HIGH RESOLUTION SPECTRA ANALYSIS: ADVANCES AND APPLICATIONS

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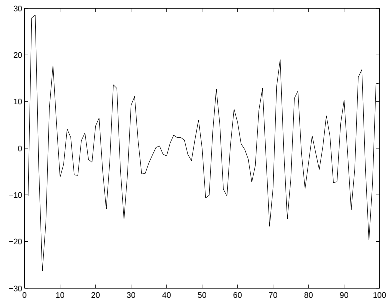
IN HONOR OF DR. MARC JACOBS

Collaboration: Chris Byrnes and Anders Lindquist

AFOSR Grant No. AF/F49620-00-1-0078

Given time-series data:

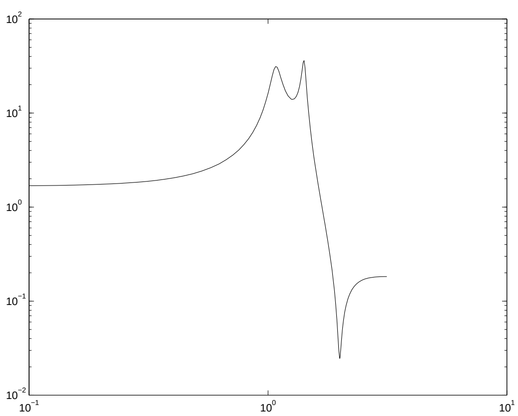
$$\{u_0, u_1, u_2, \ldots, u_{N-1}\}$$



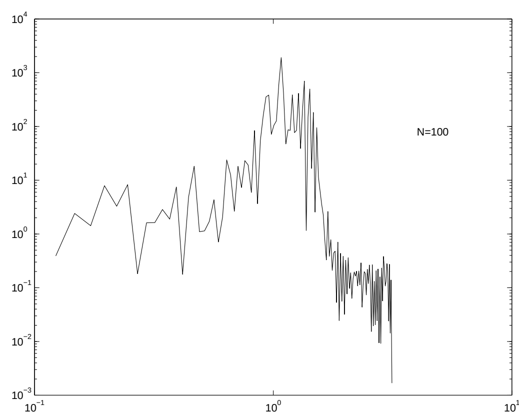
determine the **power spectrum of** y,
i.e., periodicities and "color"

Methods:

- ◦ Periodogram, FFT
- ◦ Model based (ARMA,....)
- ◦ Modern nonlinear (Maximum-entropy, maximum-likelihood,...)

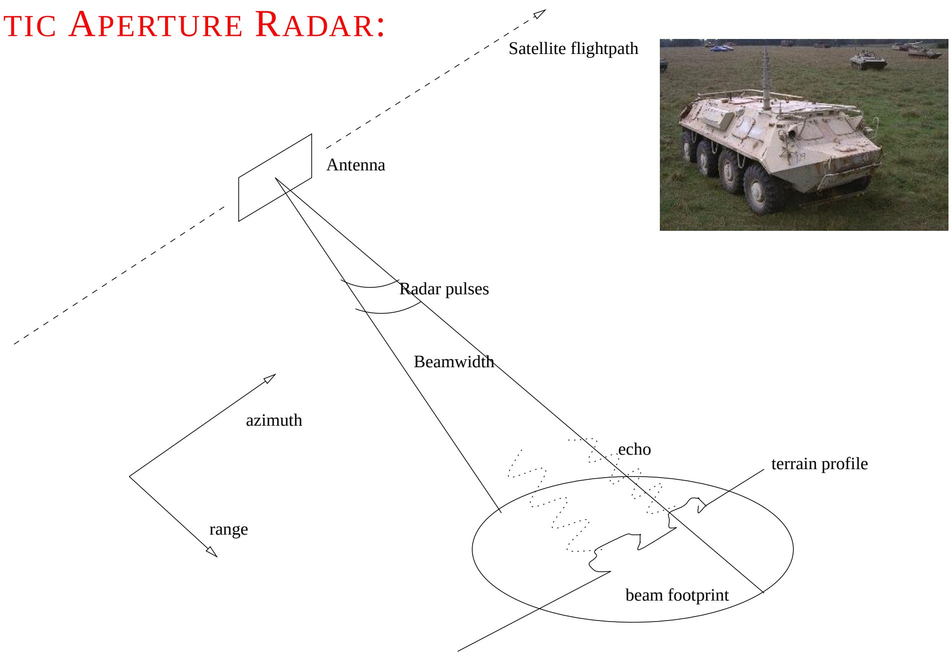


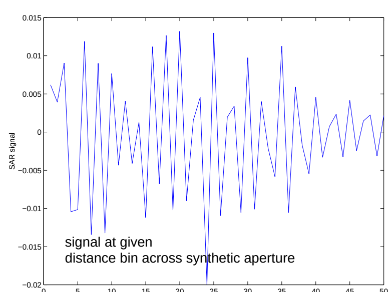
True spectrum



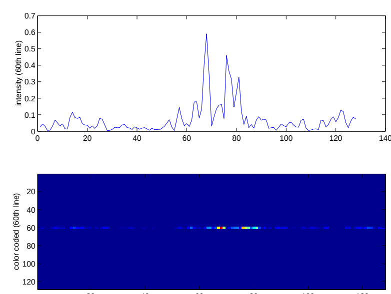
Estimated via periodogram

SYNTHETIC APERTURE RADAR:

