner loop rate-feedback. This is shown below (its just the PID diagram, with the integral action removed)



In terms of plant and controller parameters, the loop gain (at breaking point marked by $\times$) is

$$L(s) = \frac{A(K_D s + K_P)}{s(s+p)}$$

In other words, from a stability point of view, the system is just unity-gain, negative feedback around L.



The closed-loop transfer function from R and D to Y is

$$Y(s) = \frac{AK_P}{s^2 + (AK_D + p)s + AK_P} R(s) \ + \ \frac{1}{s^2 + (AK_D + p)s + AK_P} D(s)$$

The characteristic equation is

$$\text{CE}: \qquad s^2 + (AK_D + p)s + AK_P$$

Clearly, with two controller parameters, and a 2nd order closed-loop system, the poles can be freely assigned. Using the (ξ, ω_n) parametrization, we set the characteristic equation to be

$$s^2 + 2\xi\omega_n s + \omega_n^2$$

giving design equations

$$K_P := \frac{\omega_n^2}{A}, \qquad K_D := \frac{2\xi\omega_n - p}{A}$$

In terms of the (ξ, ω_n) parametrization, the loop gain and transfer functions are

$$L(s) = \frac{(2\xi\omega_n - p)s + \omega_n^2}{s(s+p)}$$

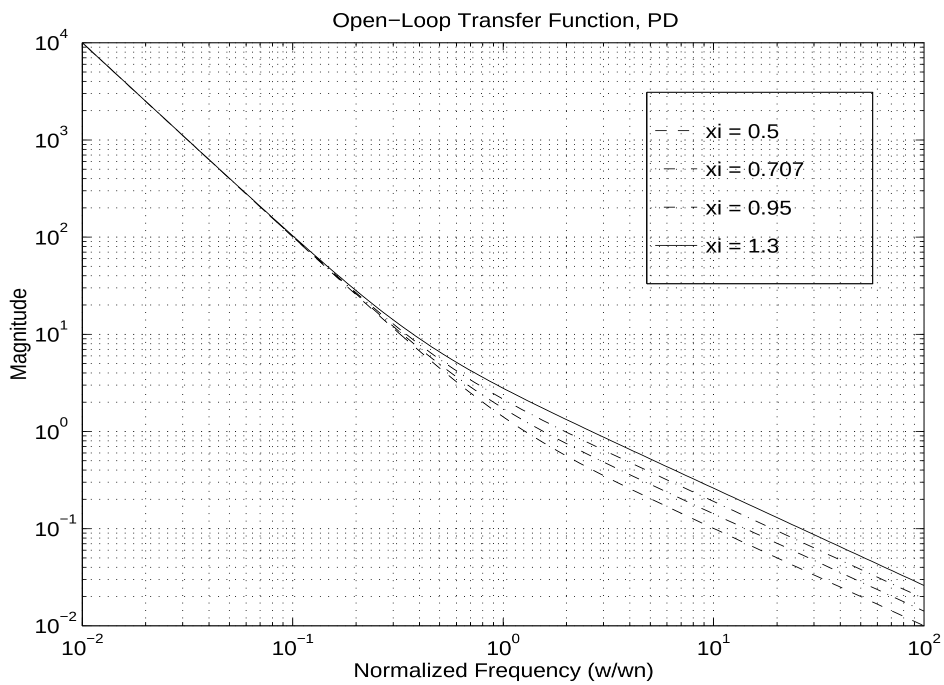
$$Y(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2} R(s) \ + \ \frac{1}{s^2 + 2\xi\omega_n + \omega_n^2} D(s) \qquad (66)$$

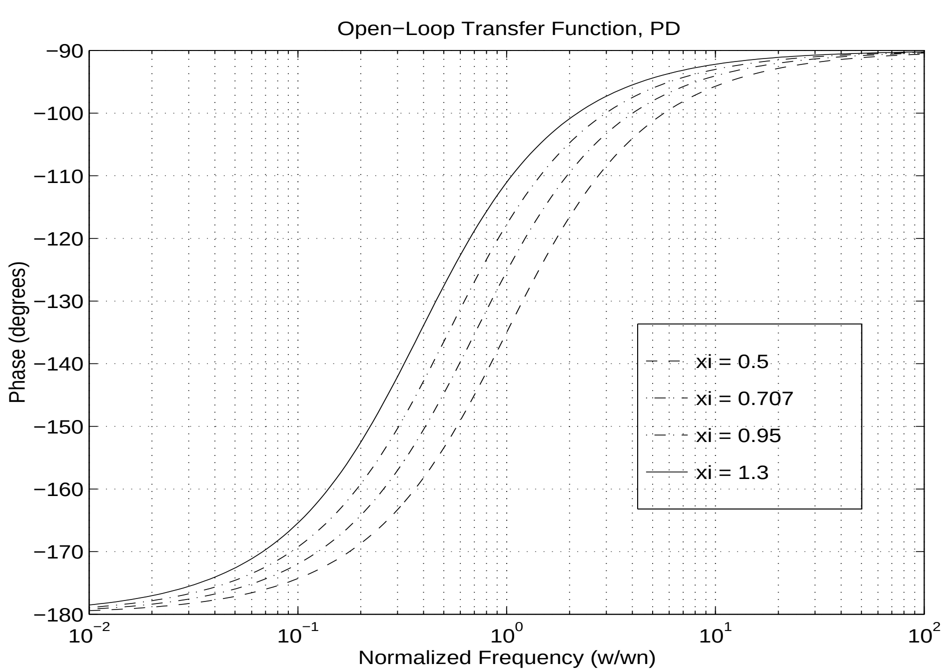
Although this is a 2nd order system, and most quantities can be computed analytically, the formulae that arise are rather messy, and interpretation ends up requiring plotting. Hence, we skip the analytic calculations, and simply numerically compute and plot interesting properties for different values of ω_n, p and ξ. Normalization is the key to displaying the data in a cohesive and minimal fashion.

For now, take $p = 0$ (you should take the time to write a MatLab script file that duplicates these results for arbitrary p). In this case, it is possible to write everything in terms of normalized frequency, all relative to ω_n. This simultaneously leads to a normalization in time (recall homework 8). Hence frequency responses are plotted $G(j\omega)$ versus $\frac{\omega}{\omega_n}$, and time responses plotted $y(t)$ versus $\omega_n t$. We consider a few typical values for ξ.

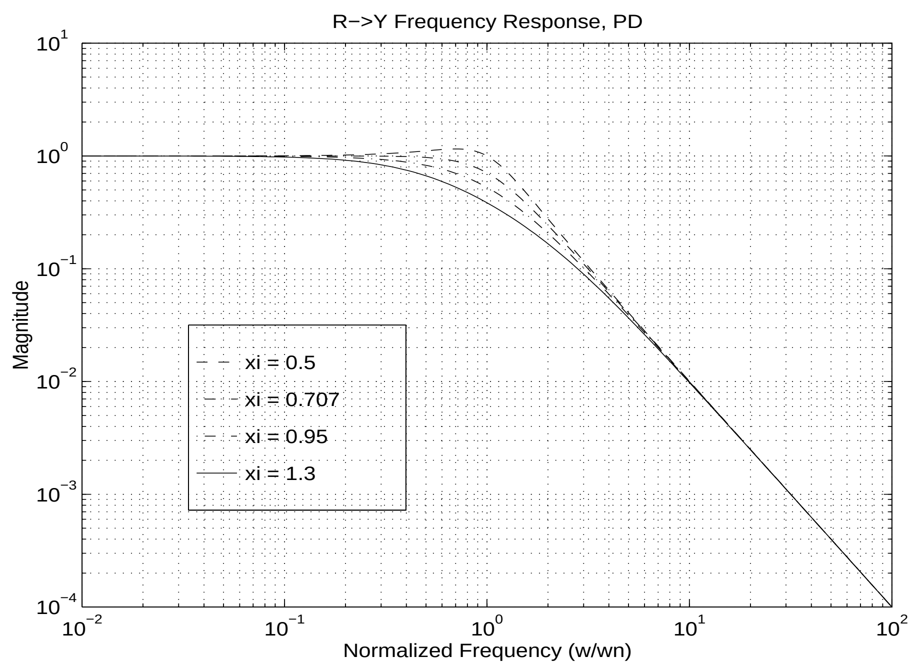
The plots below are:

