1. Let $V$ be the vector space of $2 \times 2$ matrices over $\mathbb{R}$ with the inner product defined by 
$\langle A, B \rangle = \text{trace}(AB^t)$ where $B^t$ is the transpose of $B$.

(a) Find the norm of the vector $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$.

(b) Find the projection of $A$ on the subspace spanned by $B$ and $C$ if 
$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) Define a linear transformation $T : V \to V$ by $T(X) = AXC$ (matrix multiplication). Find the adjoint of $T$.

2. (a) Let $T : V \to V$ be a linear operator on a finite dimensional real inner product space $V$. We call $T$ orthogonal when it preserves the inner product in the sense that $\langle Tu, Tv \rangle = \langle u, v \rangle$ for all $u, v \in V$. Prove that all of the eigenvalues of $T$ have absolute value 1. Give an example in which there is an eigenvalue that is not 1.

(b) Find the Jordan canonical form of a linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ that is both self-adjoint and orthogonal.

3. Let $f$ be real valued and twice continuously differentiable on an interval $[a, b]$ and let $\bar{x}$ be a simple zero of $f$ in $(a, b)$. Show that Newton’s method defined by 
$x_{n+1} = g(x_n), \quad g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$

is a contraction mapping in some neighborhood of $\bar{x}$. Show that the iterative sequence converges to $\bar{x}$ for any $x_0$ sufficiently close to $\bar{x}$.