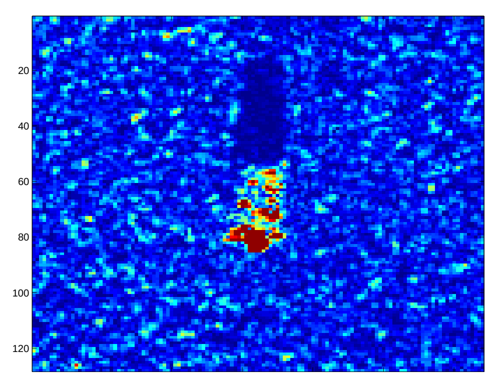


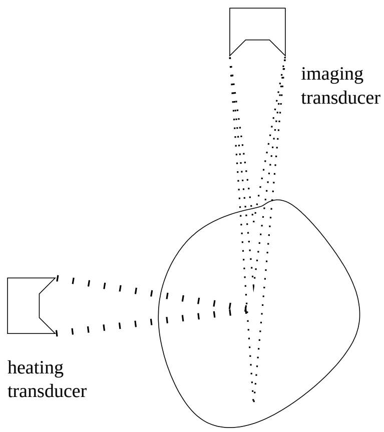


⇒

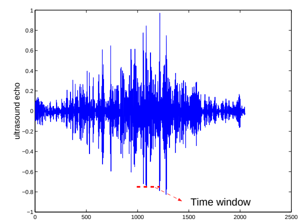


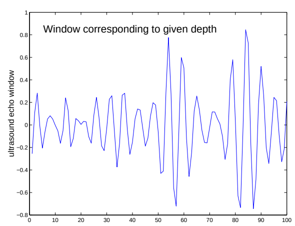
Line by line produces:



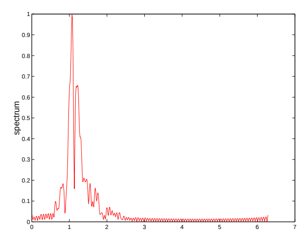


Ultrasound echo:

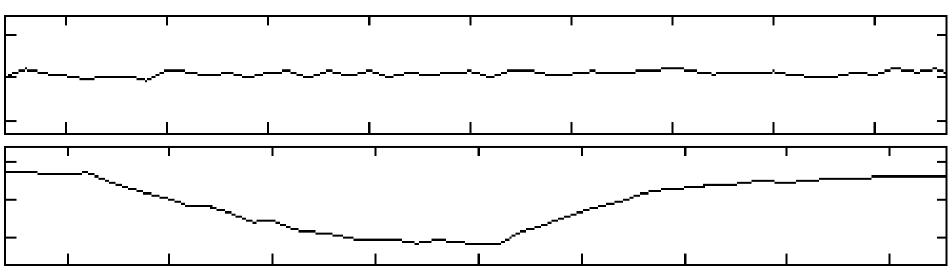




$\Rightarrow$



frequency vs. time ⇒ temperature profile



$$u_k = \int_{-\pi}^{\pi} e^{jk\theta} dv(\theta)$$

$\rho(\theta)d\theta \sim E\{dv(\theta)^2\}$ "energy density across frequencies"

Covariance statistics & spectral density

$$c_k = E\{u_t u_{t+k}\}$$

$$\Updownarrow$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jk\theta} \underbrace{\rho(\theta)d\theta}_{d\mu(\theta)}$$

$c_0, c_1, c_2, \ldots$ autocorrelation sequence

$$\Updownarrow$$

$F(z) = \frac{1}{2}c_0 + c_1 z + c_2 z^2 + \ldots$ "positive-real" function

$$\Updownarrow$$

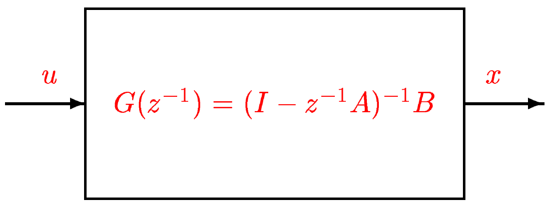
$$\begin{aligned} \rho(\theta) &= \mathrm{Re}(F(e^{j\theta})) \\ &= \ldots c_2 e^{-2j\theta} + c_1 e^{-j\theta} + c_0 + c_1 e^{j\theta} + c_2 e^{2j\theta} + \ldots \geq 0 \end{aligned}$$

Given finite data $c_0, \ldots, c_N$
all consistent spectra are given by:

$\rho = \mathrm{Re}\left(\frac{A+BQ}{C+DQ}\right)$ with Q a "free" parameter

Spectral analysis ⇔ analytic interpolation

- **Theory** ⇐
- **Academic examples** ⇐
- **Applications** ⇐

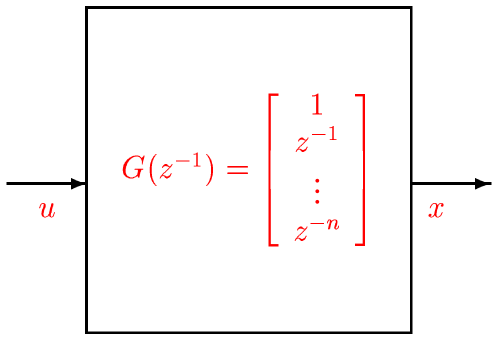


$$x_k = Ax_{k-1} + Bu_k$$

What is the structure of the state-covariance matrix $\Sigma := E\{xx^*\}$*?*

What are all spectra consistent with Σ*?*

$$\Sigma = \int_{-\pi}^{\pi} \left(G(e^{j\theta}) \frac{d\mu(\theta)}{2\pi} G(e^{j\theta})^* \right)$$



$$x_k = \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ uk - n \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} c_0 & c_1 & \dots & c_n \\ c_1^* & c_0 & \dots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_n^* & c_{n-1}^* & \dots & c_0 \end{bmatrix}$$

$$\begin{aligned}\Sigma &= \int_{-\pi}^{\pi}\left(G(e^{j\theta})\frac{d\mu(\theta)}{2\pi}G(e^{j\theta})^*\right)\\ &\vdots\\ &= BH + H^*B^* + A\Sigma A^*\end{aligned}$$

