V&V MURI Overview

Caltech, October 2008

Pablo A. Parrilo

Laboratory for Information and Decision Systems

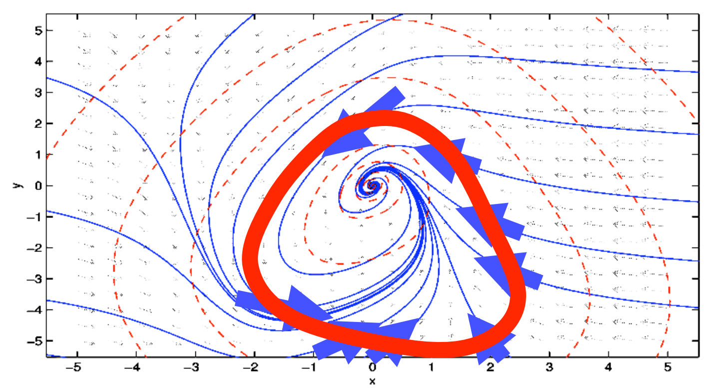
Massachusetts Institute of Technology

Goals

- Specification, design, and certification
- Coherent view and computational tools for assessment of performance and uncertainty
- Efficiency (both theoretical and practical)
- Continuous/discrete unification

$$\langle Q, \Sigma, \delta, S_0, F \rangle$$

$$\delta : Q \times \Sigma \rightarrow Q$$



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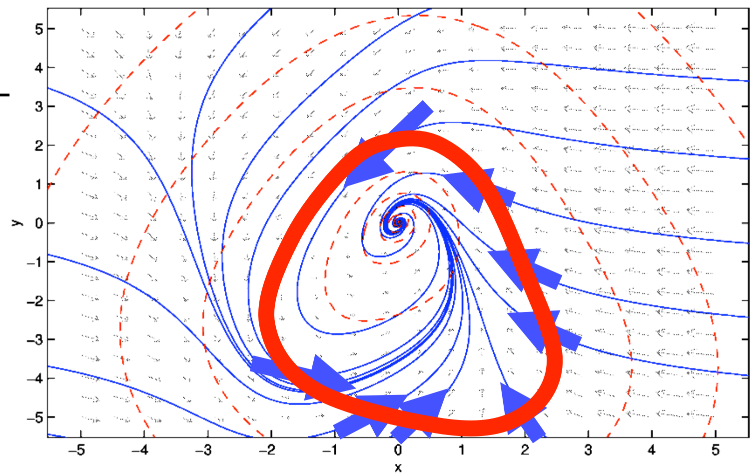


- How to reason about dynamics?
- Reduction from transitions/dynamics to propositions
  - Vector fields to inequalities via Lyapunov/dissipation: LMIs, SOS
  - Automata to satisfiability: theorem proving, bounded model checking
- Systematize and unify transition from dynamics to algebra
- Develop suitable computational techniques

Personnel at MIT

- Grad students
  - Amir Ali Ahmadi
  - Parikshit Shah
  - Noah Stein (joint w/Prof. Asu Ozdaglar)
  - Ozan Candogan

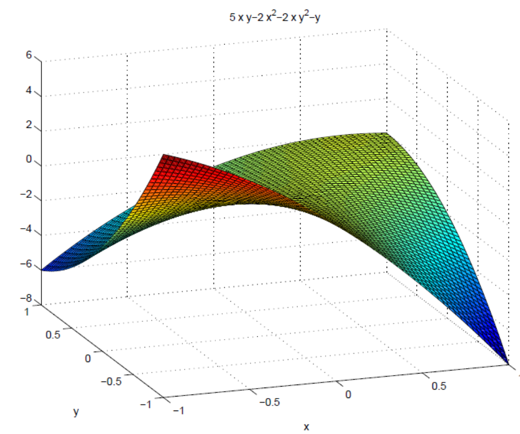
- Postdocs
  - Danielle Tarraf
    (MIT -> Caltech -> now at Johns Hopkins)

Topics

- Convex approaches to analysis, synthesis and decentralization
- Nash and correlated equilibria. Stochastic games
- Partial orders and decentralized control
- Non-monotonic Lyapunov functions
- SOS techniques and extensions

