Control Bifurcations: An example of the fusion of Linear Control and Nonlinear Dynamics

A. J. Krener
Department of Mathematics
University of California
Davis, CA 95616, USA
ajkrener@ucdavis.edu

Based on work of
Wei Kang, Dong Eui Chang, Jean-Pierre Barbot, Boumediene Hamzi and AJK.

Research supported in part by
AFOSR-49620-95-1-0409

Outline

- Local Topological Equivalence of Dynamical Systems
- Classical Bifurcation
- Local Topological Equivalence of Control Systems
- Control Bifurcation
- Classical Fold Bifurcation
- Control Fold Bifurcation
- Classical Transcritical Bifurcation
- Control Transcritical Bifurcation
- Classical Hopf Bifurcation
- Control Hopf Bifurcation
- Control via Invariant Manifolds
- Concluding Remarks

Local Topological Equivalence of Dynamical Systems

Two equilibria x_e and z_e

$$\begin{aligned} \dot{x} &= f(x) \\ 0 &= f(x_e) \end{aligned}$$

$$\begin{aligned} \dot{z} &= g(z) \\ 0 &= g(z_e) \end{aligned}$$

are *locally topologically equivalent* if there is a local homeomorphism

$$\begin{aligned} z &= h(x) \\ z_e &= h(x_e) \end{aligned}$$

which carries x-trajectories to z-trajectories preserving orientation of time but not exact time.

Weaker than locally diffeomorphic so there are fewer equivalence classes, e.g.,

$$\begin{aligned} \dot{x} &= -x \\ \dot{z} &= -z^3 \end{aligned}$$

are locally topologically equivalent.

An equilibrium is *structurally stable* if it is topologically equivalent to all nearby equilibria of all nearby systems.

Theorem: (Grobman-Hartman) A hyperbolic equilibrium

$$\begin{aligned} \dot{x} &= f(x) \\ 0 &= f(x_e) \end{aligned}$$

is topologically equivalent to its linear part

$$\dot{z} = \frac{\partial f}{\partial x}(x_e) z$$

Corollary: An equilibrium is structurally stable iff it is hyperbolic.

Theorem: (Poincaré) An equilibrium is formally equivalent to its linear part if there are no resonances.

A *normal form* is a parameterized family of representatives of the equivalence classes, usually the equivalence is local diffeomorphic.

Classical Bifurcation

Parameterized dynamics around an equilibrium

$$\dot{x} = f(x, \mu)$$

$$x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}$$

Parameterized family of equilibria

$$0 = f(x_e(\mu), \mu)$$

A *classical bifurcation* occurs at an equilibrium $x_e(\mu_c)$ which is not topologically conjugate to nearby equilibria $x_e(\mu)$.

In practice, a bifurcation occurs when an eigenvalue or pair of eigenvalues hits the imaginary axis but it doesn't have to occur, e.g.,

$$\dot{x} = -\mu^2 x - x^3$$

Normal forms (relative to diffeomorphisms) are very useful in classifying the simplest ways a parameterized dynamics can bifurcate.

Local Topological Equivalence of Control Systems

Two control systems with equilibria

$$\dot{x} = f(x,u), \qquad 0 = f(x_e,u_e)$$

$$\dot{z} = g(z,v), \qquad 0 = g(x_e,v_e)$$

are *locally feedback equivalent* if there is a local diffeomorphism

$$\begin{aligned} z &= \phi(x) \\ v &= \kappa(x,u) \end{aligned}$$

between the two systems at the equilibria.

Too fine an equivalence, too many equivalence classes.

A coarser equivalence; two equilibria are *locally, closed loop, topologically equivalent* if there are continuous feedbacks

$$
\begin{aligned}
u &= \kappa(x) \\
v &= \lambda(z)
\end{aligned}
$$

such that the closed loop systems are topologically equivalent.

But this definition is not an equivalence relation (not transitive) and it ignores the reason for closing the loop, i.e., to stabilize the system.

Therfore we shall add the requirement that the feedbacks locally asymptotically stabilize the systems.

A equilibrium of a control system is *structurally stabilizable* if it and all nearby equilibria of all nearby systems are locally asymptotically stabilizable by continuous feedbacks.

Theorem: An equilibrium x_e, u_e of a control system and its linear part

$$\dot{z} = \frac{\partial f}{\partial x}(x_e, u_e)z + \frac{\partial f}{\partial u}(x_e, u_e)v$$

are locally, closed loop, topologically equivalent if the latter is linearly stabilizable.

Corollary: An equilibrium is structurally stabilizable iff it is linearly stabilizable.

Control Bifurcation

Control systems have parameterized family of equilibria

$$0 = f(x_e(\mu), u_e(\mu))$$

$$x \in I\!R^n, \quad u \in I\!R$$

where typically $\mu = x_i$ or $\mu = u$.

A control system is *locally parameterically stabilizable* at $x_e(\mu_c), u_e(\mu_c)$ if there exists a continuous, parameterized feedback

$$u = k(x, \mu)$$

which locally, asymptotically stabilizes the system to $x_e(\mu)$ for all μ near μ_c.

A *control bifurcation* occurs at an equilibrium x_e, u_e which is not locally parameterically stabilizable.

Perhaps this should be called ”stabilizability bifurcation” but this is too awkward.

In practice, control bifurcations occur when the linear part of the system loses stabilizability but not always, e.g.,

$$\dot{x} = -u^2 x - x^3$$

If a control system is locally parameterically stabilizable then any other system that is feedback equivalent to it is also locally parameterically stabilizable.

Normal forms (relative to the smooth feedback group) are very useful in classifying the simplest ways a control bifurcation can happen.

There is a close correspondence between the simplest classical and control bifurcations.

Classical Fold Bifurcation

aka Saddle Node Bifurcation

This is one of the two generic classical bifurcations when $\mu \in \mathbb{R}$. The other is the Hopf.

Normal Form:

$$\begin{aligned} \dot{x}_1 &= \mu - x_1^2 + \dots \\ \dot{x}_2 &= A_2 x_2 + \dots \end{aligned}$$

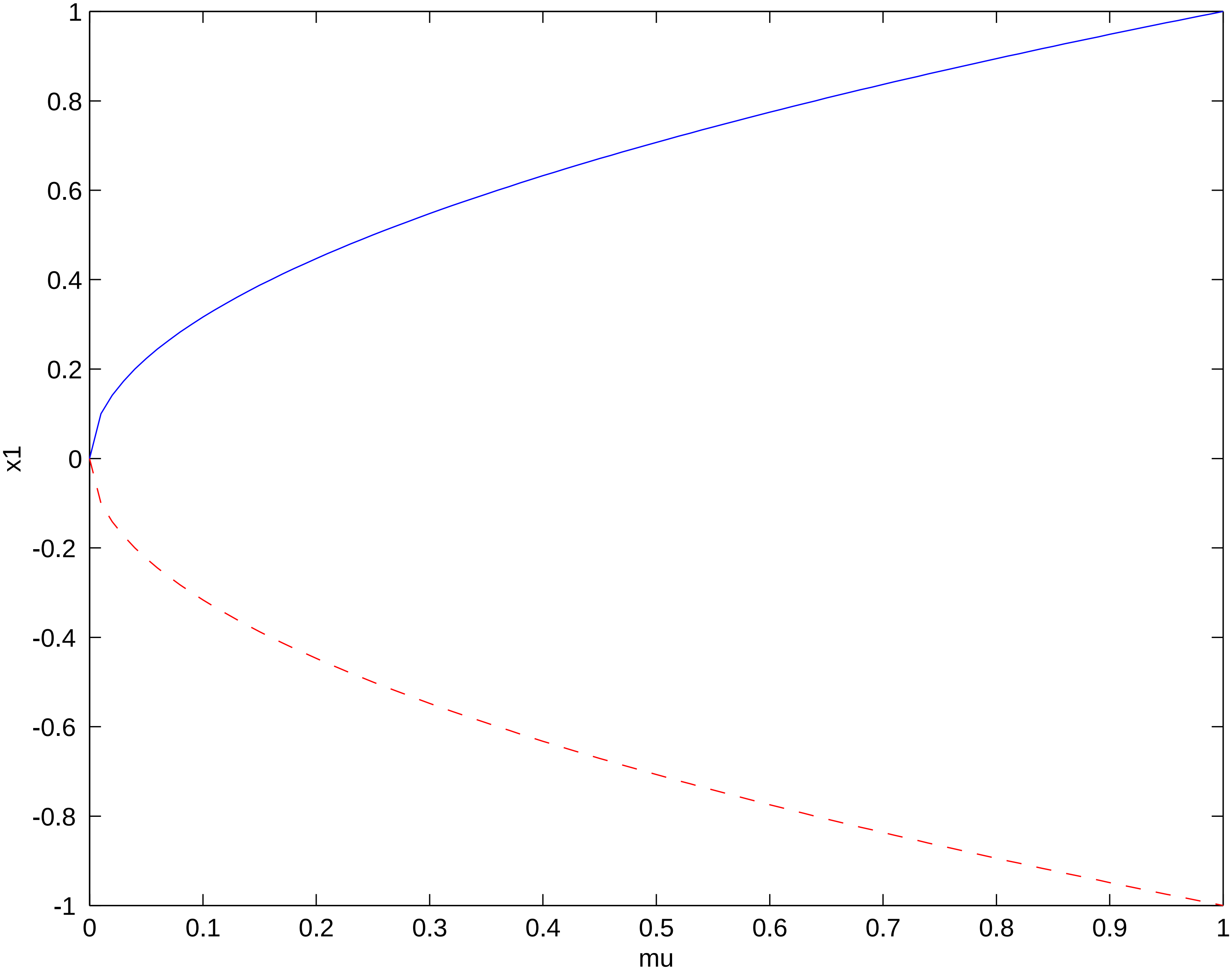
$$\mu, x_1 \in \mathbb{R}, \qquad x_2 \in \mathbb{R}^{n-1}, \qquad A_2 \text{ hyperbolic}$$