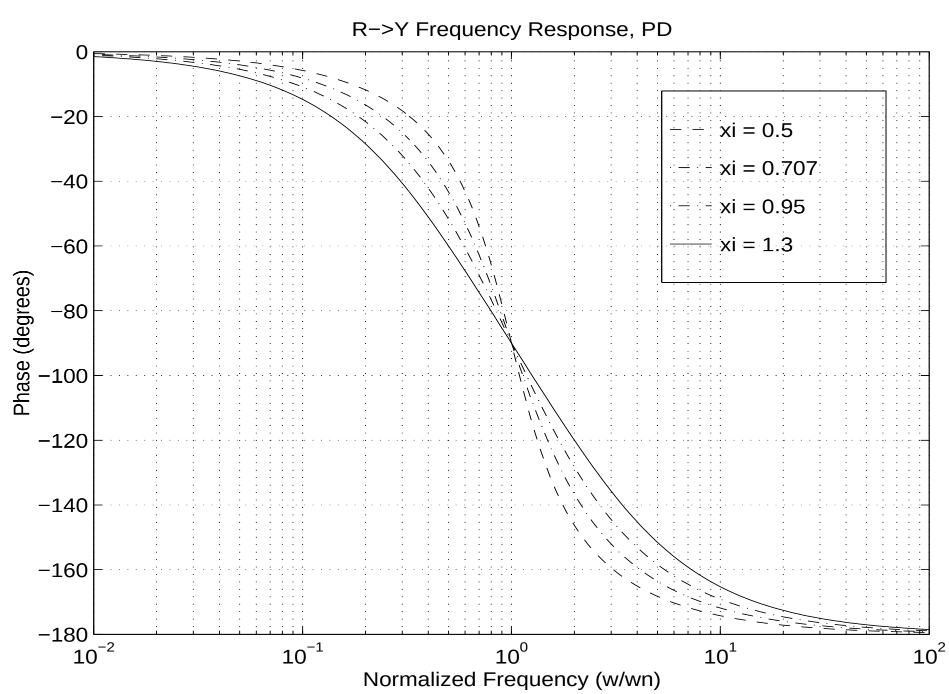
- Magnitude/Phase plots of Loop transfer function. These are normalized in frequency, and show $L(j\omega)$ versus $\frac{\omega}{\omega_n}$. From these, we can read off the crossover frequencies and margins.



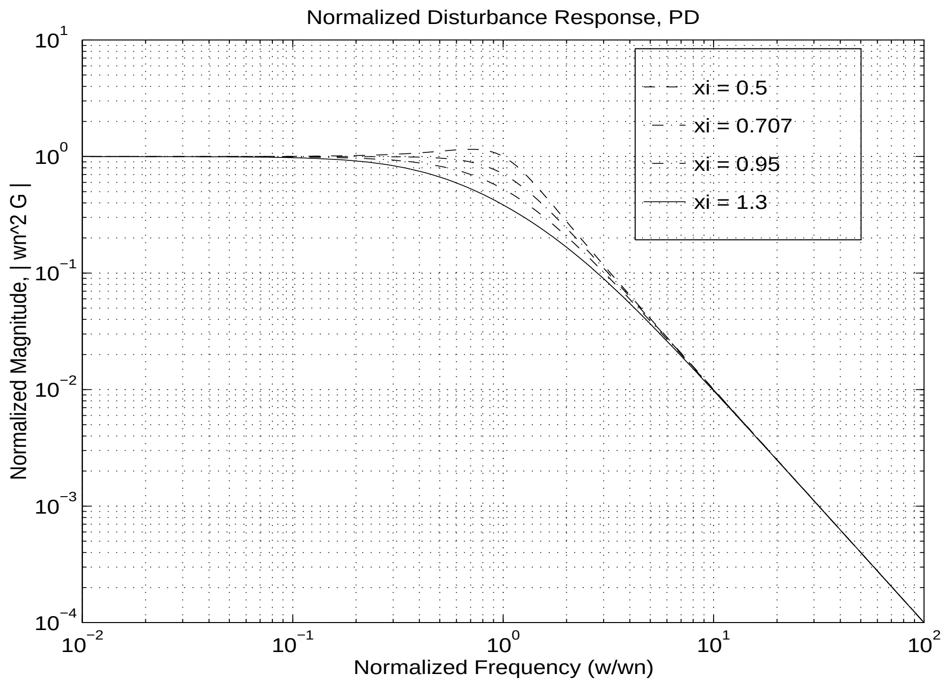


- Magnitude/Phase plots of closed-loop $R \to Y$ transfer function. These are normalized in frequency, and show $G_{R\to Y}(j\omega)$ versus $\frac{\omega}{\omega_n}$.

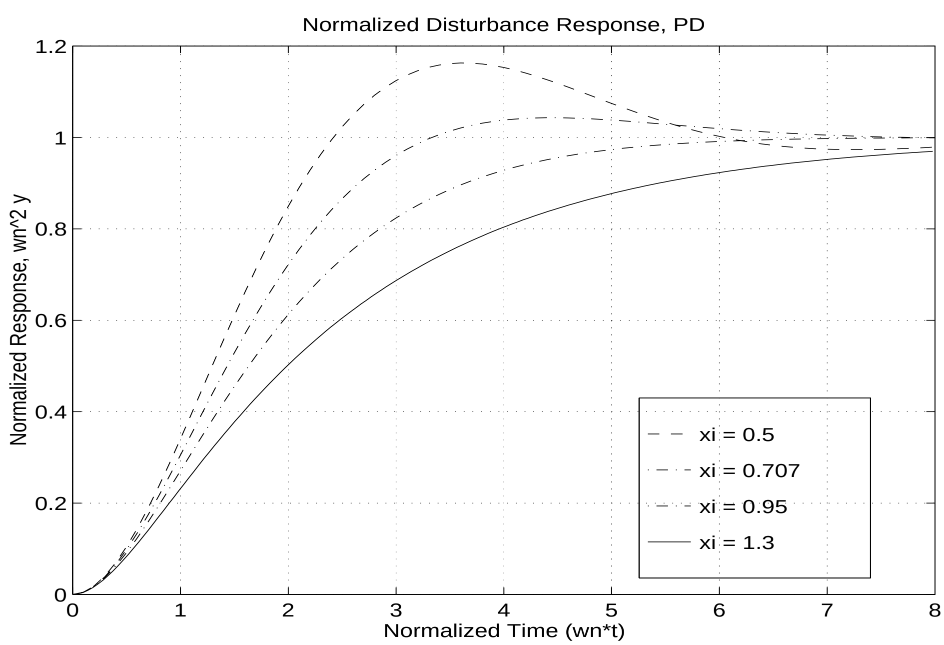




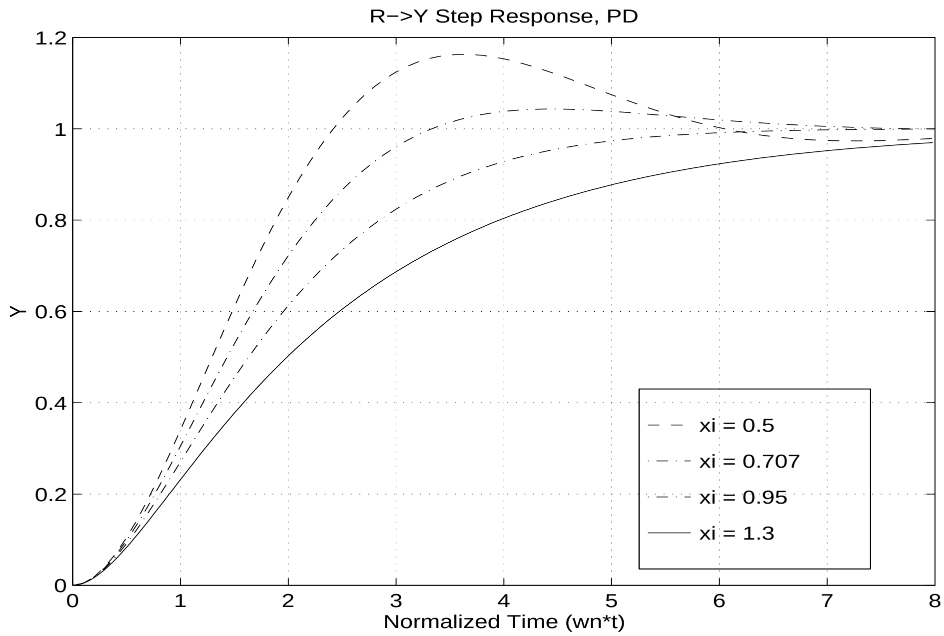
- Magnitude plot of closed-loop $D \to Y$ These are normalized in frequency and magnitude,, and show $\omega_n^2 G_{D\to Y}(j\omega)$ versus $\frac{\omega}{\omega_n}$.