Adversaries and game theory

- Interesting per se, but also necessary to address robustness
- SOS techniques not just for optimization, but also for games

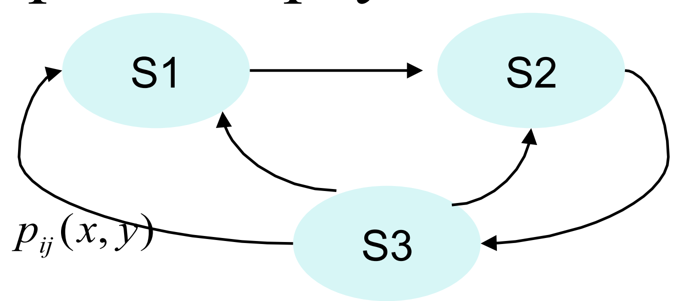


- Earlier results for semialgebraic games:
  - Two-player, zero-sum, polynomial payoffs
  - Optimal strategies and payoff computed via SOS
  - Extends (with changes) to multiplayer setting
- We can extend to *stochastic games*

N. Stein, A. Ozdaglar, P. Parrilo, "Separable and low-rank continuous games", Int. Journal of Game Theory, 2008.

Zero-sum stochastic continuous games

- Two competing players, state-dependent payoffs
- Discounted, infinite game



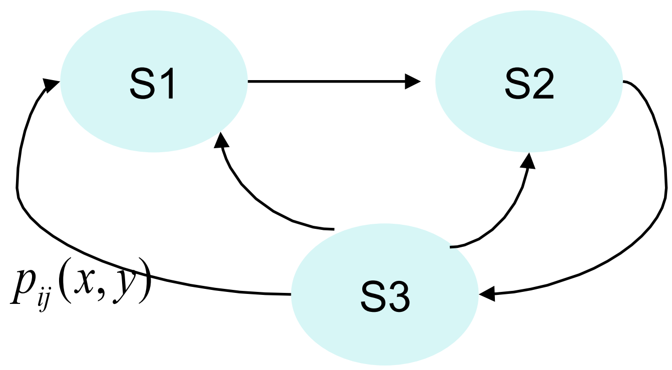
- Generalizes Markov Decision Processes (MDPs)
- Finite number of states, continuous actions
- Control action affects both immediate payoff and transition probabilities.
- Find Shapley value and optimal strategies

P. Shah and P.A. Parrilo "Polynomial stochastic games via sum of squares optimization," IEEE TAC, submitted.

Stochastic continuous games

- single controller assumption yields convexity
- exploit explicit description of moment spaces
- convex optimization – SOS and SDP
- extend techniques from the static case

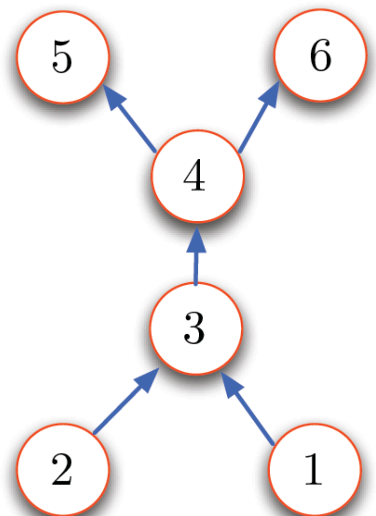
$q_i(x, y)$, $p_{ij}(x, y)$



P. Shah and P.A. Parrilo "Polynomial stochastic games via sum of squares optimization," CDC2007. IEEE TAC, submitted.

Partial orders and decentralized control

- What is a suitable mathematical language and tools to reason about *information flow*?
- Refined notions of causality:
  non-determinism, branching time,
  concurrency, n-D, etc.
- Abstract away continuous/discrete distinction
- What decision-making structures make analysis and synthesis possible?