Equilibria:

$$\begin{aligned} x_1 &= \pm\sqrt{\mu} + O(\mu), \qquad \mu \geq 0 \\ x_2 &= O(\mu) \end{aligned}$$

Linearization around μ^{th} equilibria in displacement coordinates:

$$\begin{aligned} \dot{z}_1 &= (\mp\sqrt{\mu} + O(\mu)) z_1 \\ \dot{z}_2 &= (A_2 + O(\mu)) z_2 \end{aligned}$$



Classical Fold Bifurcation
Horizontal Axis: μ
Vertical Axis: x_1
Stable Equilibria: solid, blue
Unstable Equilibria: dash, red

Control Fold Bifurcation

Only generic control bifurcation when $u \in I\!R$.

Normal Form:

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 + \gamma x_1 x_{21} + \delta x_{21}^2 + O(x,u)^3 \\ \dot{x}_2 &= A_2 x_2 + B_2 u + O(x,u)^2 \end{aligned}$$

$$x_1 \in I\!R, \qquad x_2 \in I\!R^{n-1}$$

Without loss of generality,

$$A_2, B_2 \text{ Brunovsky}, \qquad \alpha > 0, \qquad \gamma \geq 0, \qquad \delta \geq 0$$

Equilibria:

$$\begin{aligned} x_1 &= -\frac{\delta}{\alpha}\mu^2 + O(\mu)^3 \\ x_{21} &= \mu \\ x_{2i} &= O(\mu)^2, \qquad i = 2, \ldots, n-1 \\ u &= O(\mu)^2 \end{aligned}$$

Linearization around μ^{th} equilibrium in displacement coordinates:

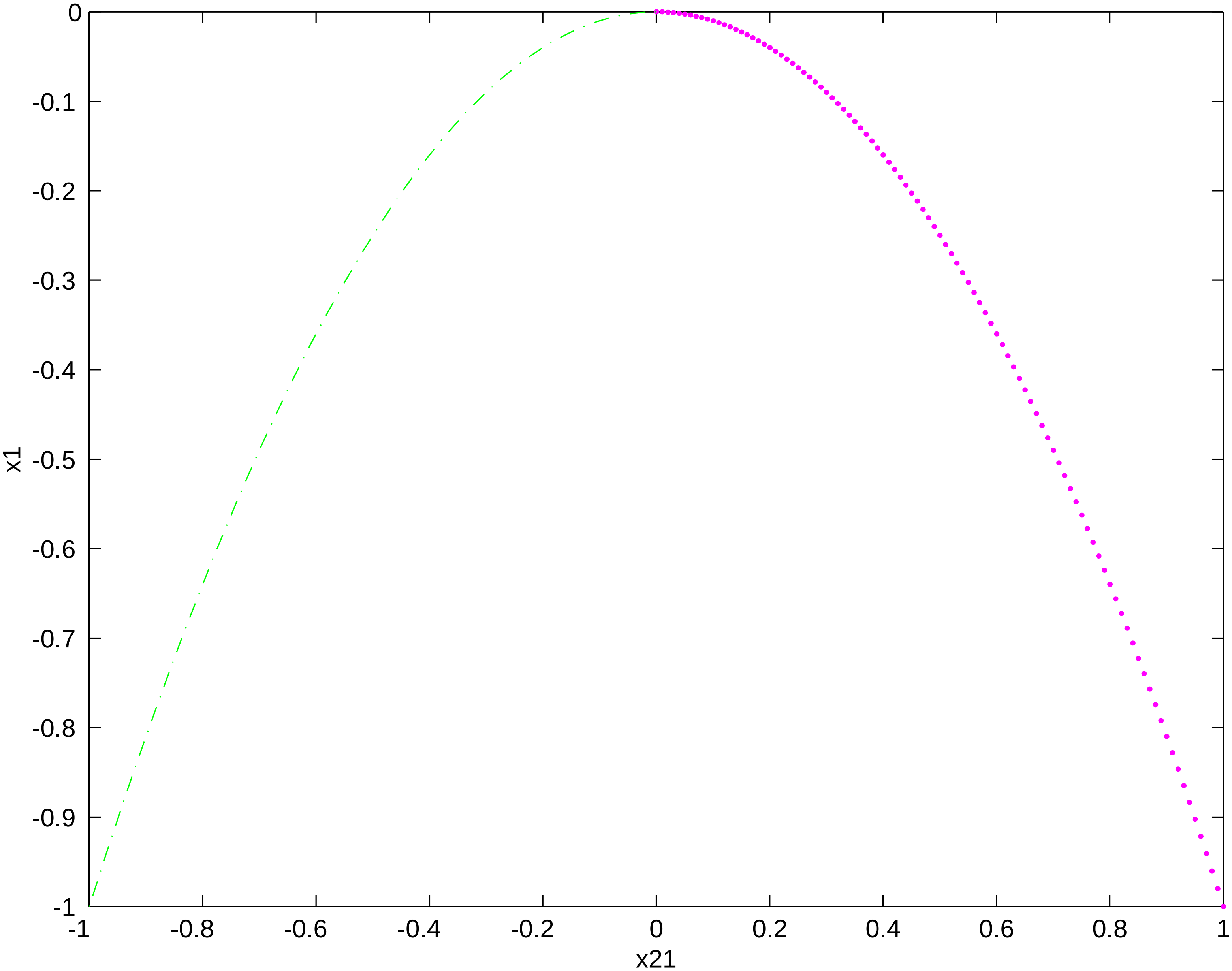
$$\dot{z} = \left(\left[\begin{array}{c|c} \alpha + \gamma\mu & 2\delta\mu \ 0 \ \ldots \ 0 \\ \hline 0 & A_2 \end{array} \right] + O(\mu)^2 \right) z$$

$$+ \left(\left[\begin{array}{c} 0 \\ \hline B_2 \end{array} \right] + O(\mu)^2 \right) v$$

$$\dot{z} = A(\mu)z + Bv + \ldots$$

$$\left[\, A^{n-1}B \ldots B \,\right] = \begin{bmatrix} 2\delta\mu & 0 \\ 0 & I \end{bmatrix}$$

The determinant of the controllability matrix changes sign at $\mu = 0$.