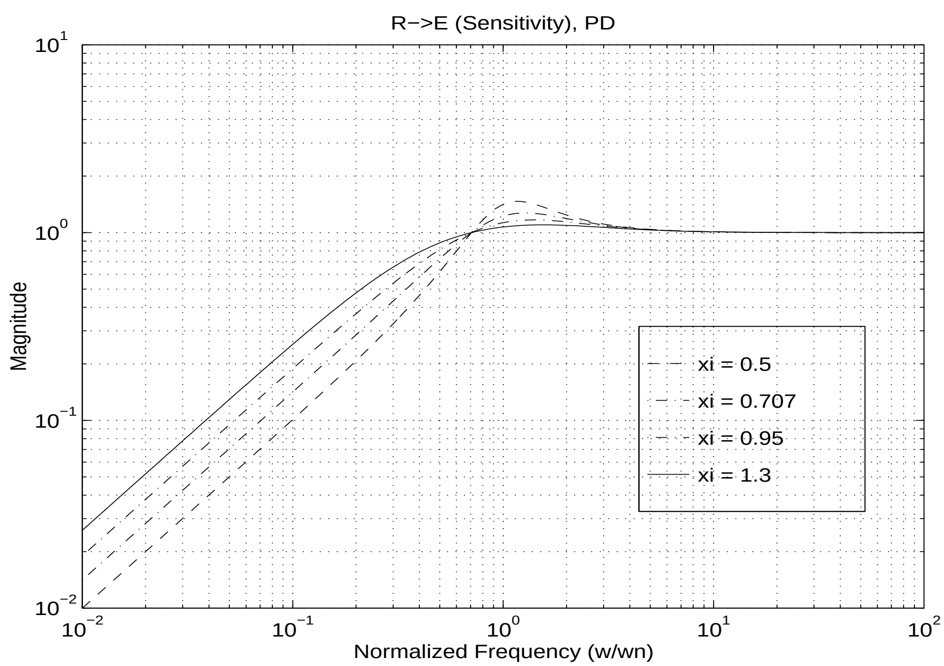
- Unit step $d \to y$ responses. These are normalized both in time, and in response. Hence the plot is $\omega_n^2 y(t)$ versus $\omega_n t$.



- Unit step $r \to y$ responses. These are normalized in time, and show $y(t)$ versus $\omega_n t$.



- Magnitude plot of closed-loop $R \to E$. These are normalized in frequency, and show $G_{R\to E}(j\omega)$ versus $\frac{\omega}{\omega_n}$.



Some things to notice.

- The $r \to y$ response has the canonical 2nd order response we have come to know and love.

- The steady-state disturbance rejection properties are dependent on ω_n. As ω_n increases, the effect of a disturbance d on the output y is decreased. Hence, in order to improve the disturbance rejection characteristics, we need to pick larger ω_n.

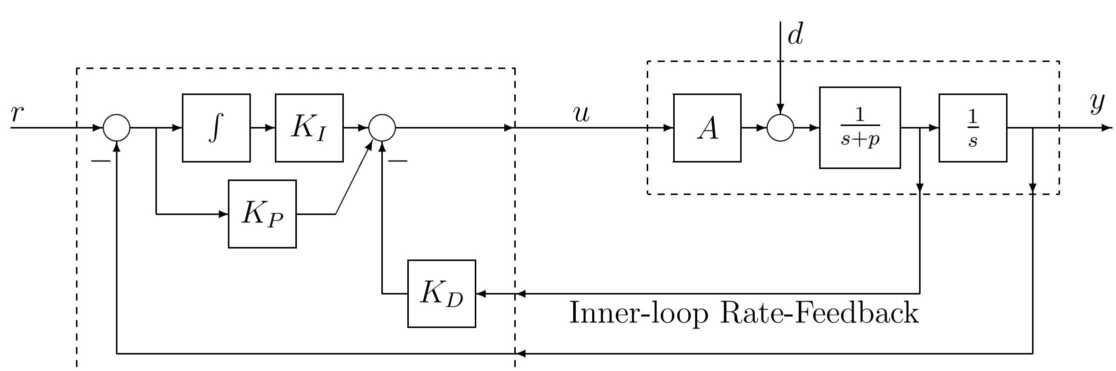
- Depending on ξ, the gain-crossover frequency is between about $1.3\omega_n$ and $2.5\omega_n$. So, using this controller architecture, the gain crossover frequency must increase when the steady-state disturbance rejection is improved. The phase margin varies between 53° and 83°.

- There is no phase-crossover frequency, so as defined, the gain margin is infinite.

- The closer that the complex frequency response remains to 1 (over a large frequency range), the better the $r \to y$ response. The term "bandwidth" is often used to mean the largest frequency ω_B such that for all ω satisfying $0 \leq \omega \leq \omega_B$,
$$|1 - G_{R\to Y}(j\omega)| \leq 0.3$$
Be careful with the word "bandwidth." Make sure whoever you are talking to agrees on exactly what you both mean. Sometimes people use it to mean the gain crossover frequency. Generally, the higher the bandwidth, the faster the response, and better the disturbance rejection. Of course, its hard to explicitly assess time-domain properties from a single number about a frequency response, so use it carefully. The same types of intuition can also be assessed by looking at the frequency range over which the transfer function is $G_{R\to E}$ small, and also verifying that it is not too large in another range.

13.2 PID Control

In order to reduce the steady-state effect of the disturbance, we next analyze PID control, namely proportional+integral, with inner loop rate feedback. This is shown below.



The open-loop transfer function is

$$L(s) = \frac{A(K_D s^2 + K_P s + K_I)}{s^2(s+p)}$$

The closed-loop transfer function is

$$Y(s) = \frac{A(K_P s + K_I)}{s^3 + (p + AK_D)s^2 + AK_P s + AK_I} R(s) + \frac{s}{s^3 + (p + AK_D)s^2 + AK_P s + AK_I} D(s)$$

The closed-loop characteristic equation is

$$s^3 + (p + AK_D)s^2 + AK_P s + AK_I$$

With three controller parameters, and a 3rd order closed-loop system, the poles can be freely assigned. Using the (ξ, ω_n) parametrization, along with a 3rd pole at $-\alpha\omega_n$, we set the characteristic equation to be

$$\text{CE}: \qquad (s^2 + 2\xi\omega_n s + \omega_n^2)(s + \alpha\omega_n)$$

Multiplied out, this gives

$$s^3 + (2\xi + \alpha)\omega_n s^2 + (2\xi\alpha + 1)\omega_n^2 s + \alpha\omega_n^3$$

Choosing specific values of ξ, ω_n and α yields appropriate controller gains, via the design equations, which are obtained by simply equating coefficients,

$$K_D = \frac{(2\xi + \alpha)\omega_n - p}{A}, \qquad K_P = \frac{(2\xi\alpha + 1)\omega_n^2}{A}, \qquad K_I = \frac{\alpha\omega_n^3}{A}$$

Note that for $\alpha \ll 1$, the design equations give K_D and K_P as in the PD case, along with a *very small* integral control term. Hence, for a given pair (ξ, ω_n), picking α small and doing the full PID design is equivalent to doing the PD design for ξ and ω_n, and then simply adding a small amount of integral control as an afterthought. That approach will leave a closed-loop pole near the origin, approximately at $s = -\frac{AK_I}{\omega_n^2} (= -\alpha\omega_n)$.

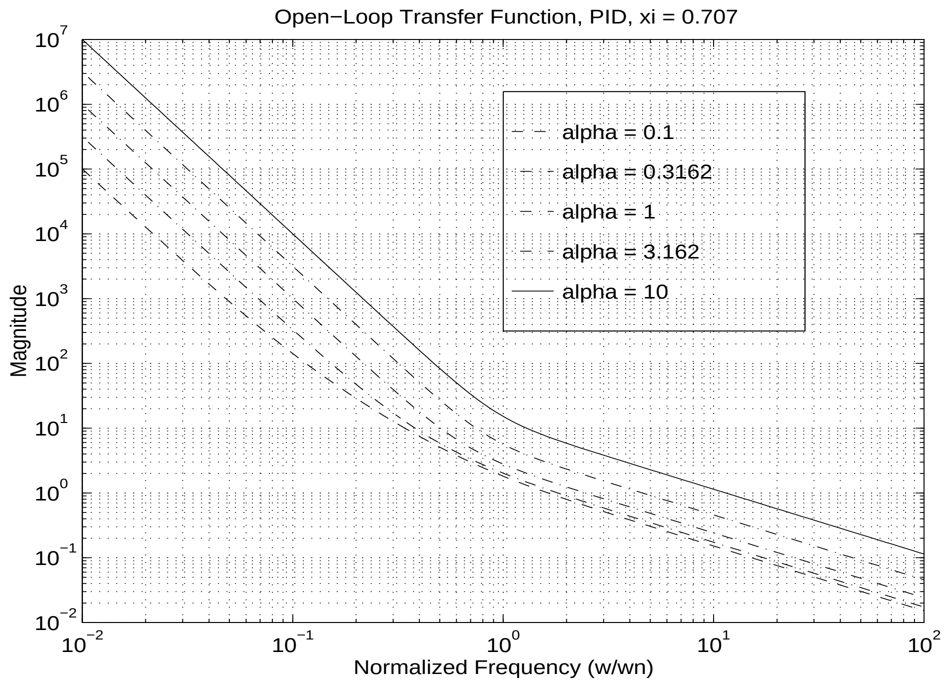
In terms of the parameters, the closed-loop transfer function is

