1. Generalized eigenvectors
Consider a planar dynamical system,
\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 7 & -4 \\ 1 & 11 \end{pmatrix}.
\]
(a) Find a matrix \( T \) such that in transformed coordinates
\[
\begin{pmatrix} u \\ v \end{pmatrix} = T^{-1} \begin{pmatrix} x \\ y \end{pmatrix},
\]
the dynamics are transformed to
\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix},
\]
where \( J \) is the Jordan canonical form of \( A \).
(b) Find an explicit solution for the trajectory \((u(t), v(t))\) that passes through \((u_0, v_0)\) at time \( t = 0 \).

2. Lagrangian mechanics
Consider the mechanical system consisting of two “beads” on a hoop, connected by a spring. Let the beads have equal mass \( m \), let the radius of the hoop be \( R \), let the spring constant be \( k \), and let the equilibrium length of the spring be zero. Assume that the only external force is gravity, ignore friction, and assume that the beads are somehow able to pass through each other freely.

(a) Write down the Lagrangian for this system and derive the Euler-Lagrange equations of motion.
(b) Find six fixed points of the system.

3. Invariant manifolds
Consider the planar dynamical system,
\[
\begin{align*}
\dot{x} &= y - x + (y - x)^3,
\dot{y} &= x - y + (x - y)^3, \quad (x, y) \in \mathbb{R}^2.
\end{align*}
\]
(a) Determine the stable, unstable, and center subspaces of the origin.
(b) Determine the set of all points that flow to the origin as \( t \to +\infty \).