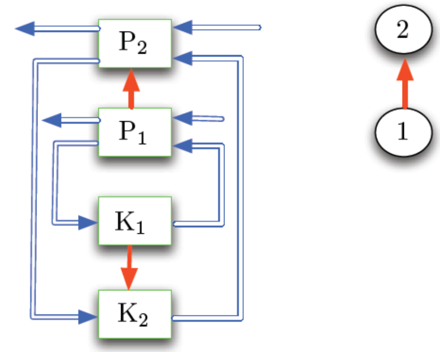
- **Propose**: partially ordered sets (posets), incidence algebras, and Galois connections

P. Shah and P.A. Parrilo "A Partial Order Approach to Decentralized Control," CDC2008.

Posets and incidence algebras

Definition 1. *A poset $\mathcal{P} = (P, \preceq)$ is a set P along with a binary relation $\preceq$ which satisfies for all $a, b, c \in P$:*

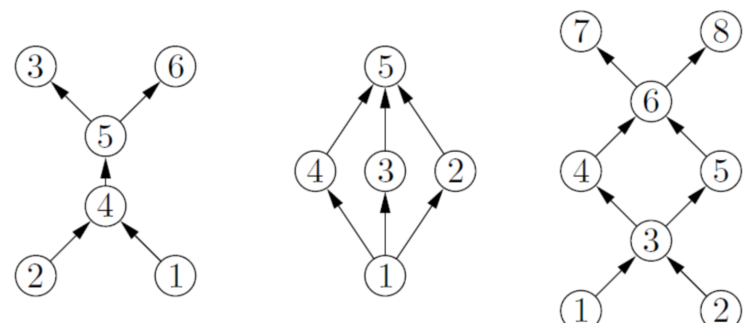
1. *$a \preceq a$ (reflexivity)*
2. *$a \preceq b$ and $b \preceq a$ implies $a = b$ (antisymmetry)*
3. *$a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity).*



Definition 2. *The set of functions $f : \mathcal{P} \times \mathcal{P} \to \mathbb{Q}$ with the property that $f(x, y) = 0$ whenever $x \not\preceq y$ is called the incidence algebra $\mathcal{I}$.*

Posets and incidence algebras

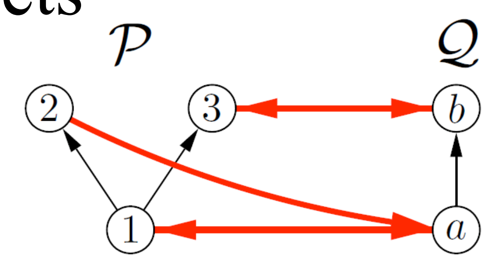
- Posets can be used to model the spatial and/or temporal dependence among subsystems
- Incidence algebras describe order-preserving maps (e.g., for linearly ordered sets, lower triangular matrices)



(a) (b) (c)

- *Galois connections* can be used to describe order-preserving maps between *different* posets

$$\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$



Results:

- Unifies most of previous formulations (e.g., partially nested control)
- Poset framework automatically yields formulations that are *quadratically invariant*
- Thus, amenable to *convex optimization*
- Coordinate-free interpretation, via structural matrix algebras and the associated lattice of invariant subspaces
- Galois connections provide a natural way of modeling communications-constrained control

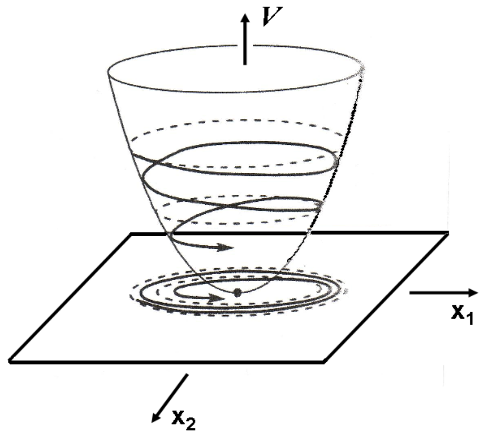
P. Shah and P.A. Parrilo "A Partial Order Approach to Decentralized Control," CDC2008.