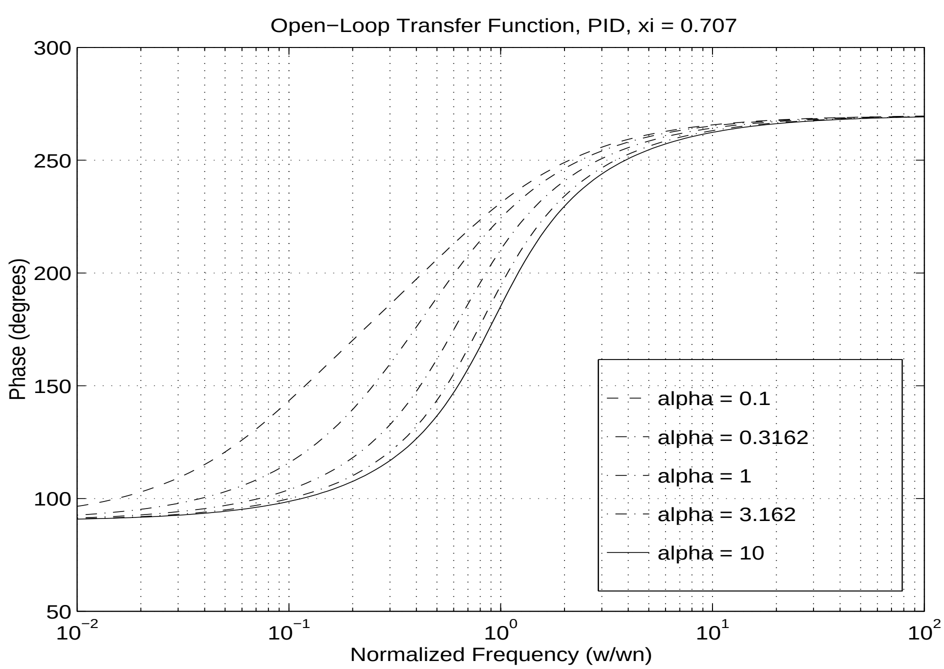
$$Y(s) = \frac{(2\xi\alpha + 1)\omega_n^2 s + \alpha\omega_n^3}{(s^2 + 2\xi\omega_n s + \omega_n^2)(s + \alpha\omega_n)} R(s) \;+\; \frac{s}{(s^2 + 2\xi\omega_n s + \omega_n^2)(s + \alpha\omega_n)} D(s)$$

The steady-state gain from d to y is zero, due to the integral term. Again, take the case $p = 0$. For clarity, let's also pick $\xi = 0.707$, and only study the variation in responses due to our choice of α. Again, the normalization with ω_n is complete, in both time and frequency, with frequency responses plotted versus $\frac{\omega}{\omega_n}$, and time responses plotted versus $\omega_n t$.

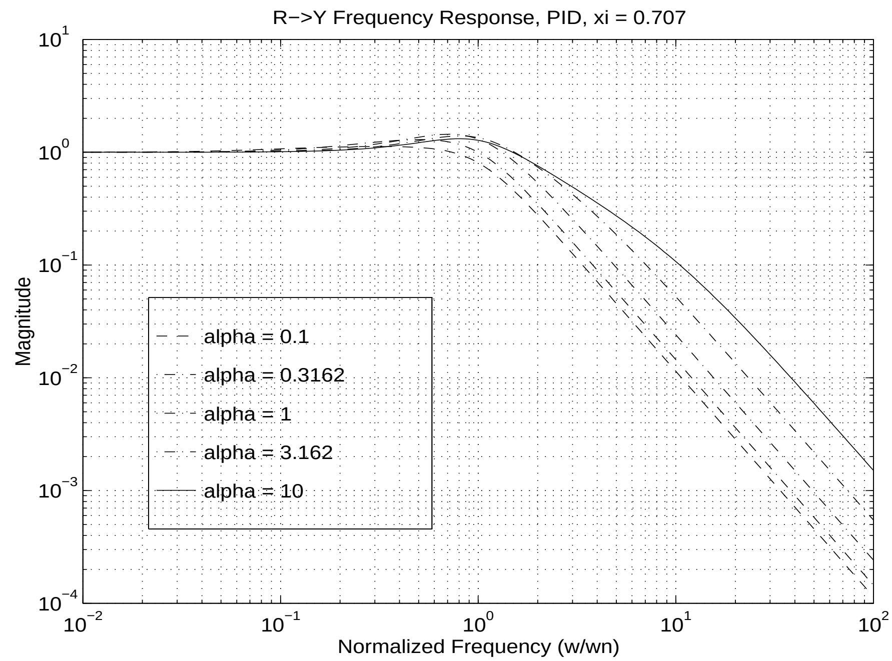
The plots below are:

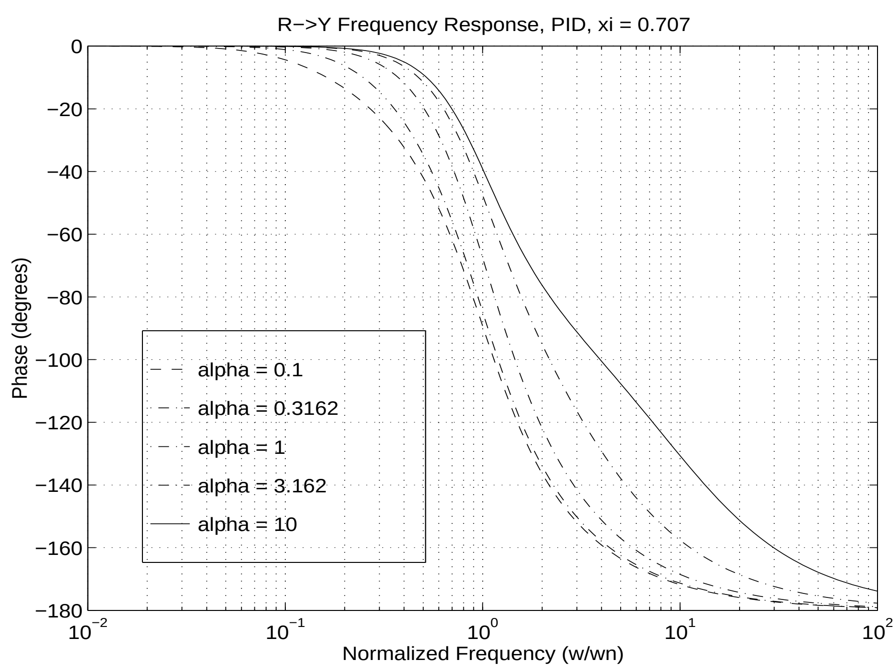
- Magnitude/Phase plots of Loop transfer function





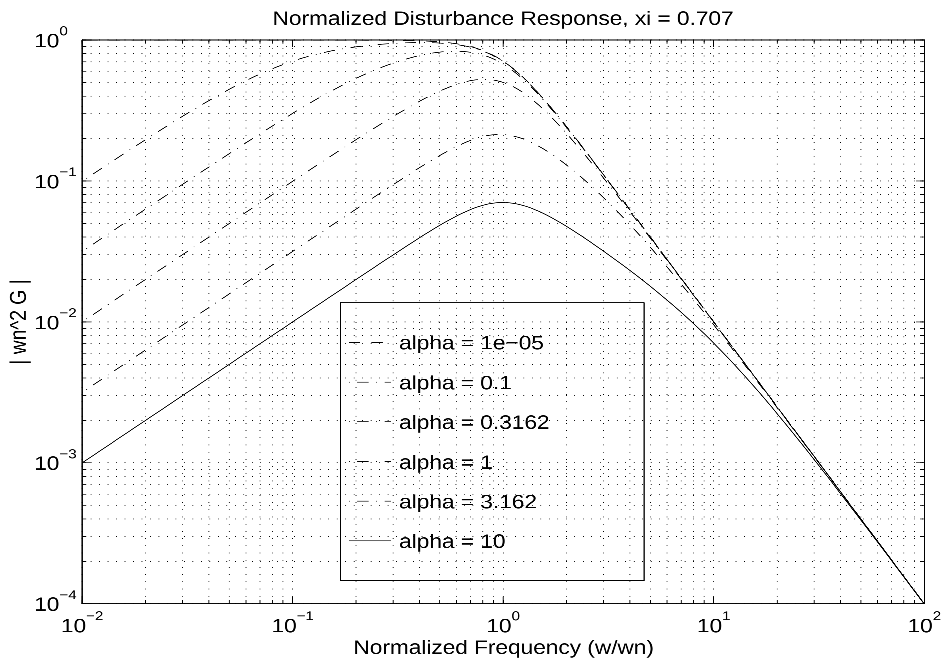
- Magnitude/Phase plots of closed-loop $R \rightarrow Y$ transfer function