THM: **With (A, B) controllable pair, A stable, $\Sigma \geq 0$:**

Σ is a covariance of $x_k = Ax_{k-1} + Bu_k$

$$\Leftrightarrow$$

$\Sigma = BH + H^*B^* + A\Sigma A^*$ has a solution H

$$\Leftrightarrow$$

$$\text{rank}\begin{bmatrix}\Sigma - A\Sigma A^* & B\\ B^* & 0\end{bmatrix} = \text{rank}\begin{bmatrix}0 & B\\ B^* & 0\end{bmatrix}$$

THM: All input spectra consistent with Σ are

$$d\mu(\theta) \sim \lim_{r\to 1} \mathrm{Re}\left(F(re^{j\theta})\right) d\theta$$

where

$$F(\lambda) = F_0(\lambda) + Q(\lambda)V(\lambda) \quad \textbf{is positive-real}$$

Data A, B, Σ, H**:**
$F_0(\lambda) = H(I - \lambda A)^{-1}B$**,**
$V(\lambda) = D + C\lambda(I - \lambda A)^{-1}B$ **all-pass,**
$Q(\lambda)$ **free parameter.**

$\Rightarrow$ **LFT parametrization of all spectra consistent with** Σ

Entropy:

$$\mathbb{I}(\mu) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det(\dot{\mu}(\theta))d\theta$$

THM: A unique maximizing spectrum is

$$d\mu_o(\theta) := \left(\Phi(e^{j\theta})^{-1}\Omega^{-1}\left(\Phi(e^{j\theta})^{-1}\right)^*\right)d\theta$$

Where:

$\Phi(\lambda) := (B^*\Sigma^{-1}B)^{-1}B^*\Sigma^{-1}(I-\lambda A)^{-1}B,$

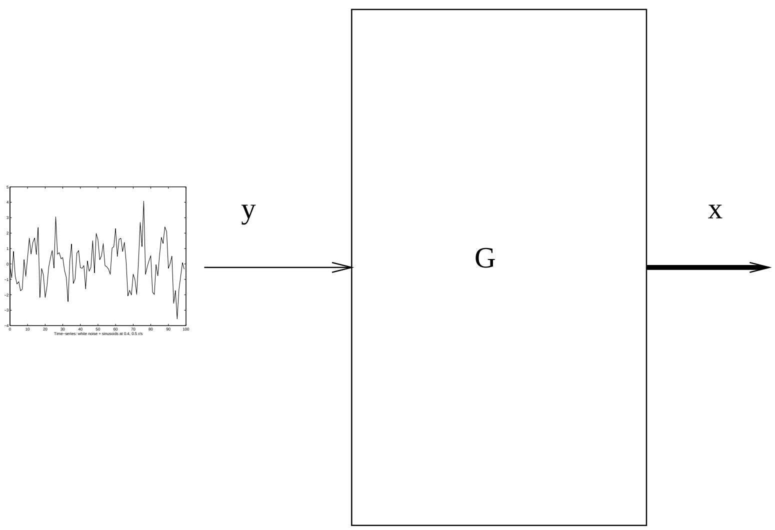
$\Omega := (B^*\Sigma^{-1}B)^{-1}.$

If $u_\ell =$ "white-noise" $+ \sum_{k=1}^{m} \sqrt{\rho_k} e^{j\ell\omega_k}$, then

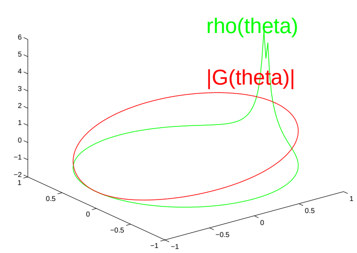
$$\Sigma = \rho_0 I + \sum_{i=1}^{m} \rho_i G(e^{j\omega_i}) G(e^{j\omega_i})^*$$

THM: There exists a unique minimal decomposition of Σ corresponding to a sum of sinusoids

- minimal complexity spectra
- Kullback-Leibler distance & approximation of spectra