Control Fold Bifurcation
Horizontal Axis: x_{21}
Vertical Axis: x_1
Positively Oriented Controllability: dot, magenta
Negatively Oriented Controllability: dash-dot, green

Closed Loop leads to Fold Bifurcation

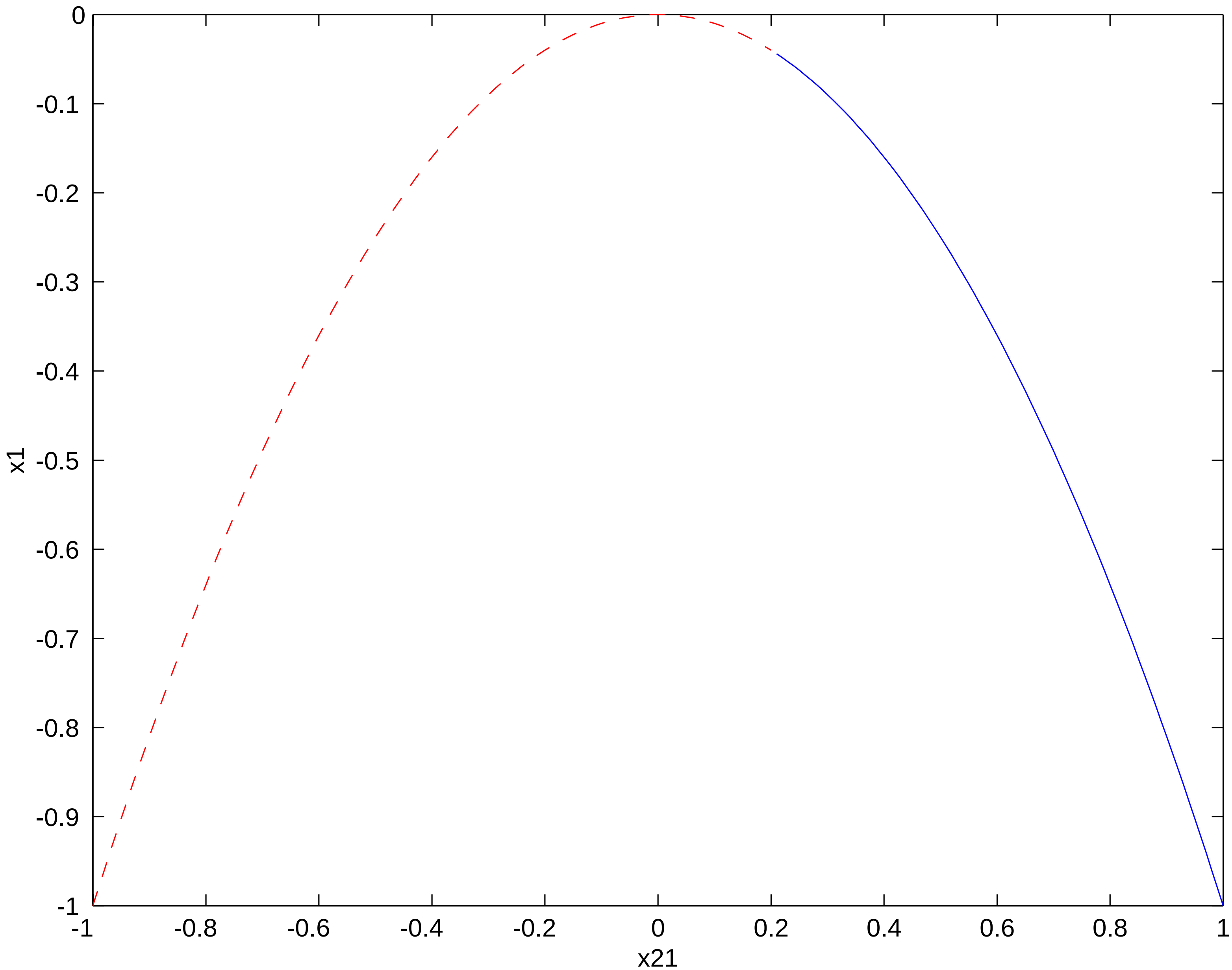
In displacement coordinates the feedback is

$$v = K_1(\mu)z_1 + K_2(\mu)z_2 + \dots$$

The linear part of the closed loop system at the μ^{th} equilibrium is

$$\dot{z} = \left(\left[\begin{array}{c|c} \alpha + \gamma\mu & 2\delta\mu \;\; 0 \;\; \dots \;\; 0 \\ \hline B_2 K_1 & A_2 + B_2 K_2 \end{array} \right] + O(\mu)^2 \right) z$$

and so the closed loop system undergoes a classical fold bifurcation near $\mu = 0$.



Closed Loop Fold Bifurcation

Horizontal Axis: x_{21}

Vertical Axis: x_1

Stable Equilibria: solid, blue

Unstable Equilibria: dash, red

Classical Transcritical Bifurcation

Normal Form:

$$\begin{aligned}\dot{x}_1 &= \mu x_1 - x_1^2 + \dots \\ \dot{x}_2 &= A_2 x_2 + \dots\end{aligned}$$

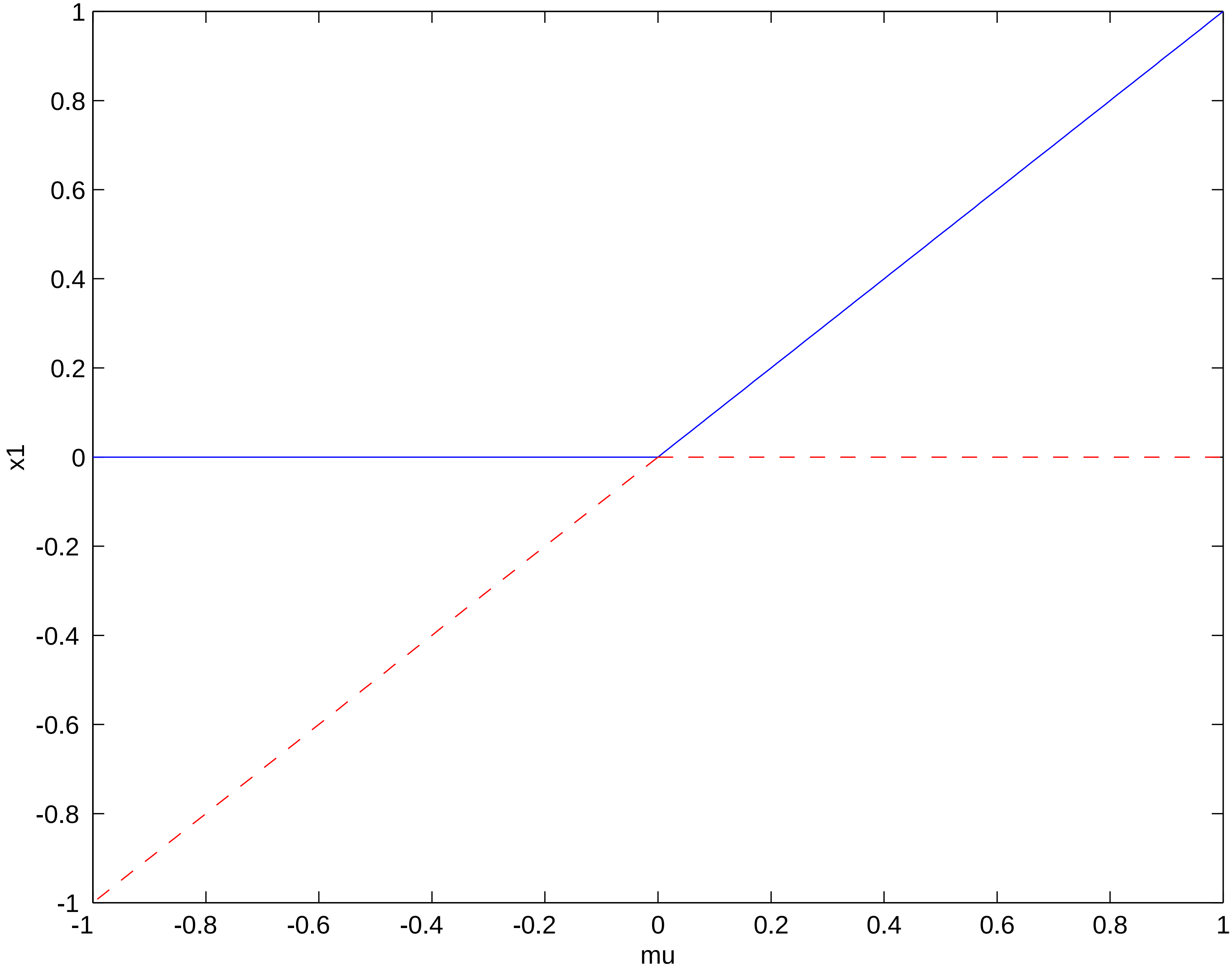
$$\mu, x_1 \in I\!R, \qquad x_2 \in I\!R^{n-1}, \qquad A_2 \text{ hyperbolic}$$