Non-monotonic Lyapunov functions

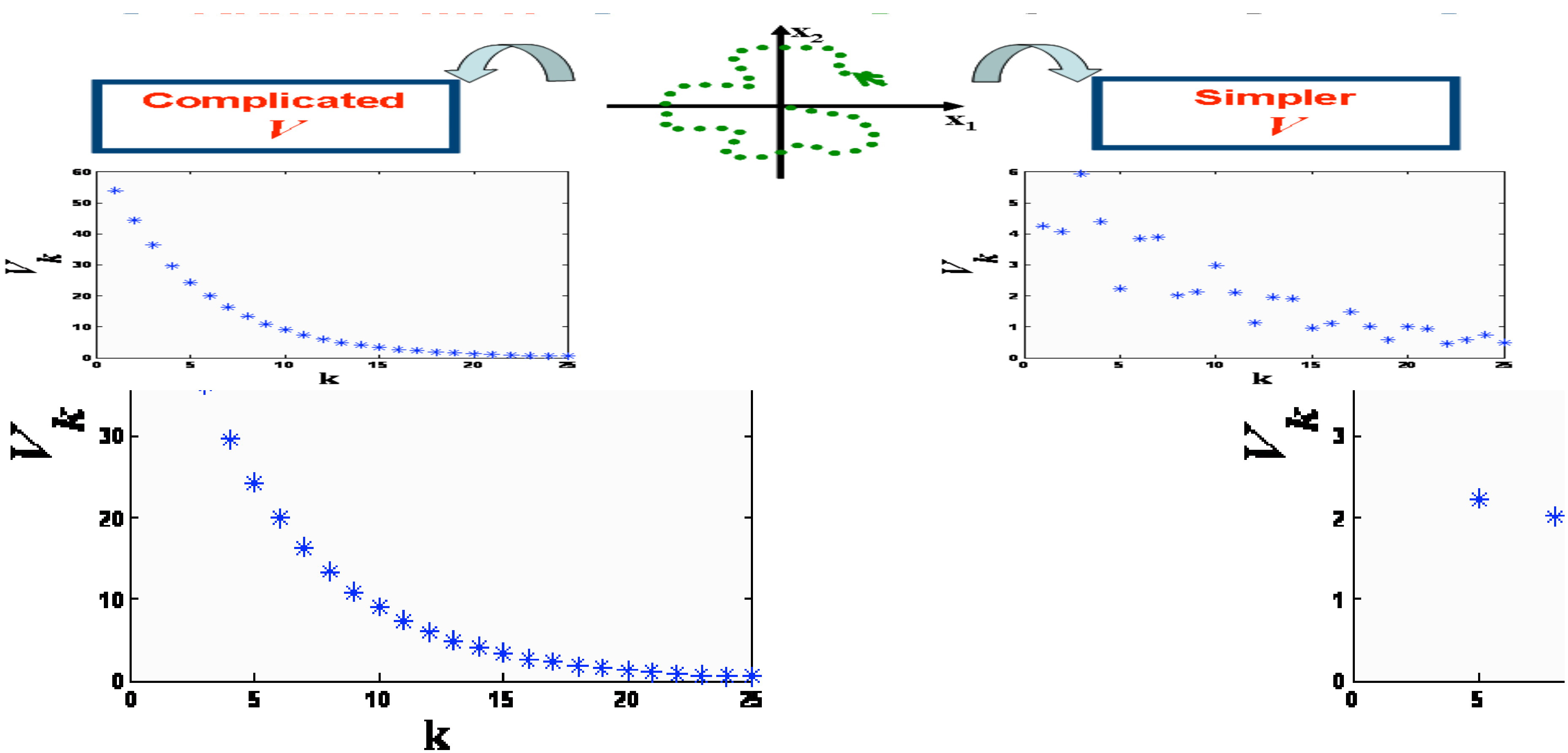
Lyapunov's direct method plays a central role in the analysis and control of dynamical systems

- Proving stability
- Synthesis via control Lyapunov functions
- Performance (e.g., rate of convergence analysis)
- Robustness and uncertainty

Why require a monotonic decrease?



A. A. Ahmadi and P.A. Parrilo "Non-monotonic Lyapunov Functions for Stability of Discrete Time Nonlinear and Switched Systems," CDC2008.