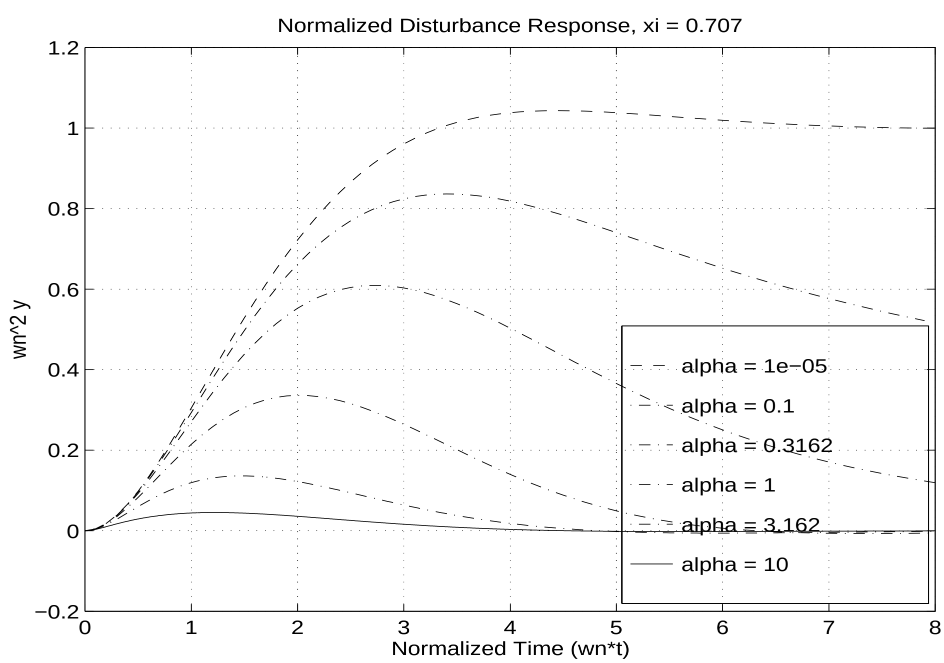




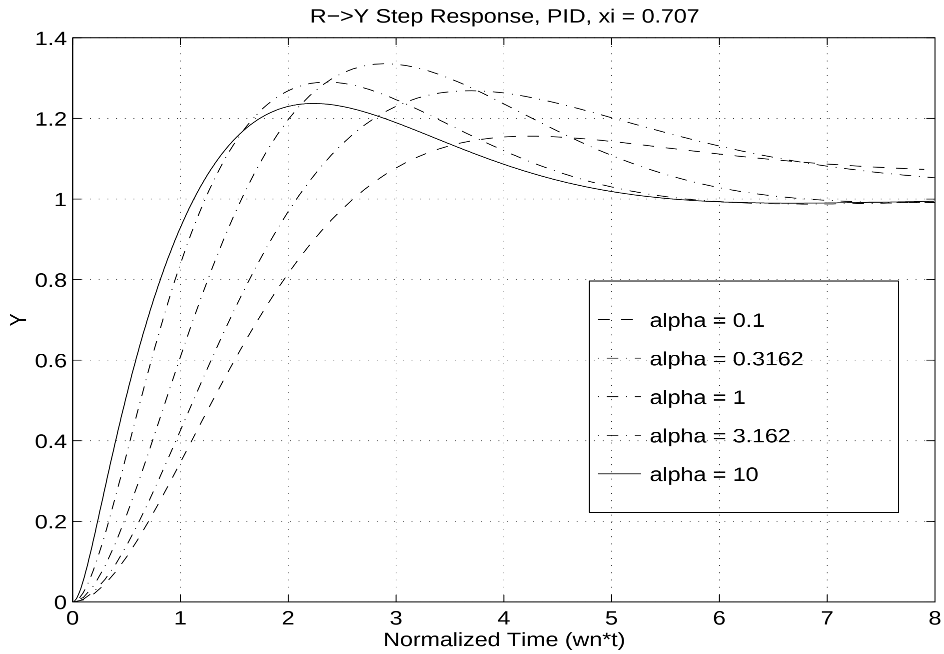
- Magnitude plot of closed-loop $R \rightarrow E$

- Normalized Disturbance-to-output response

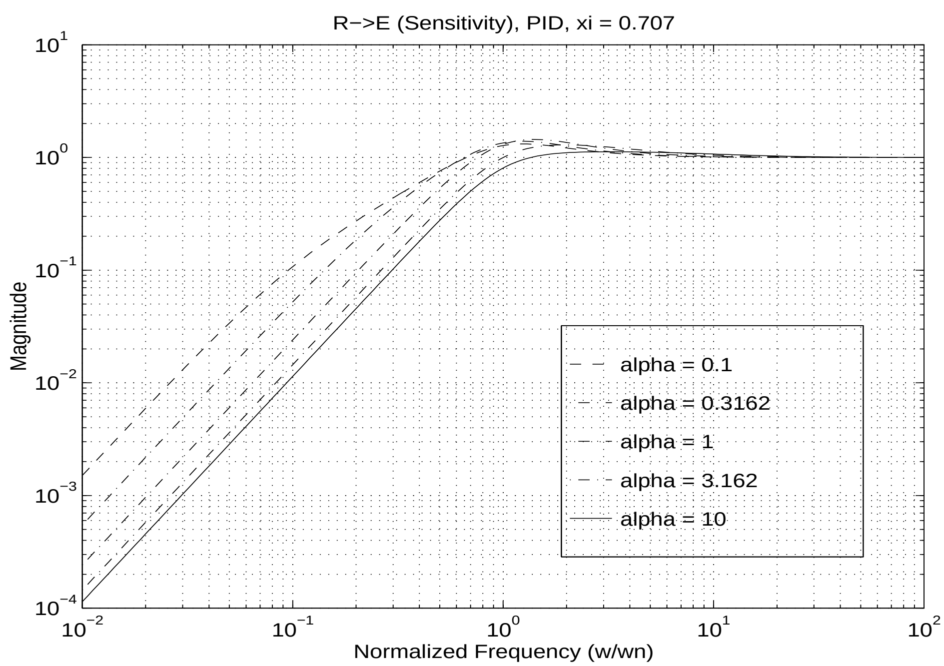




- Unit step $r \to y$ responses



- $R \to E$ magnitude plots



Some things to notice.

- For a given ω_n and $\alpha > 1$, both the crossover frequencies, and bandwidth (look at $G_{R \to E}$) are much higher than the PD case. This is somewhat reflected in quicker rise times and comparable settling times.

- The gain crossover frequency increases significantly with increasing α. For instance, at $\alpha \approx 0$ (which is the same as the PD control) the crossover frequency is about $1.7\omega_n$, whereas for the crossover frequency jumps to approximately $\approx 5\omega_n$ at $\alpha \approx 3.1$. At the respective crossover frequencies, the phase margins of eth PD and PID designs are similar.

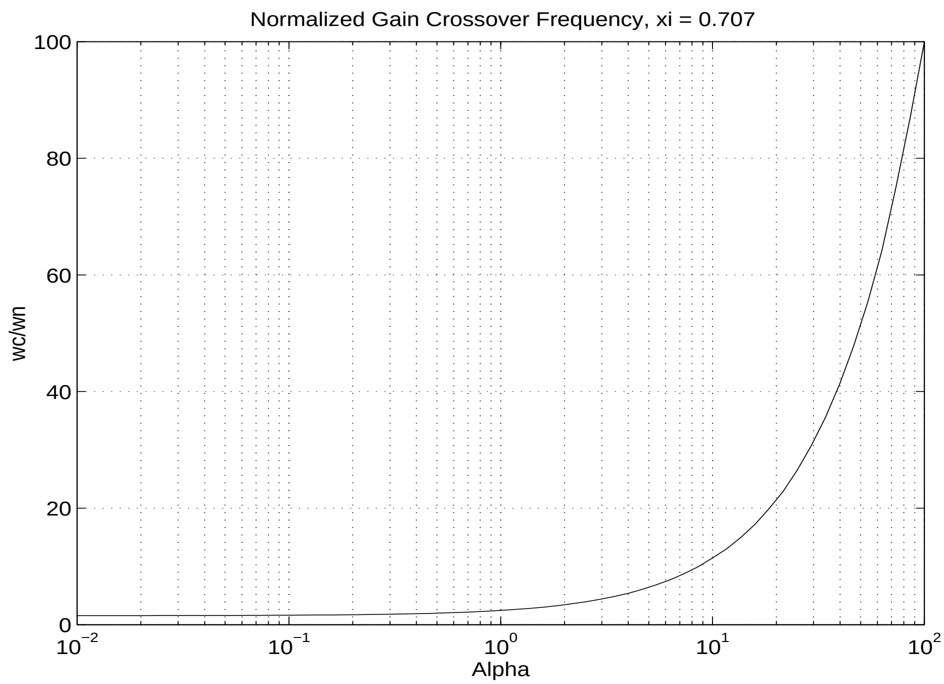
- As α increases, the disturbance rejection properties change. Any (and every) $\alpha > 0$ gives perfect steady-state disturbance rejection, but the time-domain and frequency domain properties for different α are quite different.