Equilibria:

$$\begin{aligned}x_1 &= 0 & \text{or} \quad x_1 &= \mu + O(\mu)^2 \\ x_2 &= O(\mu) & x_2 &= O(\mu)\end{aligned}$$

Linearization around μ^{th} equilibria in displacement coordinates:

$$\begin{aligned}\dot{z}_1 &= \mu z_1 & \dot{z}_1 &= (-\mu + O(\mu))z_1 \\ \dot{z}_2 &= (A_2 + O(\mu))z_2 & \dot{z}_2 &= (A_2 + O(\mu))z_2\end{aligned}$$



Classical Transcritical Bifurcation
Horizontal Axis: μ
Vertical Axis: x_1
Stable Equilibria: solid, blue
Unstable Equilibria: dash, red

Control Transcritical Bifurcation

Normal Form:

$$\begin{aligned}\dot{x}_1 &= \beta x_1^2 + \gamma x_1 x_{21} + \delta x_{21}^2 + O(x,u)^3 \\ \dot{x}_2 &= A_2 x_2 + B_2 u + O(x,u)^2\end{aligned}$$

$$x_1, u \in I\!R, \qquad x_2 \in I\!R^{n-1}, \qquad A_2, B_2 \text{ Brunovsky}$$

Equilibria:
Assume $\beta x_1^2 + \gamma x_1 x_{21} + \delta x_{21}^2$ is nondegenerate.
If it is sign definite then $x_1 = 0, x_2 = 0, u = 0$ is an isolated equilibrium.
Otherwise there are two curves of equilibria crossing at $x_1 = 0, x_2 = 0, u = 0$.

Example: $\beta = 1, \gamma = 0, \delta = -1$.

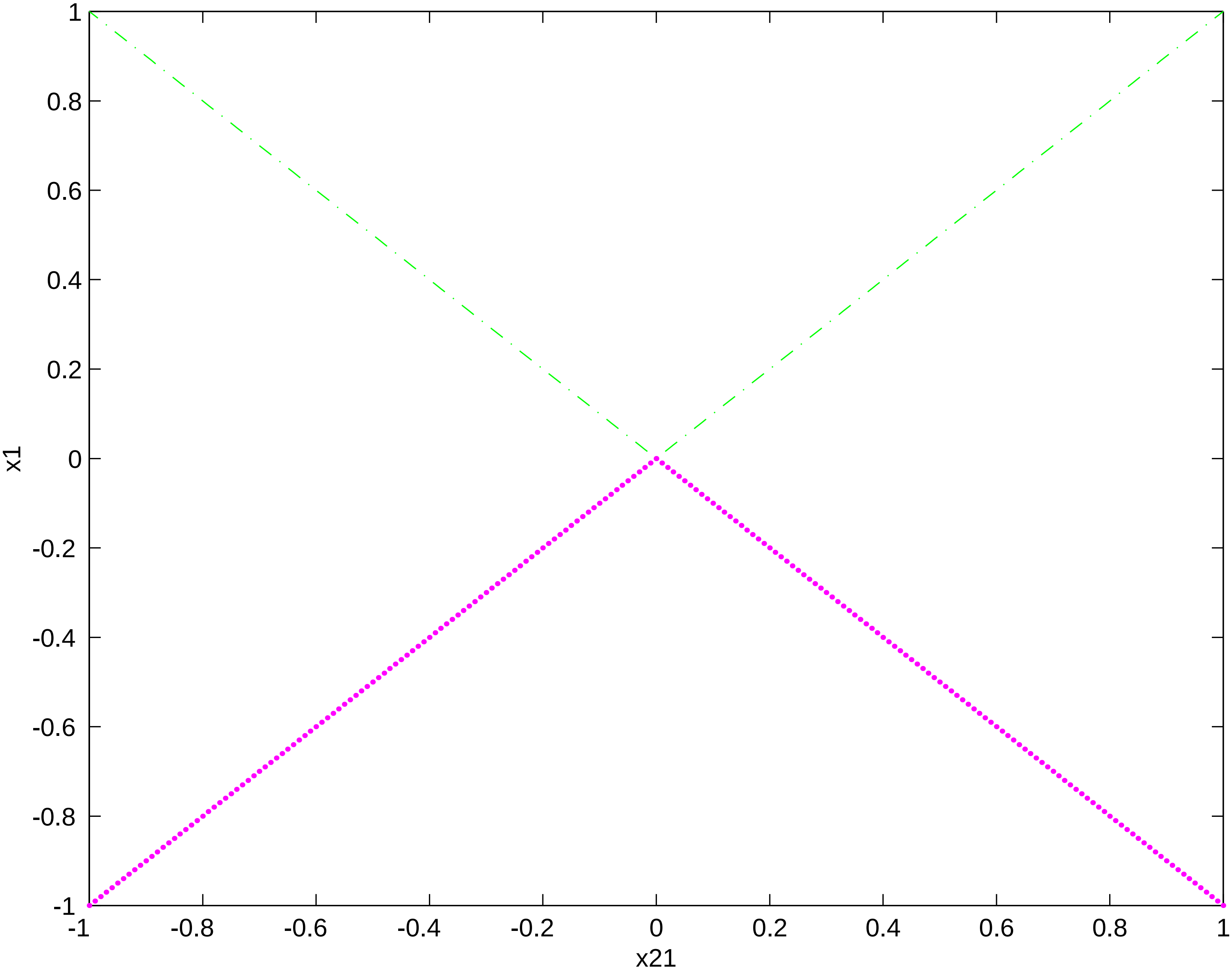
$$\begin{aligned}x_1 &= \pm\mu + O(\mu)^2 \\ x_{21} &= \mu \\ x_{2i} &= O(\mu)^2, \qquad i = 2, \dots, n-1 \\ u &= O(\mu)^2\end{aligned}$$

Linearization around μ^{th} equilibria in displacement coordinates:

$$\dot{z} = \left(\left[\begin{array}{c|c} \pm 2\mu & -2\mu \ \ 0 \ \ \ldots \ \ 0 \\ \hline 0 & A_2 \end{array} \right] + O(\mu)^2 \right) z + \left(\left[\begin{array}{c} 0 \\ \hline B_2 \end{array} \right] + O(\mu)^2 \right) v$$

$$\left[\, A^{n-1} B \ldots B \, \right] = \left[\begin{array}{cc} \mp 2\mu & 0 \\ 0 & I \end{array} \right]$$

This could be called a "Transcontrollable Bifurcation" as the determinant of the controllability matrix changes sign at $\mu = 0$ on each of the branches of equilibria.



Control Transcritical Bifurcation
Horizontal Axis: x_{21}
Vertical Axis: x_1
Positively Oriented Controllability: dot, magenta
Negatively Oriented Controllability: dash-dot, green