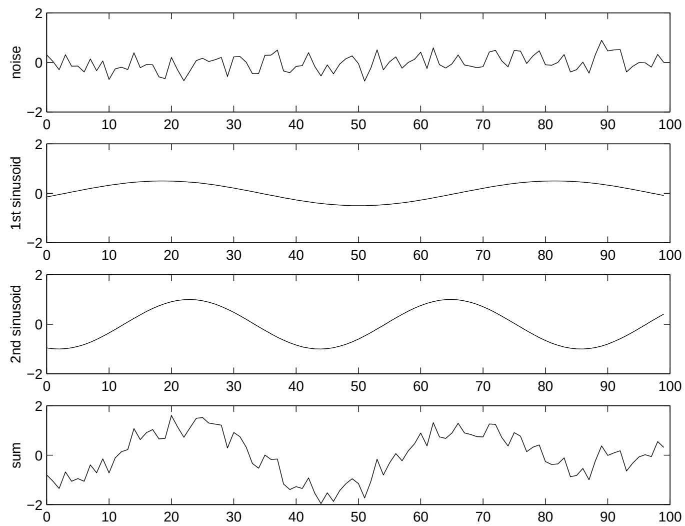


$$\Sigma \sim \sum x_k x_k^*$$



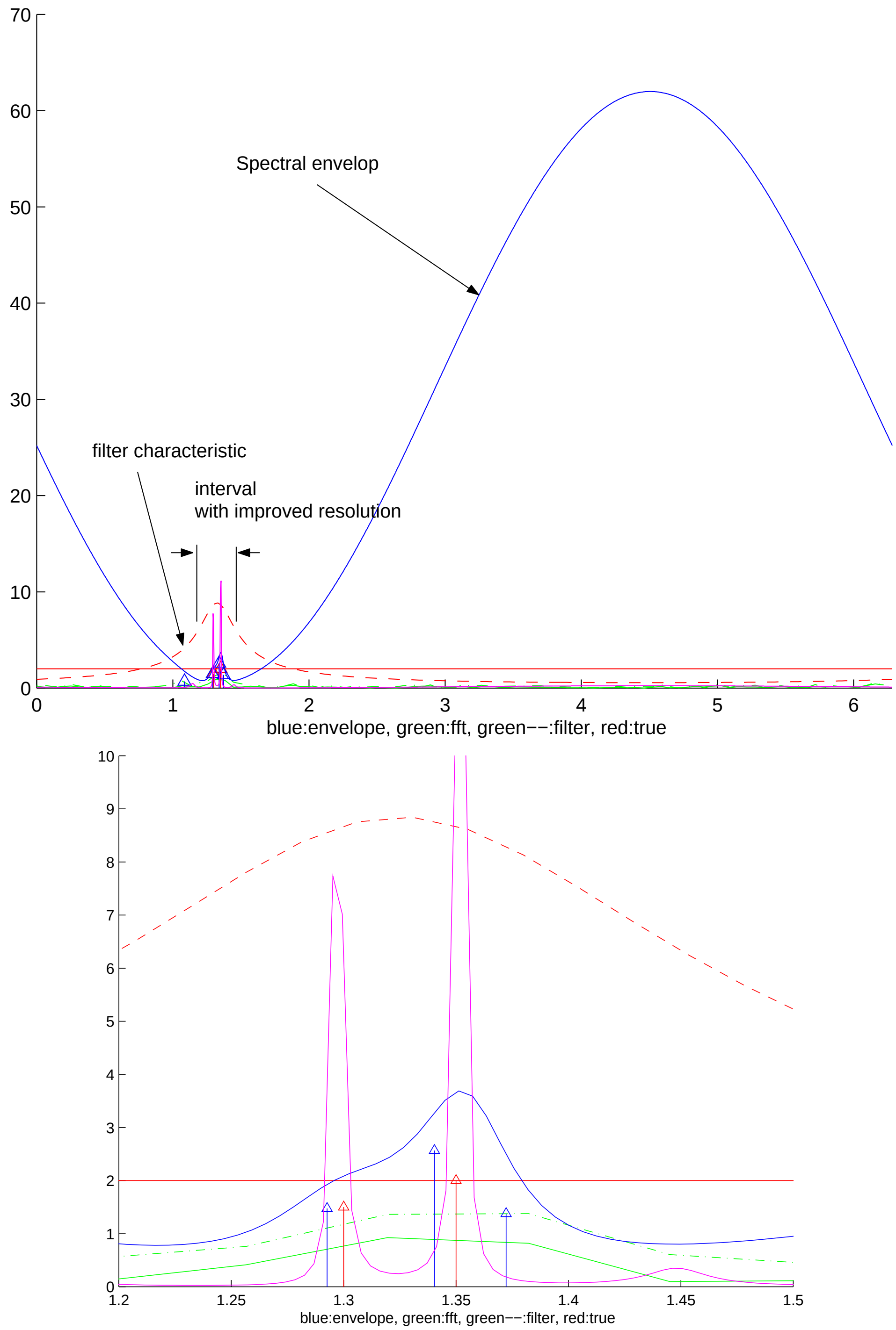
$$\Sigma = \frac{1}{2\pi}\int (G(e^{j\theta})d\mu(\theta)G(e^{j\theta})^*)$$

$$\mathbf{u}_k = \nu_k + A_1 \sin(\omega_1 k + \phi_1) + A_2 \sin(\omega_2 k + \phi_2),\ k = 1, \ldots, n,$$