- It is instructive to calculate the residue associated with the pole at $-\alpha\omega_n$ when $r(t)$ is a unit step. It is then fairly easy to explain the slow settling times that occur for the intermediate values of α.

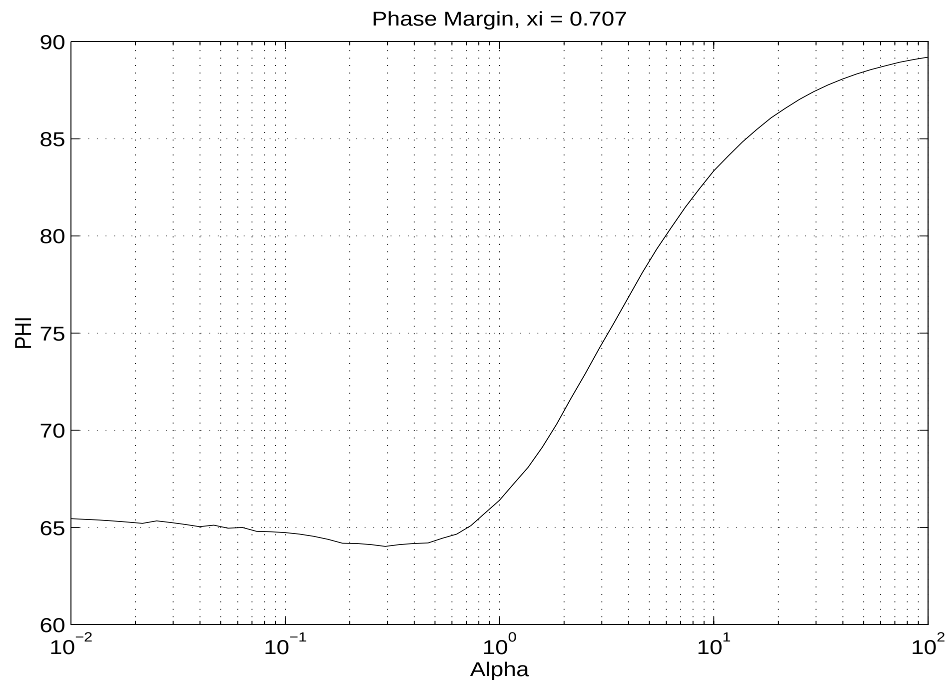
Remember, in typical applications, the uncertainty in the plant's behavior increases with increasing frequency, so designs that lead to higher crossover frequencies usually are required (for confidence) to have significantly larger phase margins. Usually, for a given problem, modeling innaccuracies and unknown dynamics typically impose a maximum allowable crossover frequency, regardless of phase margin.

Since some normalization is possible (using ω_n), brute-force repeated simulation allows us to approximately compute several functions. They are functions of p, ξ and α. Here, we imagine that p is known, and fixed. We also propose to fix $\xi = 0.707$, leaving only functions of α. In any given design situation, it may be necessary to modify the choice of ξ, and recompute. The functions are plotted below.

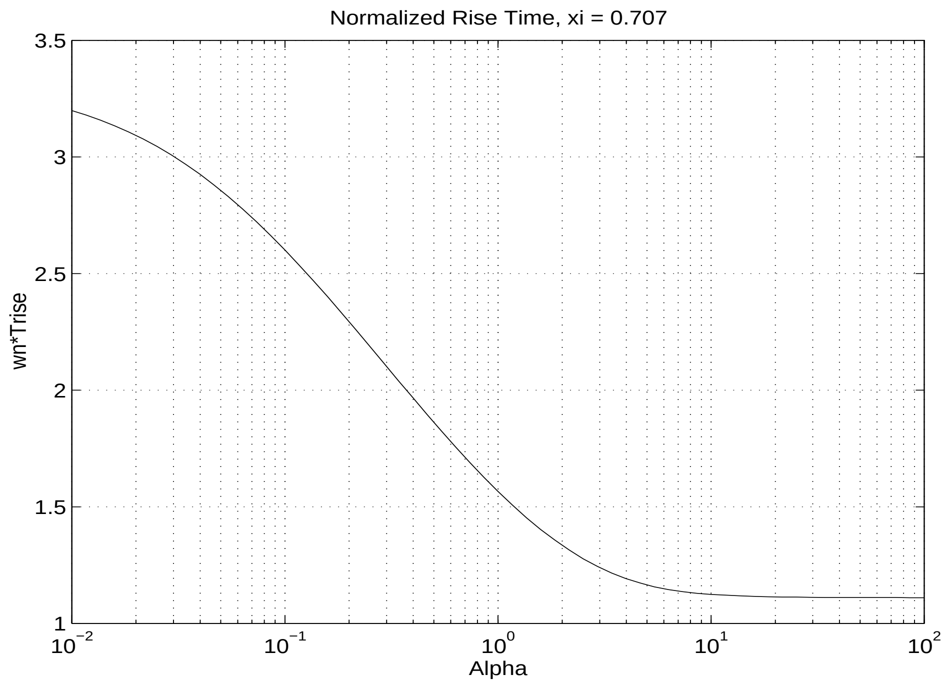
- normalized crossover frequency versus α