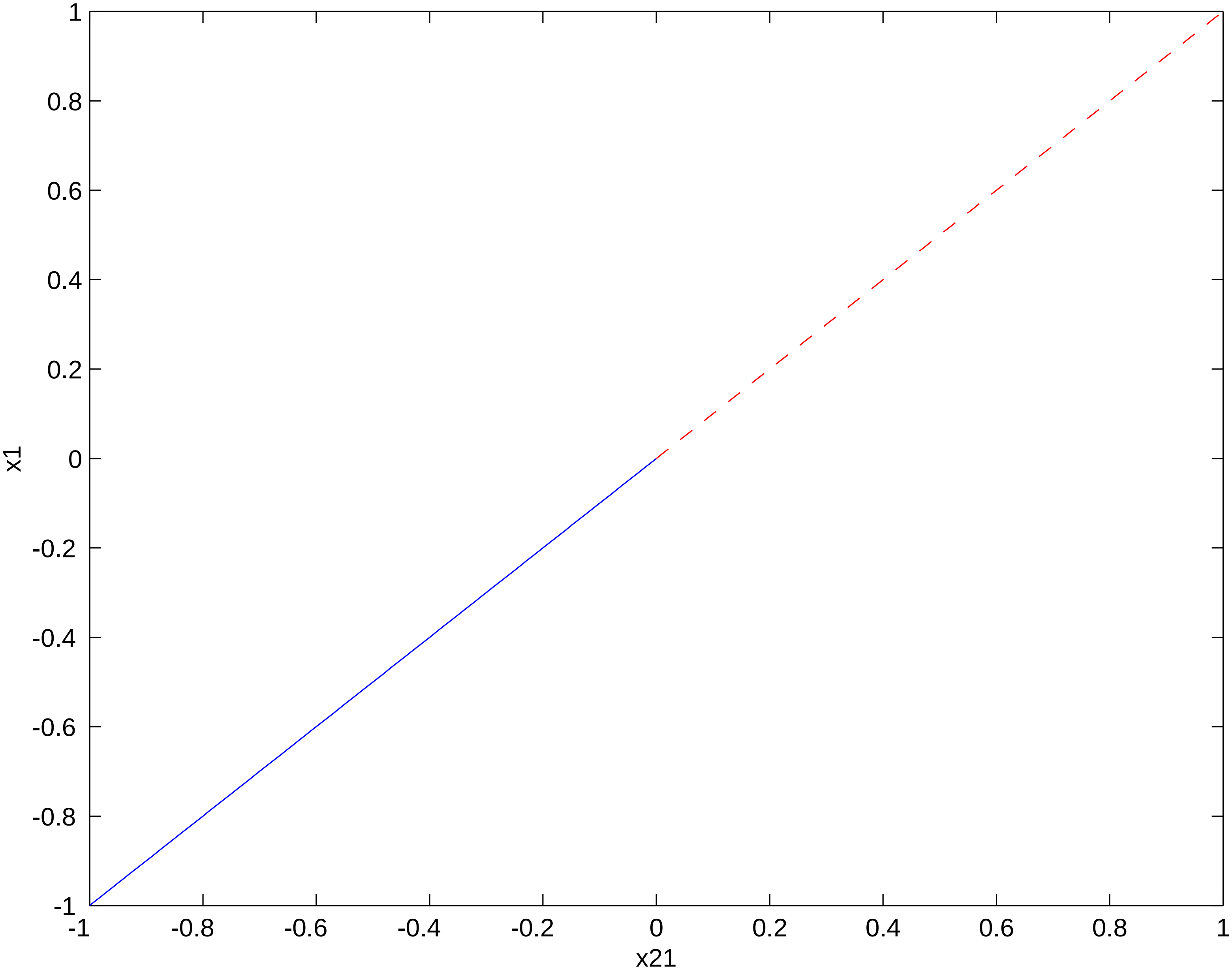
Closed Loop leads to Transcritical Bifurcation

In displacement coordinates around the branch $x_1 = \mu + O(\mu)^2$

$$v = K_1(\mu) z_1 + K_2(\mu) z_2 + \dots$$

$$\dot{z} = \left(\left[\begin{array}{c|c} 2\mu & -2\mu \;\; 0 \;\; \dots \;\; 0 \\ \hline B_2 K_1 & A_2 + B_2 K_2 \end{array} \right] + O(\mu)^2 \right) z$$

Changes stability at $\mu = 0$.



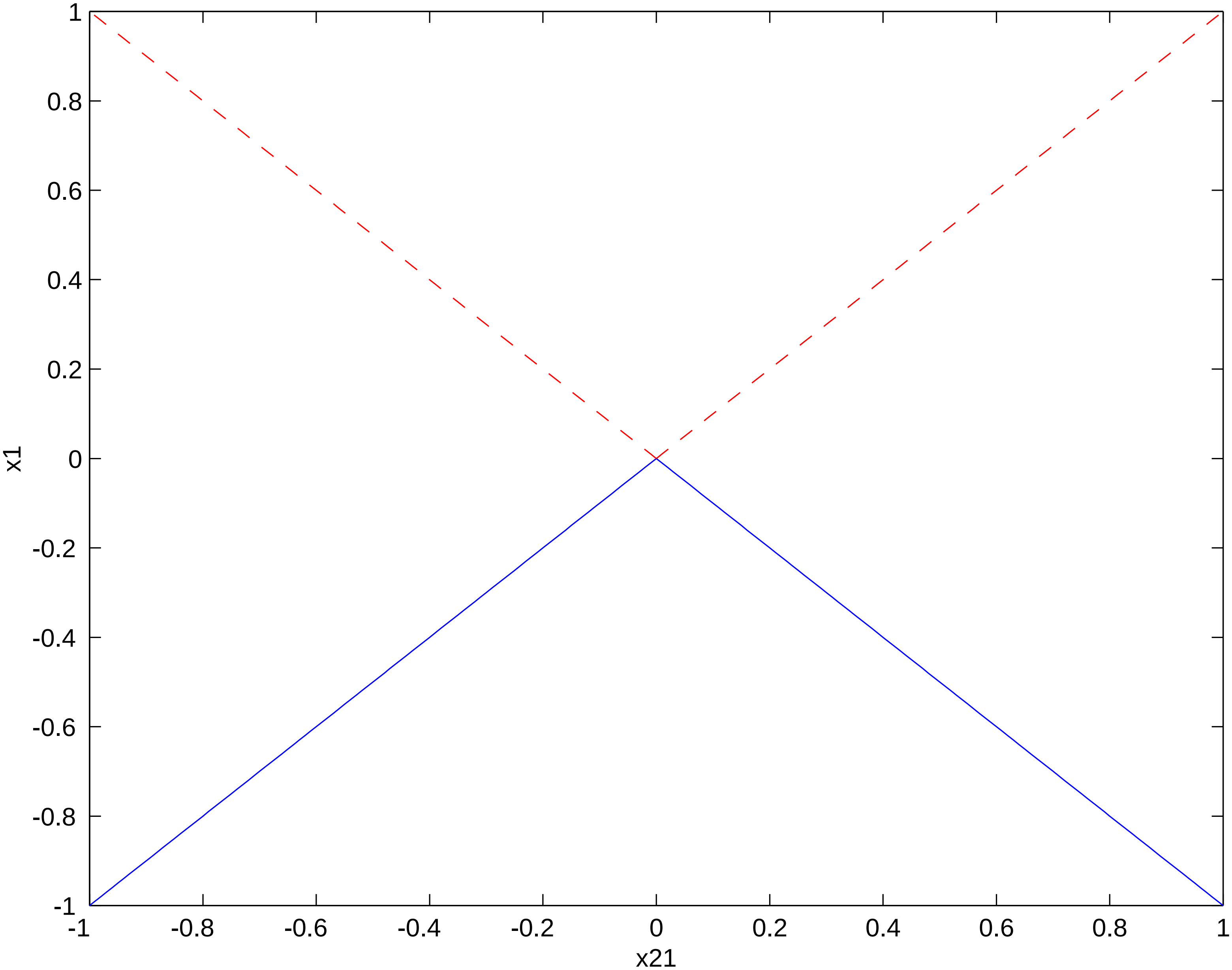
Closed Loop Transcritical Bifurcation
Horizontal Axis: x_{21}
Vertical Axis: x_1
Stable Equilibria: solid, blue
Unstable Equilibria: dash, red

Assume $|K_1| \neq |K_{21}|$ then there is a second branch of closed loop equilibria,

$$\begin{aligned} x_1 &= -\kappa\mu + O(\mu)^2 \\ x_{21} &= \kappa\mu \\ x_{2i} &= O(\mu)^2, \qquad i = 2, \ldots, n-1 \\ u &= O(\mu)^2 \end{aligned}$$

$$\kappa = \frac{K_{21} + K_1}{K_{21} - K_1}$$

$$\dot{z} = \left(\left[\begin{array}{c|c} -2\kappa\mu & 2\kappa\mu \; 0 \; \ldots \; 0 \\ \hline B_2 K_1 & A_2 + B_2 K_2 \end{array} \right] + O(\mu)^2 \right) z$$



Both Branches of Equilibria

Horizontal Axis: x_{21}

Vertical Axis: x_1

Stable Equilibria: solid, blue

Unstable Equilibria: dash, red

Classical Hopf Bifurcation

Normal Form:

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} x_1 + \lambda |x_1|^2 x_1 + \dots \\ \dot{x}_2 &= A_2 x_2 + \dots \end{aligned}$$

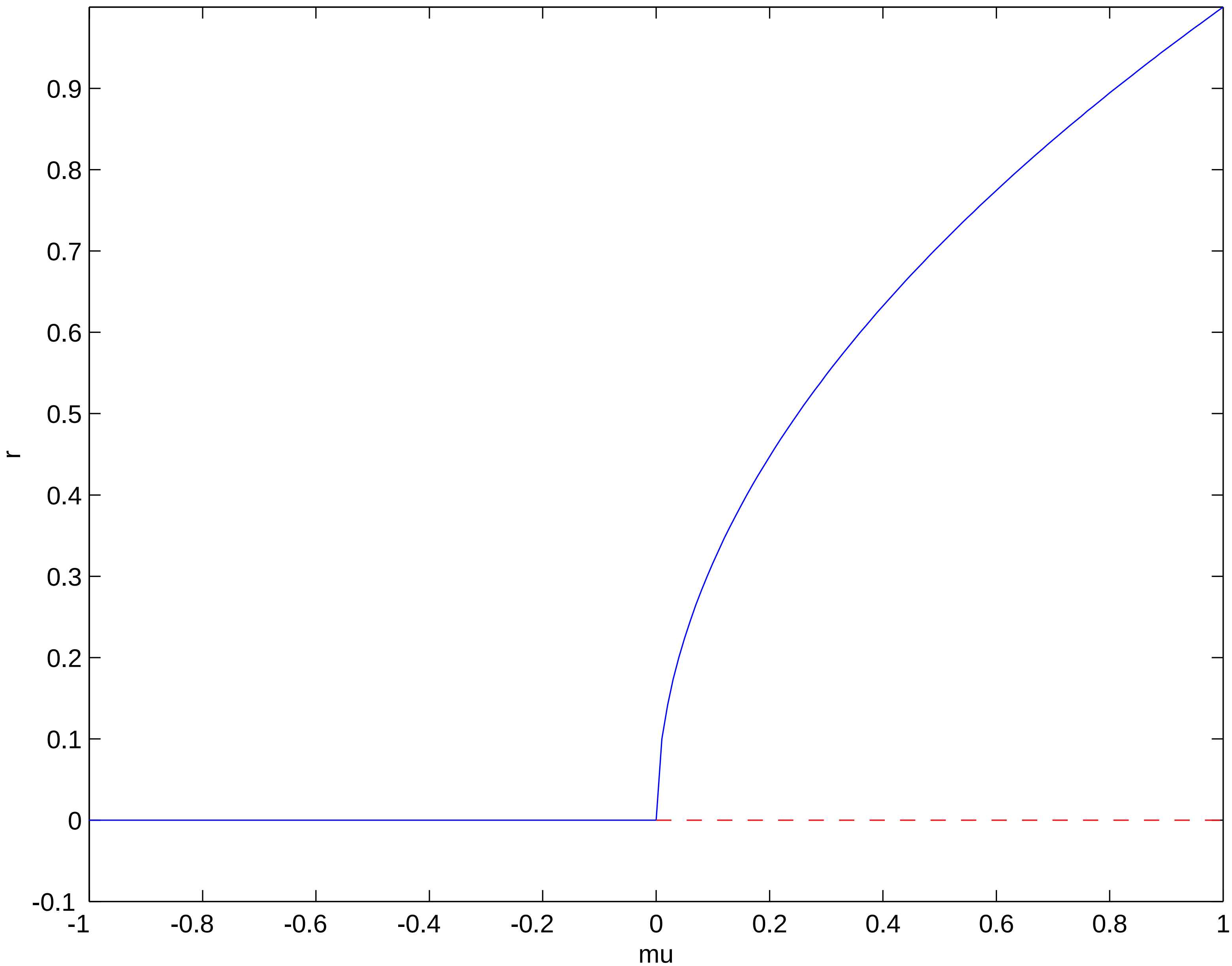
$\mu \in I\!R,\ x_1 \in I\!R^2,\ x_2 \in I\!R^{n-2},\ A_2$ hyperbolic, $\nu \neq 0$

Equilibria:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

λ is called the first Lyapunov coefficient.

The bifurcation is supercritical if $\lambda < 0$
and is subcritical if $\lambda > 0$.



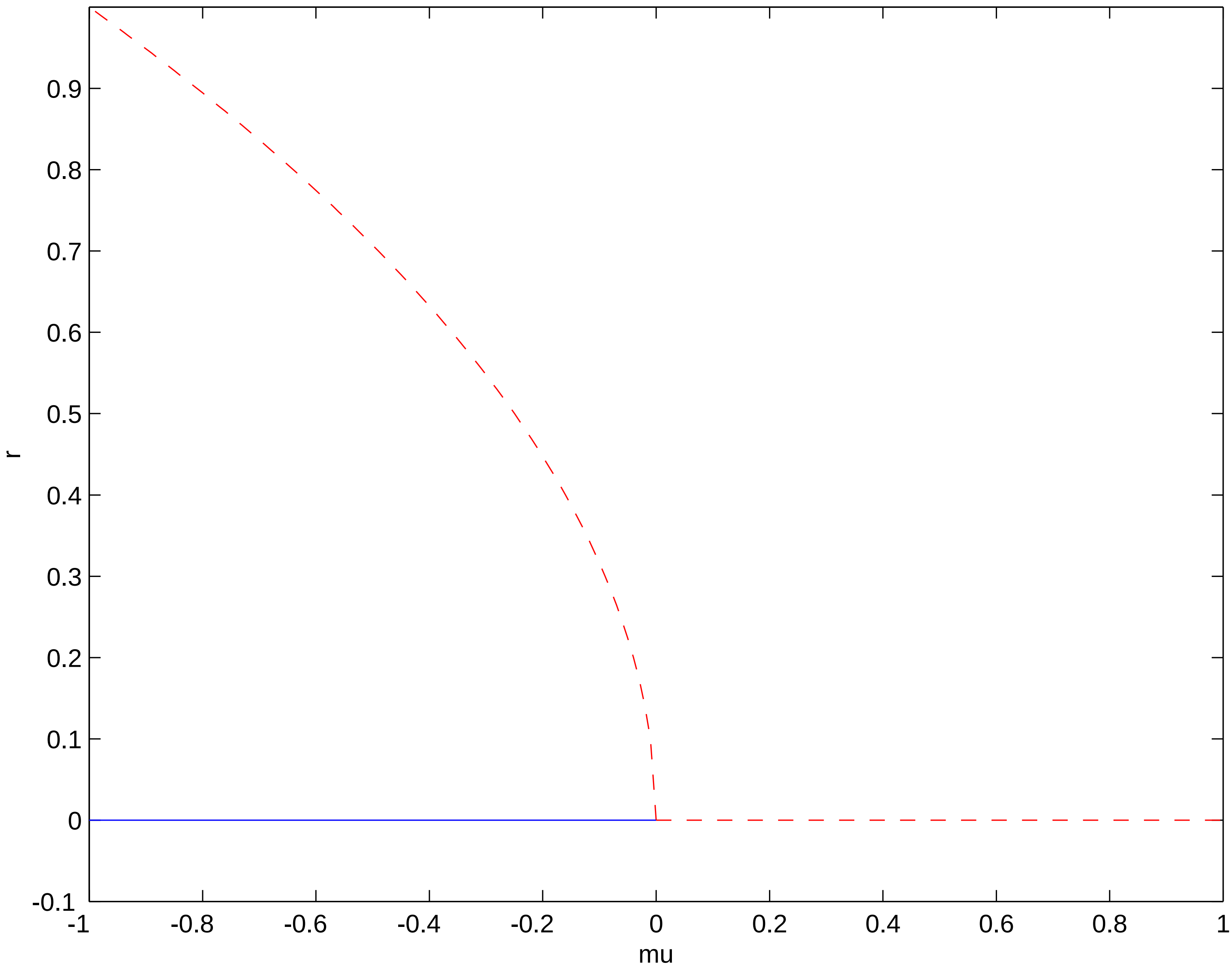
Supercritical Hopf Bifurcation

Horizontal Axis: μ

Vertical Axis: $r = \sqrt{x_{11}^2 + x_{12}^2}$

Stable Equilibria and Limit Cycles: solid, blue

Unstable Equilibria: dash, red



Subcritical Hopf Bifurcation

Horizontal Axis: μ

Vertical Axis: $r = \sqrt{x_{11}^2 + x_{12}^2}$

Stable Equilibria: solid, blue

Unstable Equilibria and Limit Cycles: dash, red

Control Hopf Bifurcation

Normal Form:

$$\begin{aligned}\dot{x}_1 &= A_1 x_1 + \Gamma x_1 x_{21} + \Delta x_{21}^2 \ldots \\ \dot{x}_2 &= A_2 x_2 + B_2 u + O(x,u)^2\end{aligned}$$

$u \in I\!R, \quad x_1 \in I\!R^2, \quad x_2 \in I\!R^{n-2}, \quad A_2, B_2$ Brunovsky

$$A_1 = \begin{bmatrix} \alpha & -\nu \\ \nu & \alpha \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_1^1 & \gamma_1^2 \\ \gamma_2^1 & \gamma_2^2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \ \nu \neq 0$$

This is a control bifurcation if $\alpha > 0$.