- Simpler Lyapunov functions (e.g. polynomials of lower degree) can decrease in a non-monotonic fashion along trajectories

If you can find $V^1, V^2 : \Re^n \to \Re$

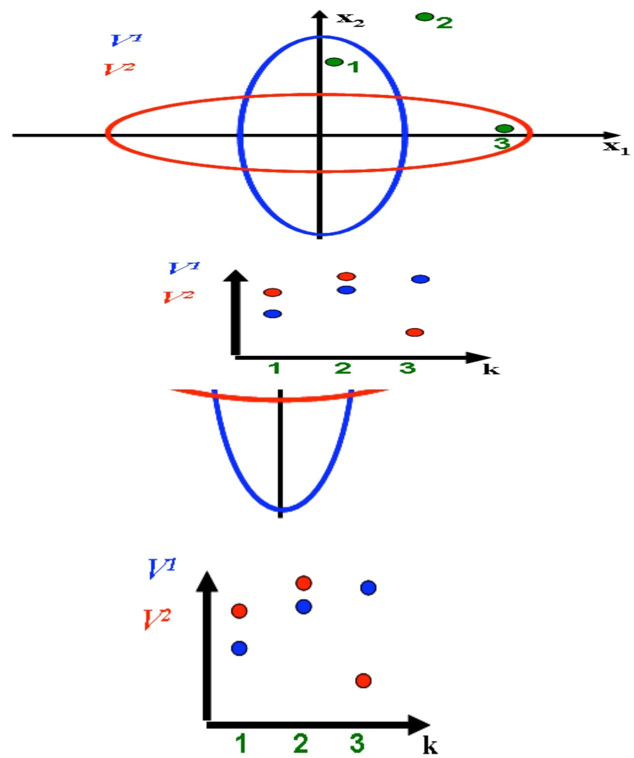
$$V^2 > 0,\ V^1 + V^2 > 0,\ V^1(0) + 2V^2(0) = 0,$$

such that

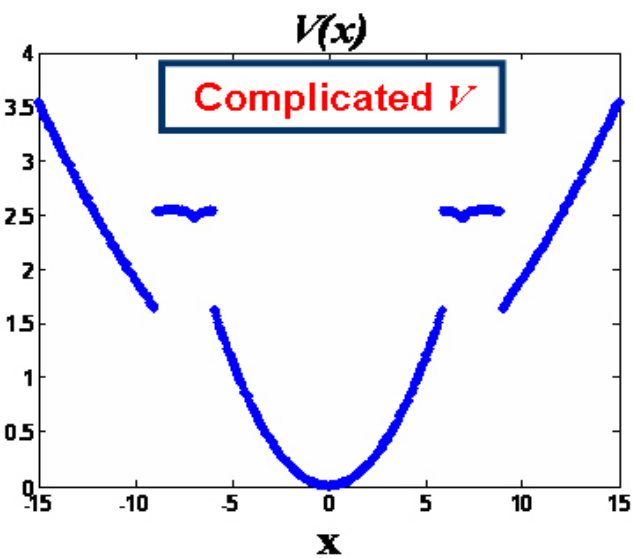
$$(V^2{}_{k+2} - V^2{}_k) + (V^1{}_{k+1} - V^1{}_k) < 0\ ,$$

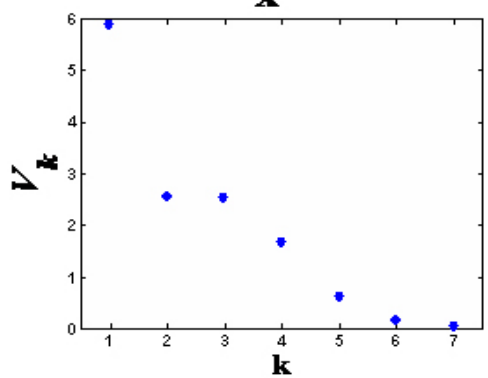
then $V^1 \to 0$, $V^2 \to 0$, which implies $x \to 0$.