Noise, sinusoid 1, sinusoid 2, and their sum

$\omega_2 - \omega_1 < \frac{2\pi}{n}$ = Fourier uncertainty bound


Spectral envelop
filter characteristic
interval
with improved resolution
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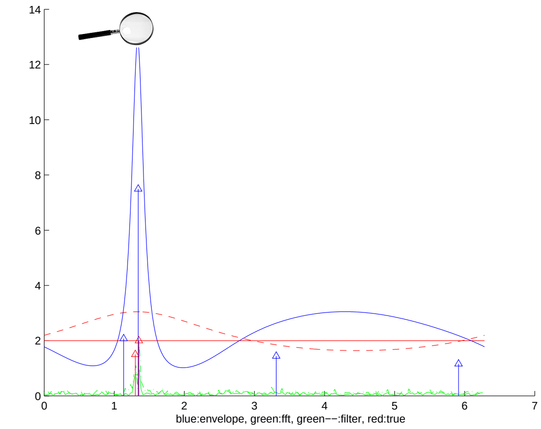
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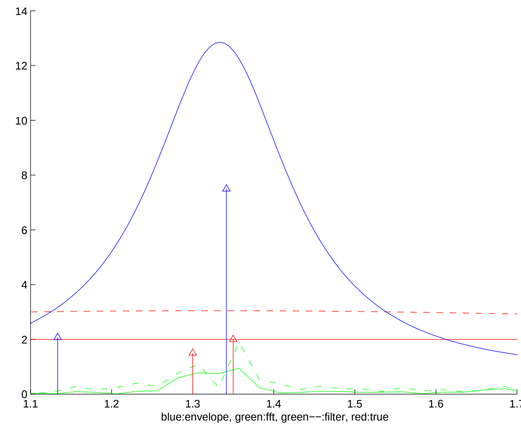
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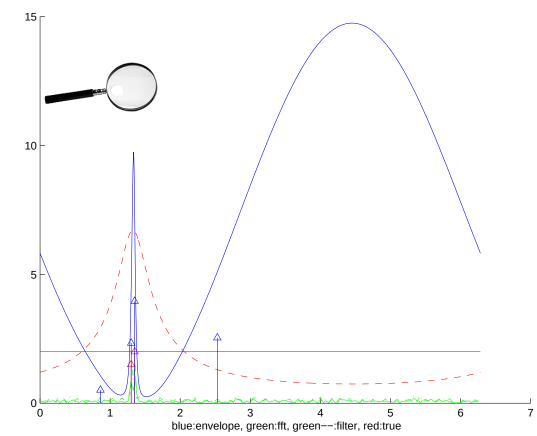
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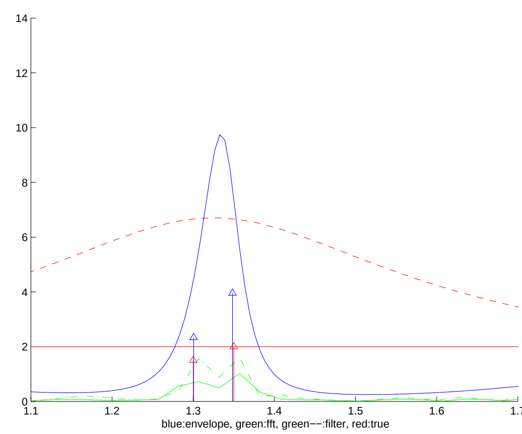
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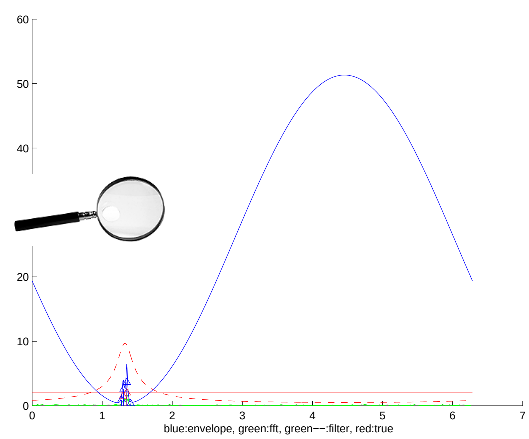
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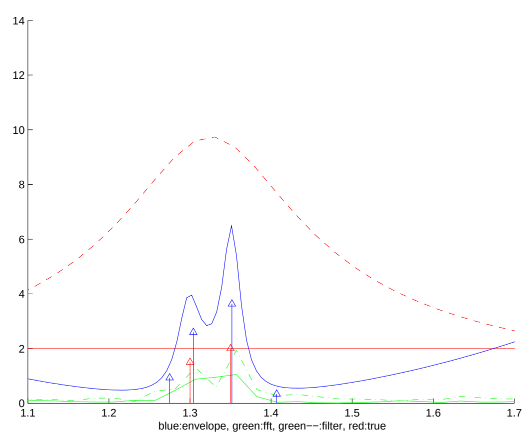
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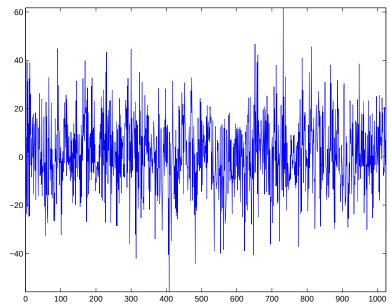
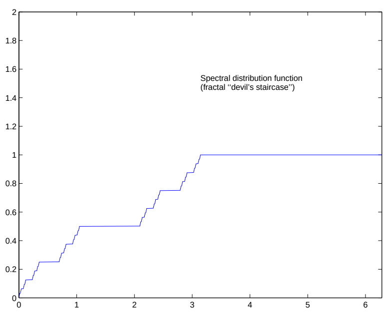
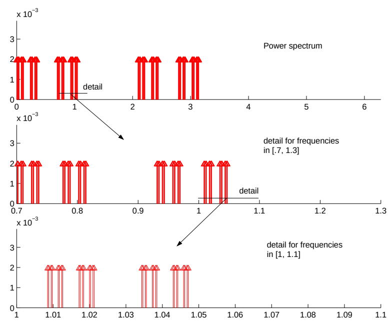
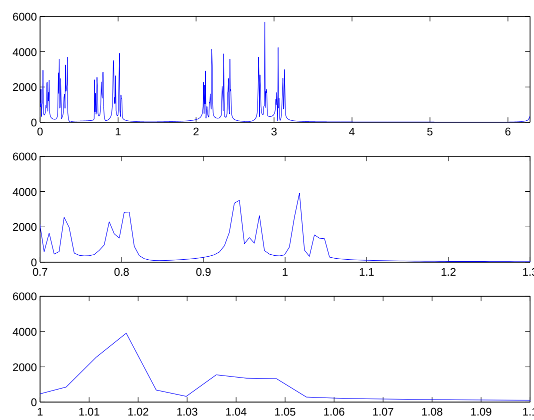
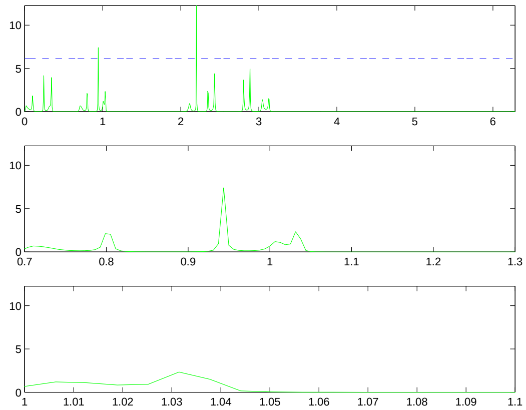