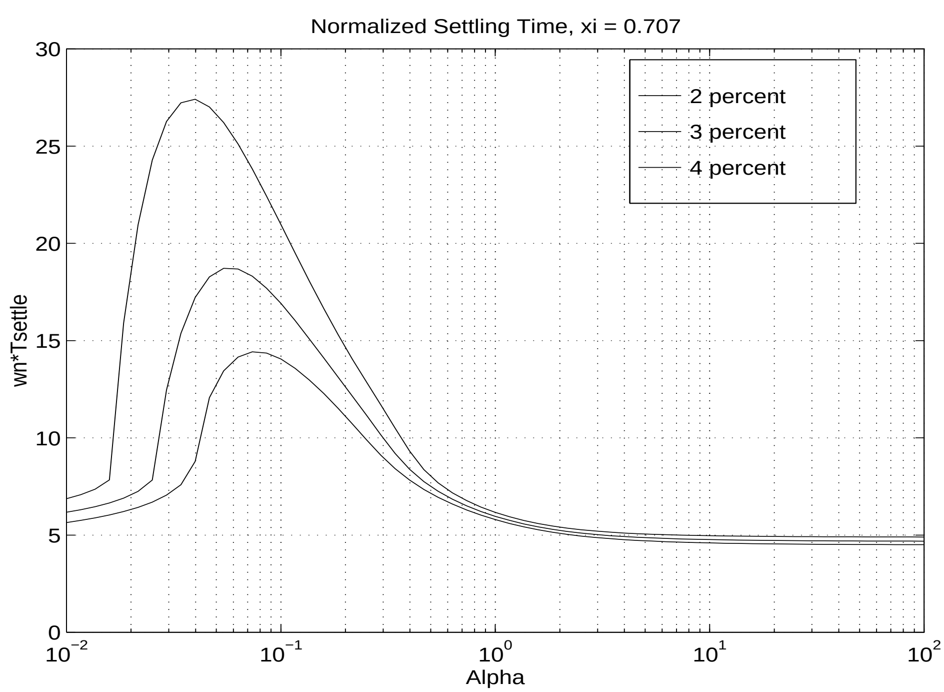
- phase margin versus α



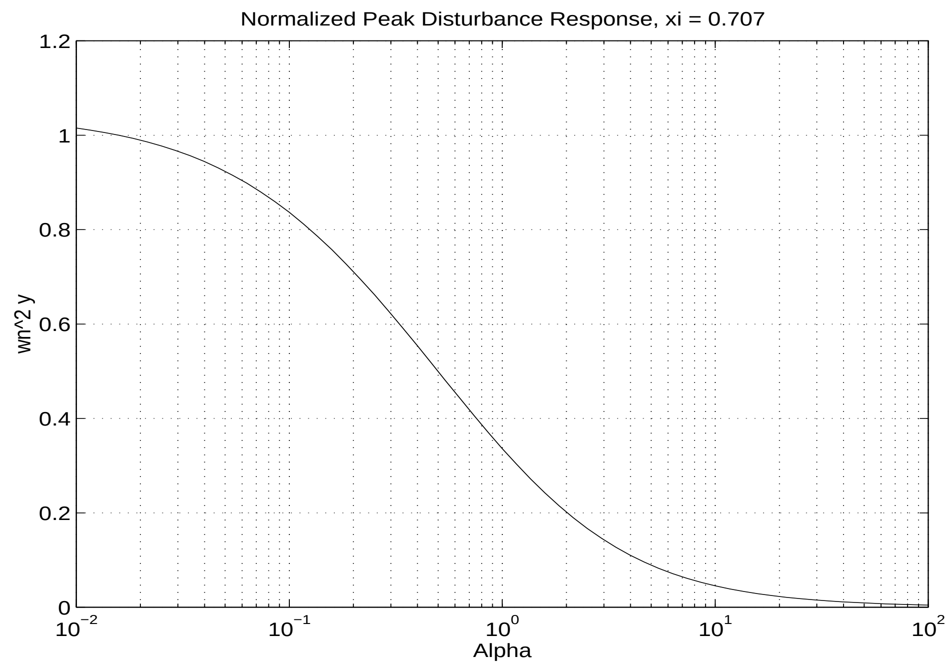
- normalized rise time versus α



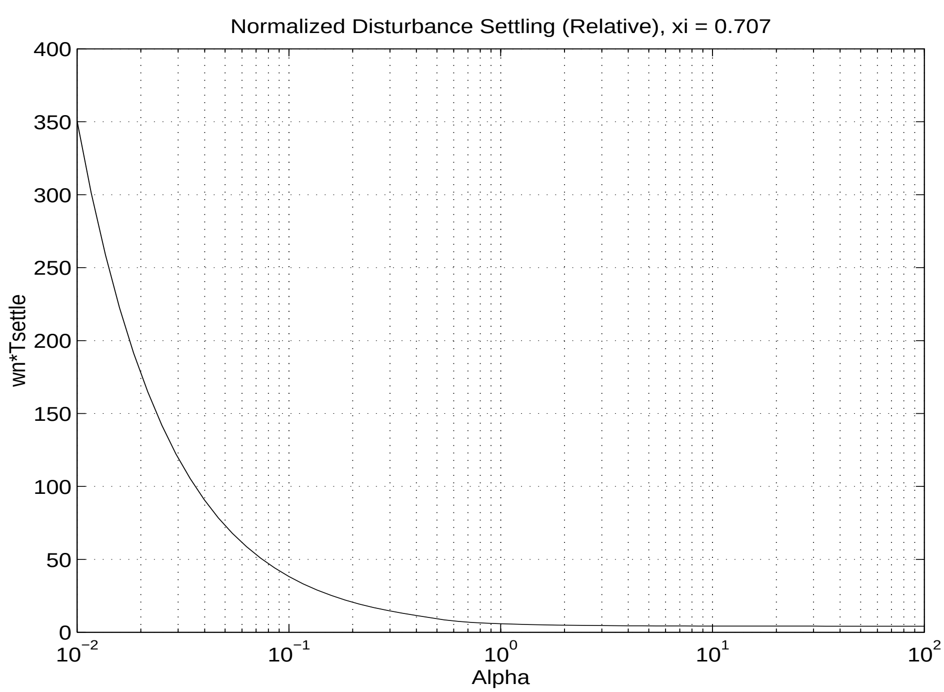
- normalized settling time versus α



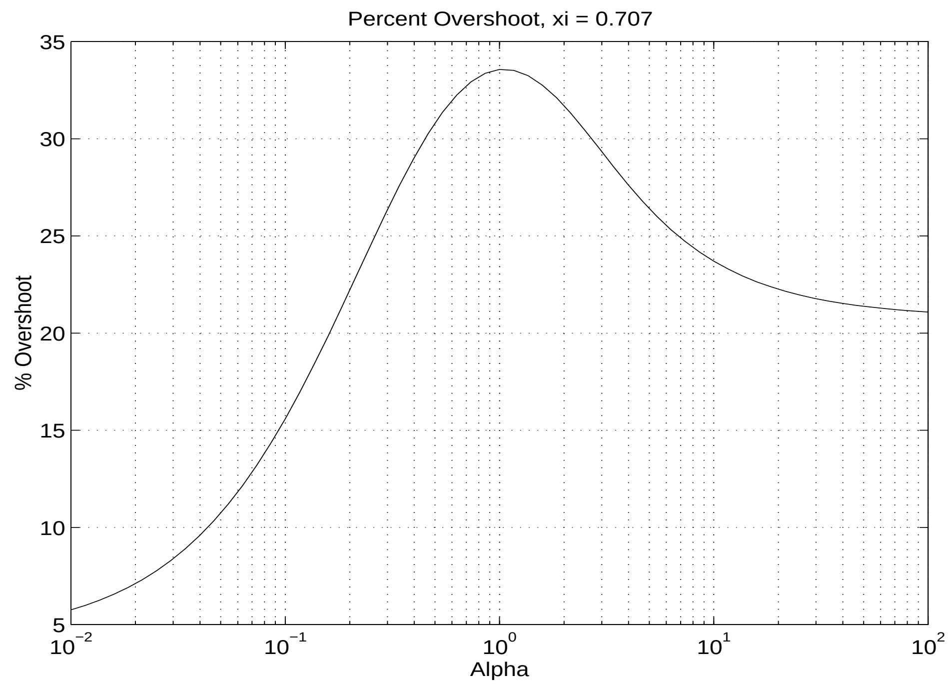
- normalized peak response due to step disturbance versus α



- normalized settling time due to step disturbance versus α



- percentage overshoot versus α



Suppose that these have been computed reasonably accurately, at sufficiently large numbers of α values. Can we use all of this data to develop a foolproof design method?

13.3 A Brute-Force Design Method

In designing the PID controller gains, the free parameters (at this point) to be chosen are ξ, ω_n and α. For fixed choice of ξ, we can precompute functions $f_1, f_2, \ldots, f_5$ of α such that

1. Gain crossover frequency (ω_c) equals $\omega_n f_1(\alpha)$
2. Rise Time (t_R) equals $f_2(\alpha)/\omega_n$
3. Settling Time (t_S) equals $f_3(\alpha)/\omega_n$
4. Peak response to step disturbance ($y_{d,max}$) equals $f_4(\alpha)/\omega_n^2$
5. Settling time of step disturbance response ($t_{S,d}$) equals $f_5(\alpha)/\omega_n$

So, given target requirements, we can fairly easily determine if there is a PID controller which satisfies the objectives. Specifically, take objectives as

$$\omega_c \leq \bar{\omega}_c, \quad t_R \leq \bar{t}_R, \quad t_S \leq \bar{t}_S, \quad y_{d,max} \leq \bar{y}_{d,max}, \quad t_{S,d} \leq \bar{t}_{S,d}$$

where the over-bar quantities are targets. Hence we search for values of ω_n and α which satisfy

$$g_L(\alpha) := \max\left\{\frac{f_2(\alpha)}{\bar{t}_R},\ \frac{f_3(\alpha)}{\bar{t}_S},\ \sqrt{\frac{f_4(\alpha)}{\bar{y}_{d,max}}},\ \frac{f_5(\alpha)}{\bar{t}_{S,d}}\right\} \ \leq\ \omega_n\ \leq\ \frac{\bar{\omega}_c}{f_1(\alpha)}\ =:\ g_U(\alpha)$$

Hence, we simply graph the two functions g_L and g_U, and see if there is any value of α where $g_l(\alpha) \leq g_u(\alpha)$. If so, then simply pick an α^* for which the inequality it true, and pick any ω_n^* such that