Equilibria:

$$\begin{aligned}x_1 &= -\mu^2 A_1^{-1} \Delta + O(\mu)^3 \\ x_{21} &= \mu \\ x_{2i} &= O(\mu)^2, \qquad i = 2, \ldots, n-1 \\ u &= O(\mu)^2\end{aligned}$$

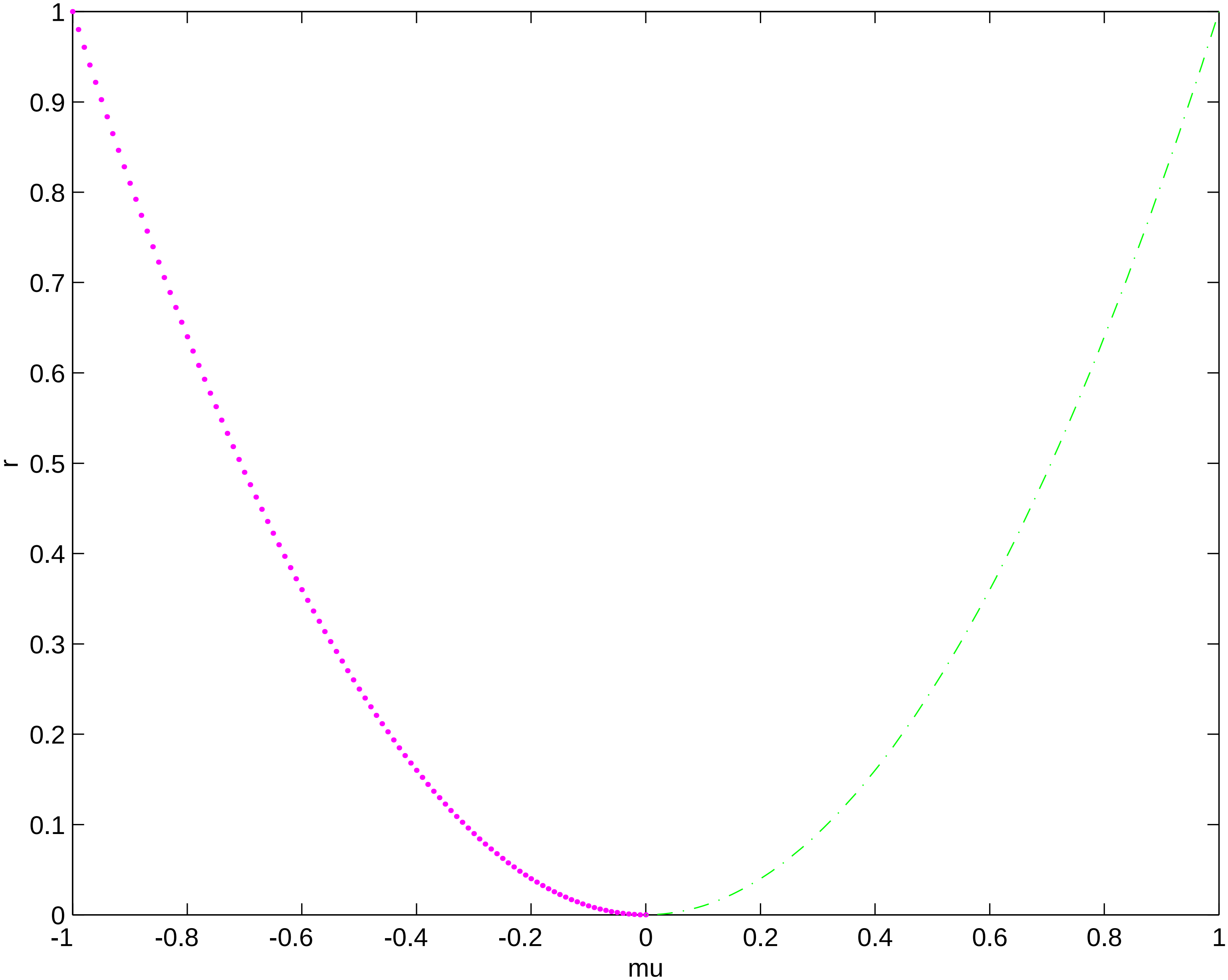
Linearization around μ^{th} equilibrium in displacement coordinates:

$$
\begin{aligned}
\dot{z} &= \left(\left[\begin{array}{c|c} A_1 + \Gamma\mu & 2\Delta\mu \ 0 \ \ldots \ 0 \\ \hline 0 & A_2 \end{array} \right] + O(\mu)^2 \right) z \\
&\quad + \left(\left[\begin{array}{c} 0 \\ \hline B_2 \end{array} \right] + O(\mu)^2 \right) v \\
\dot{z} &= A(\mu) z + B v + \ldots
\end{aligned}
$$

$$
\left[\, A^{n-1} B \ldots B \, \right] = \left[\begin{array}{ccc} 2A_1\Delta\mu & 2\Delta\mu & 0 \\ 0 & 0 & I \end{array} \right]
$$

Exchange of controllability at $\mu = 0$.

Closing the loop can lead to a classical fold, transcritical or Hopf bifurcation.



Control Hopf Bifurcation

Horizontal Axis: x_{21}

Vertical Axis: $r = \sqrt{x_{11}^2 + x_{12}^2}$

Positively Oriented Controllability: dot, magenta

Negatively Oriented Controllability: dash-dot, green

Control via Invariant Manifolds

Just as the center manifold theorem allows a reduction of dimension, one can use feedback to linearly stabilize the linearly controllable coordinates and also use feedback to nonlinearly stabilize the uncontrollable coordinates.

Example: Fold Bifurcation

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 + \gamma x_1 x_{21} + \delta x_{21}^2 + O(x,u)^3 \\ \dot{x}_2 &= A_2 x_2 + B_2 u + O(x,u)^2 \end{aligned}$$

Since there is an exchange of controllability at $\mu = 0$ we choose the piecewise linear feedback

$$u = K_1|x_1| + K_2 x_2$$

where $A_2 + B_2 K_2$ is Hurwitz. We seek an invariant manifold of the piecewise linear part of the closed loop dynamics

$$x_2 = G|x_1| = \begin{bmatrix} g_1 \\ \vdots \\ g_{n-1} \end{bmatrix} |x_1|$$

so that

$$\frac{d}{dt}x_2 = G\frac{d}{dt}|x_1| + O(x)^2$$

$$(A_2 + B_2K_2)\,G|x_1| + B_2K_1|x_1| = G\alpha|x_1|$$

This reduces to

$$\begin{aligned} g_i &= g_1\alpha^{i-1}, \qquad i = 1, \ldots, n-1 \\ K_1 &= g_1 p_2(\alpha) \end{aligned}$$