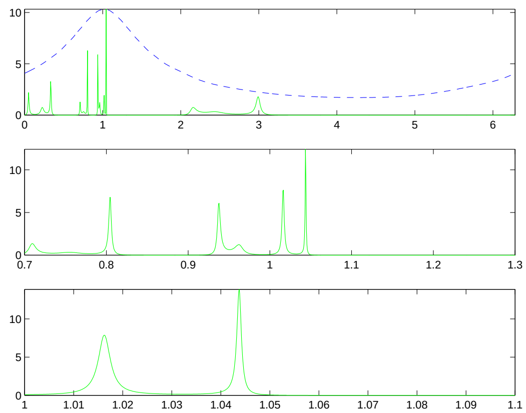


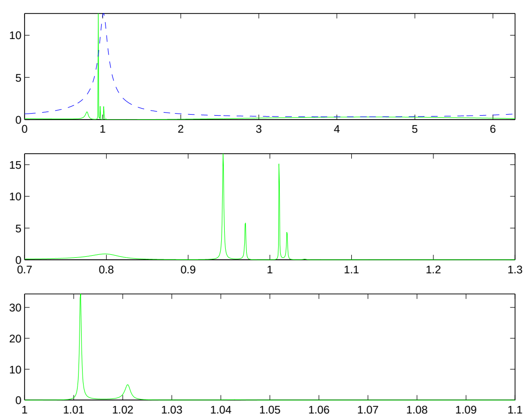




periodogram

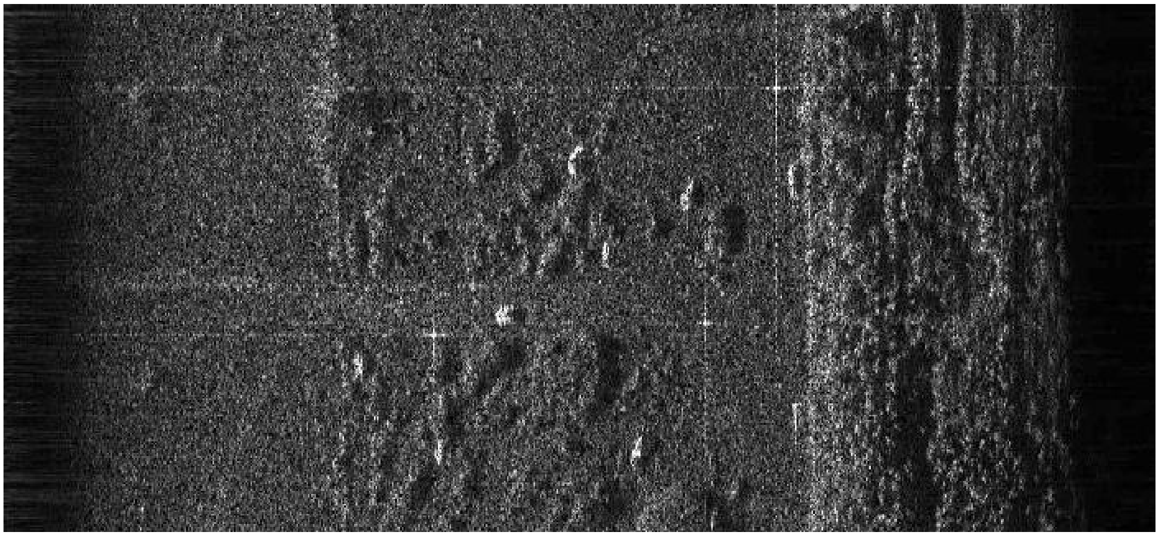


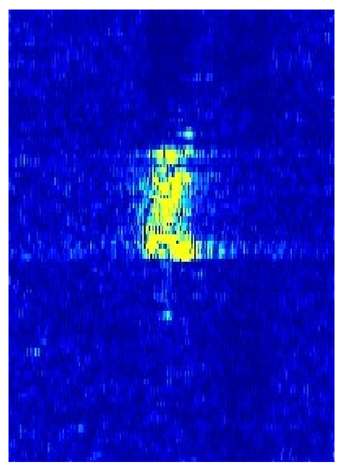




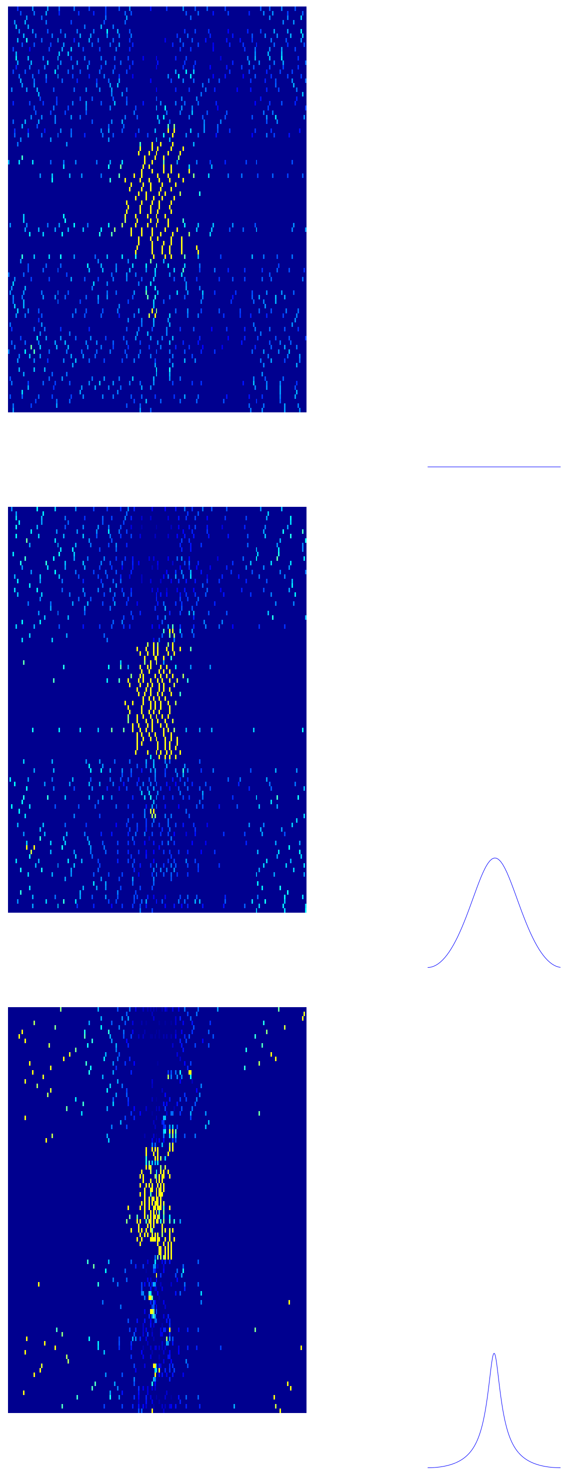
— Maximum entropy spectra

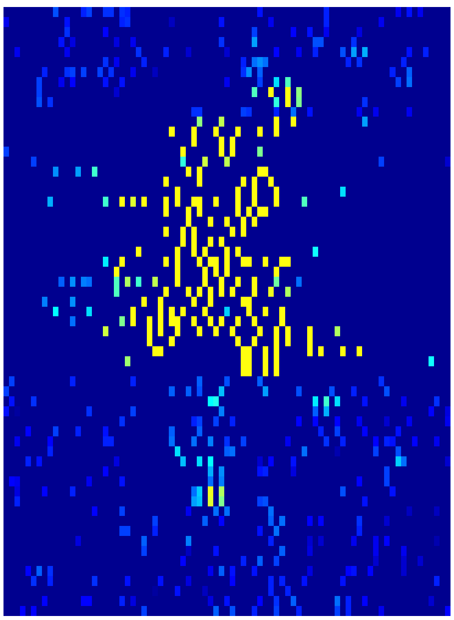
- - - Focusing filter

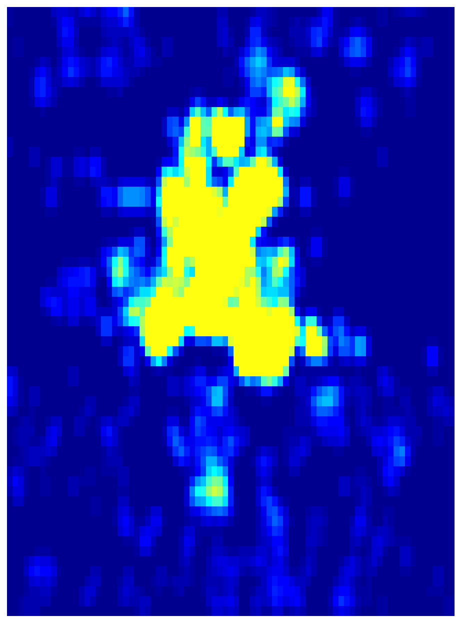




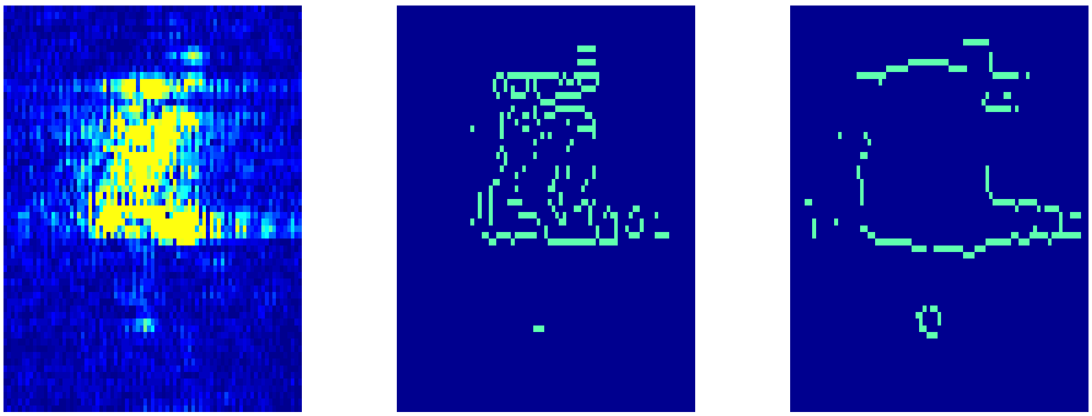
T-72



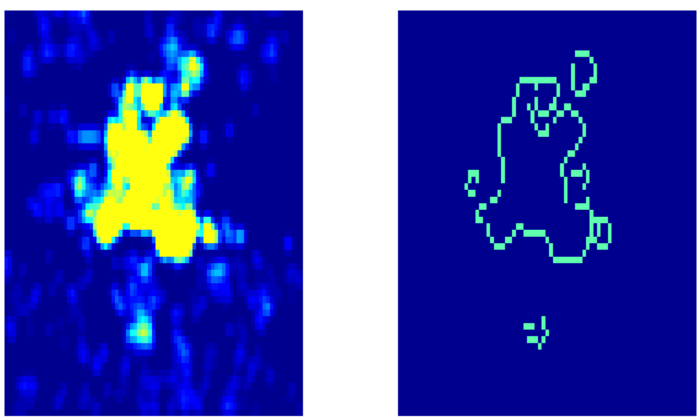