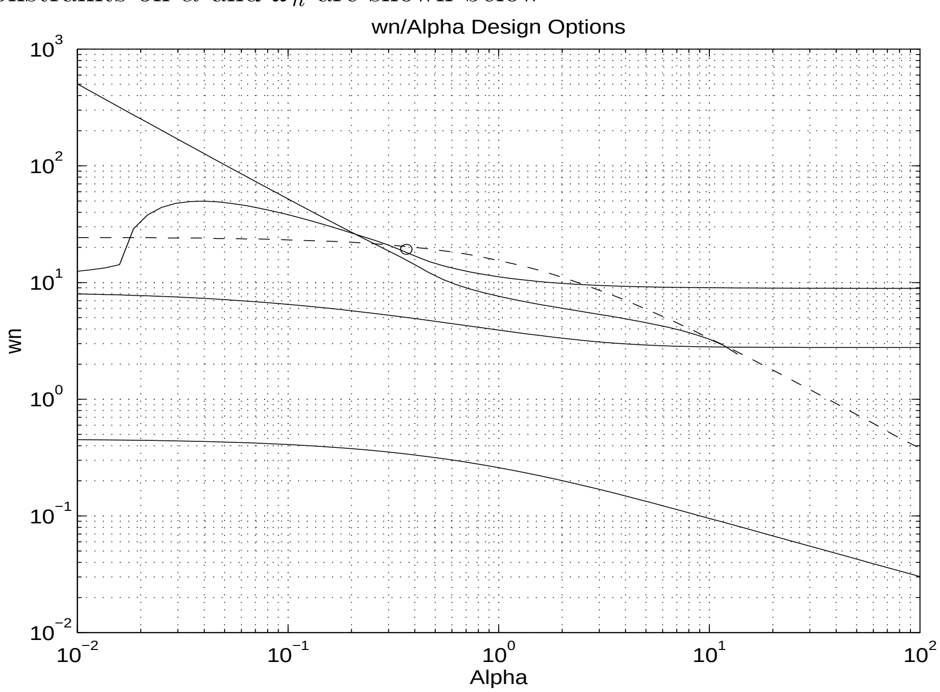
$$g_L(\alpha^*) \leq \omega_n^* \leq g_U(\alpha^*)$$

Moreover, since the overshoot reaches a maximum at $\alpha = 1$, and falls off on both sides, you can easily reduce the overshoot by moving to one side of the feasible region.

13.4 Design Example

Consider the Lab, with single inertia, and pulley. The PD control worked reasonably well with $\omega_n = 25$, and $\xi = 0.707$. This implies that a phase margin of 65° at a crossover frequency of 38 is adequate for stability robustness. So, in designing a PID controller, let's aim for a crossover frequency of 38, a rise time of 0.4 seconds, settling time of 0.55 seconds, and a disturbance response settling time of 0.7 seconds. We'll set the peak disturbance response at 5, which essentially makes it not relevant, and then we could tighten down on it if we wanted.

The constraints on α and ω_n are shown below

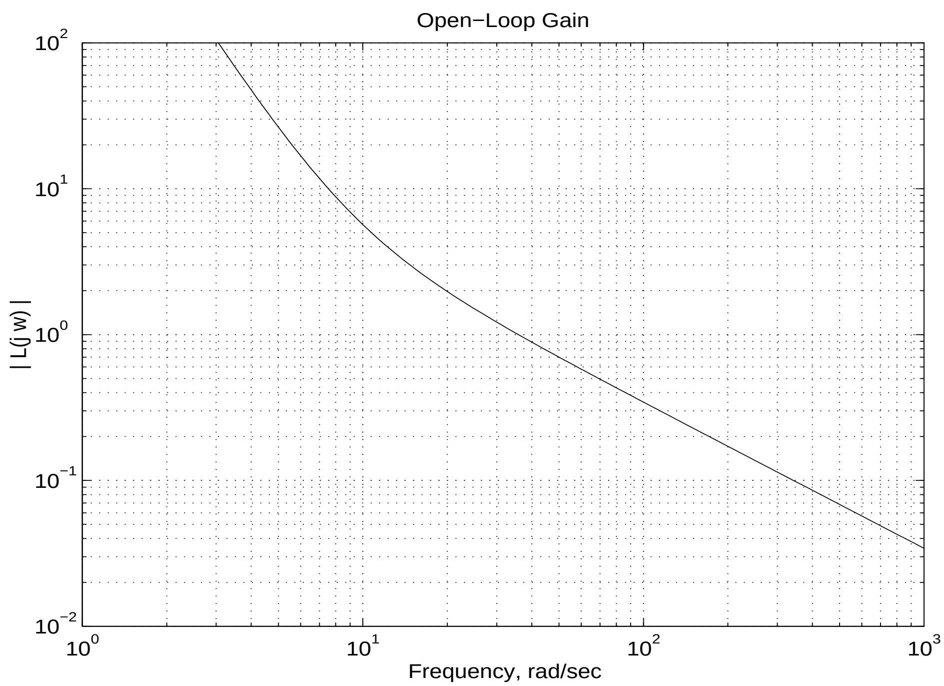


The circle is the point I chose, which MatLab tells me is

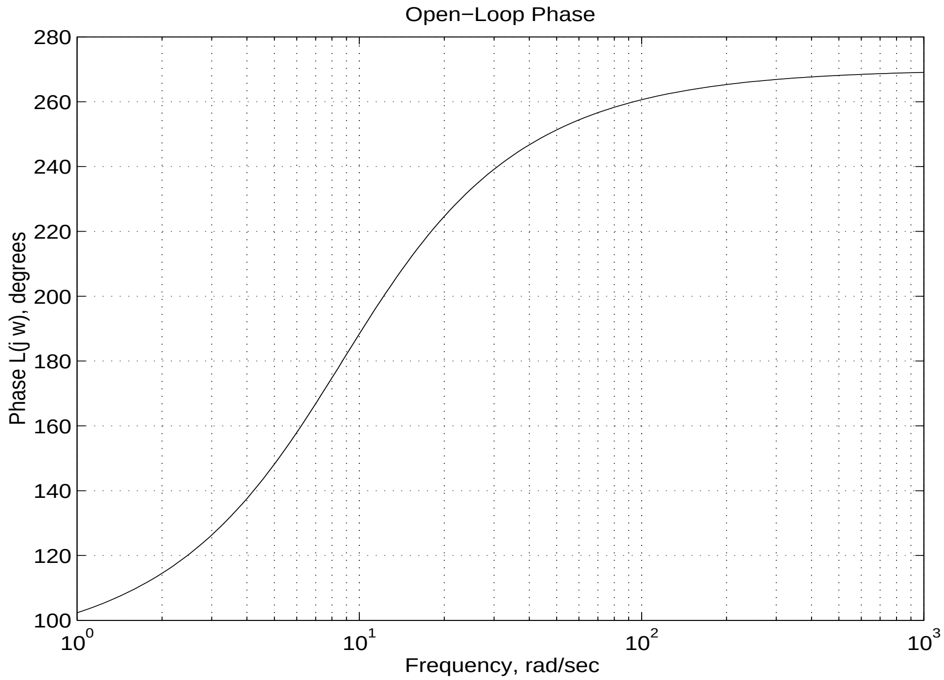
$$\alpha = 0.36, \quad \omega_n = 19.2$$

which is pretty similar to what we had working in the lab. Plots of the various relevant quantities

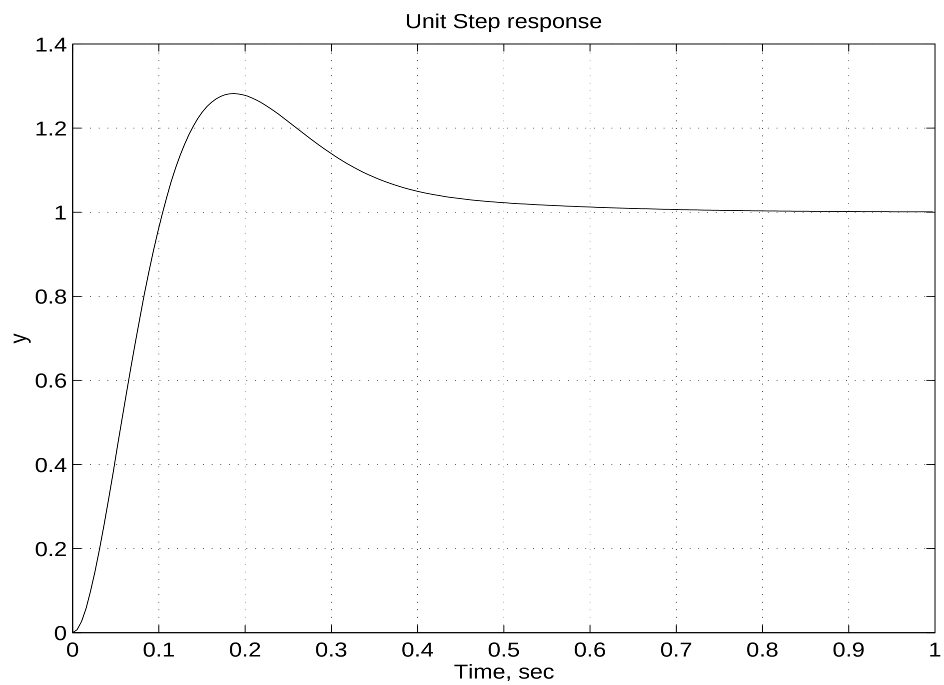
- Open-Loop gain



- Open-Loop Phase



- Response to unit-step reference



- Response to unit-step disturbance