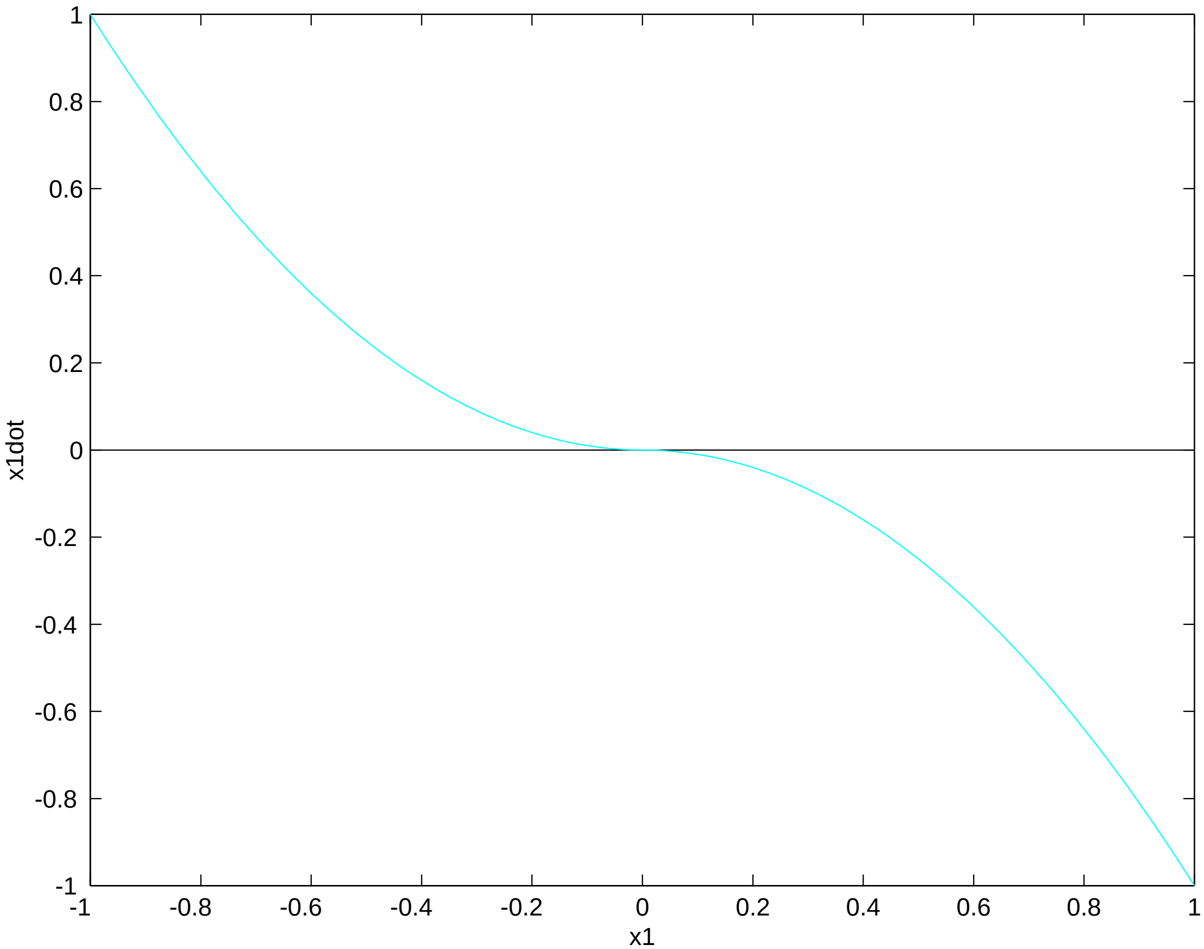
where $p_2(s)$ is the characteristic polynomial of $A_2 + B_2K_2$.

Since $\alpha > 0$, $p_2(\alpha) \neq 0$ so we can parameterize the first part of the feedback by g_1 instead of K_1.

The dynamics on this manifold is

$$\dot{x}_1 = \alpha x_1 + \gamma g_1 x_1 |x_1| + \delta g_1^2 |x_1|^2 + O(x)^3$$

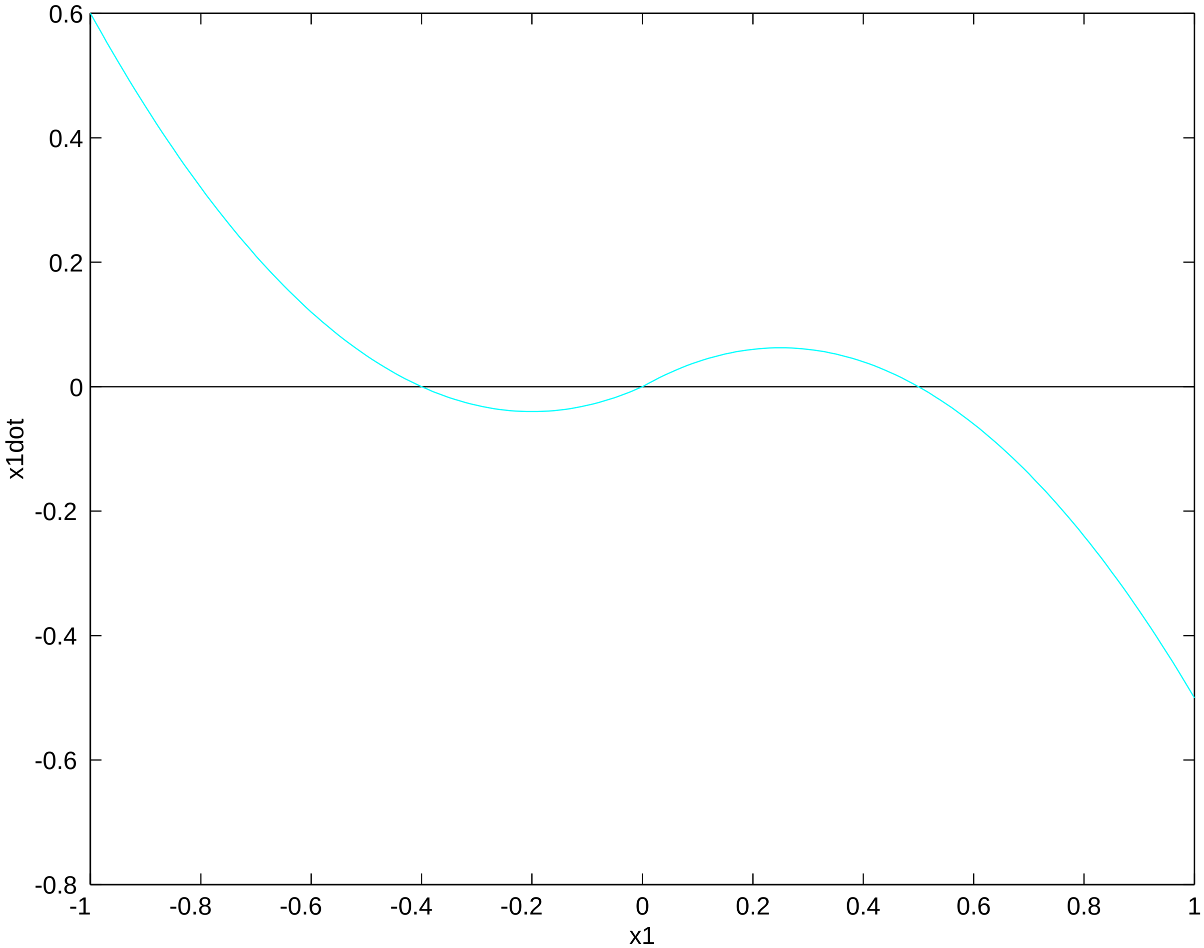
For $\alpha = 0$ and small $g_1 < 0$ we have stability.



Nonlinearly Stable Equilibrium $x_1 = 0$
Horizontal Axis: x_1
Vertical Axis: $\dot{x}_1$

For small $\alpha > 0$ and small $g_1 < 0$ we have practical stability. There are three critical points,

$$x_1 \approx \frac{\pm\alpha}{\gamma g_1 \mp \delta g_1^2}, \qquad x_1^0 = 0$$



Practically Stable Equilibrium $x_1 = 0$
Horizontal Axis: x_1
Vertical Axis: $\dot{x}_1$

Concluding Remarks

• Most of the above control bifurcations occur in the various versions of the Moore-Greitzer equations for an axial flow compressor.

• The above theory arose in abstracting and generalizing Moore-Greitzer.

• The theory of control bifurcations goes beyond the paradigms of linear control theory by incorporating paradigms from nonlinear dynamics.

• This theory of control bifurcations is a local theory.