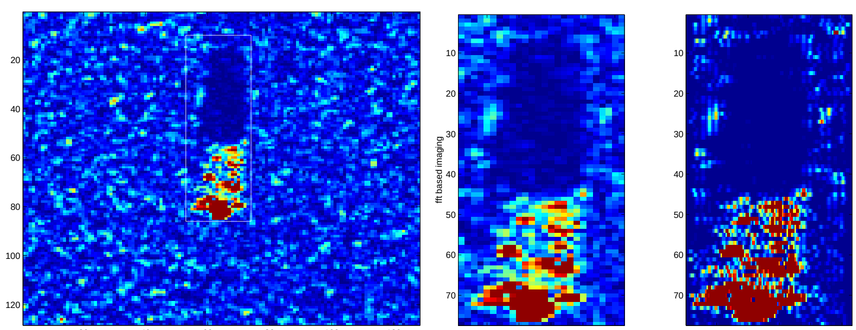


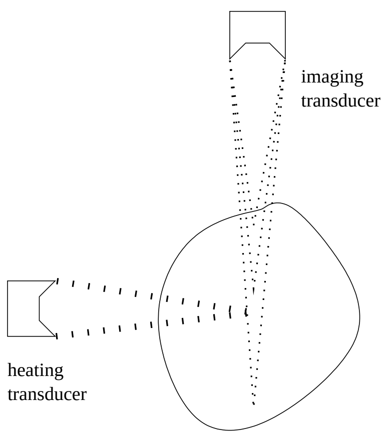
fft



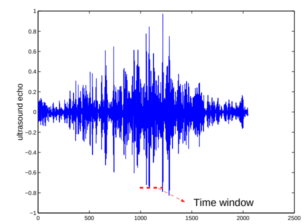
high resolution



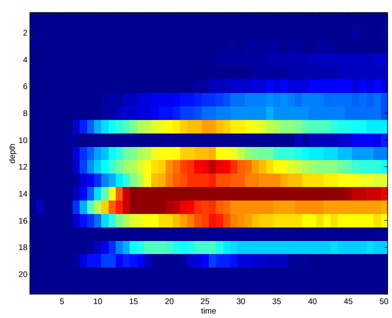

fft based imaging

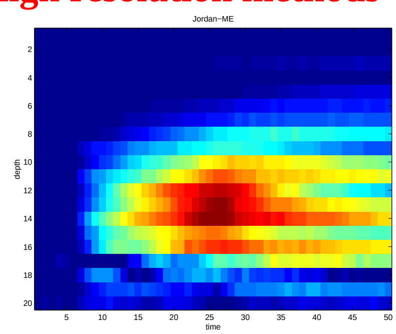


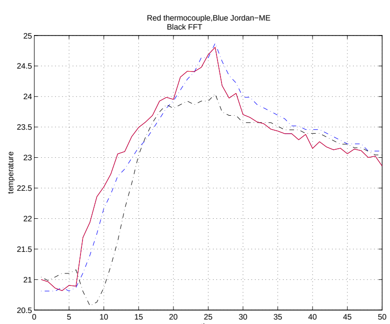
Ultrasound echo:



periodogram analysis vs. high resolution methods







Comparison with thermocouple
(thermocouple, periodogram, high resolution)

- **Collaboration: E. Ebbini**

SUMMARY

- generalized statistics $\sim$ analytic interpolation
- high resolution, applications

QUESTIONS AND ON-GOING RESEARCH PROGRAM:

- ¿how can we quantify resolution?
- tradeoffs between variance and resolution
  seeking an "H_∞-like paradigm"
- spacio-temporal dynamics and non-uniform arrays
- applications: SAR, medical imaging, polarimetry

Matlab code and references at:

`http://www.ece.umn.edu/users/georgiou`