- State space mapped to more than one Lyapunov function
- Improvements in different steps measured according different functions
- Convex parametrization, can use SOS to search for candidate functions
- Generally “simpler” (e.g., lower degree) than if monotonicity is required



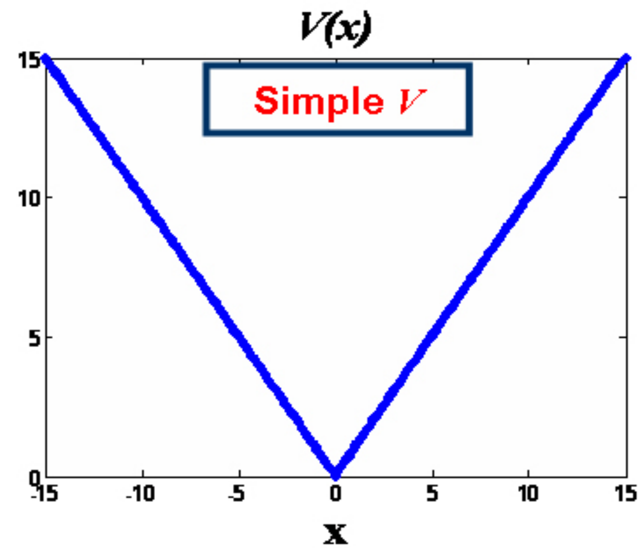
Standard Lyapunov fn.

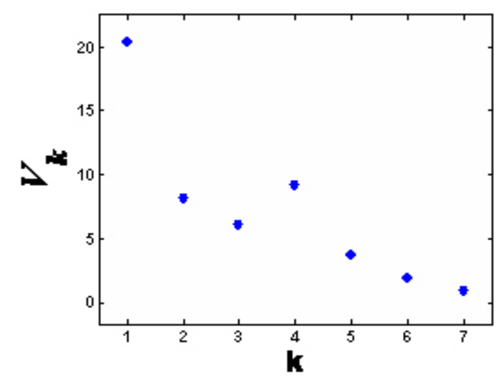




$V_{k+1} < V_k$

Non-monotonic Lyapunov fn.





$\tau(V_{k+2} - V_k) + (V_{k+1} - V_k) < 0 \quad \tau = 1.125$

A. A. Ahmadi and P.A. Parrilo "Non-monotonic Lyapunov Functions for Stability of Discrete Time Nonlinear and Switched Systems," CDC2008.

Related progress

- Guaranteed bounds on joint spectral radius via SOS (w/Ali Jadbabaie, UPenn)

- Code for SDP relaxations QP + Branch/Bound
  - Parallel, runs under MPI
  - Fully portable code (uses CSDP solver)
  - Written by Sha Hu (S.M. student)

- Ongoing work: SOS on lattices and semigroups (w/Rekha Thomas, UW)
  Characterization of "theta bodies" of polynomial ideals (arXiv:0809.3480)

Related outside developments

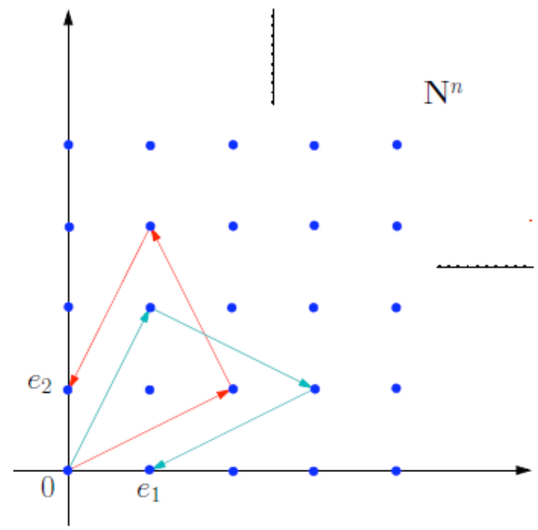
- Incorporation of SOS methods in HOL Light theorem prover (`hol.sosa`, John Harrison, Intel)
- Ongoing collaboration with Henry Cohn (Microsoft Research) on computation of bounds on density of lattice packings via SOS methods
- Sum of squares package for Macaulay 2 (`SOS.m2`), a software for commutative algebra and algebraic geometry (H. Peyrl, ETH Zurich)

Where things are going

- Dynamics on string and graph grammars
- Sparsity and proofs (L1 and nuclear norms), connections to compressed sensing
- Structure, structure, structure: graphical models + BDDs
- Rewrite and extend SOSTOOLS. Python-based? Interface w/CVX?

Relaxations for reachability and word problems

- Goal: efficient tests
  - Can we transition between two states, using only moves from a given finite set? (word problem for finite semi-Thue systems, generally undecidable)

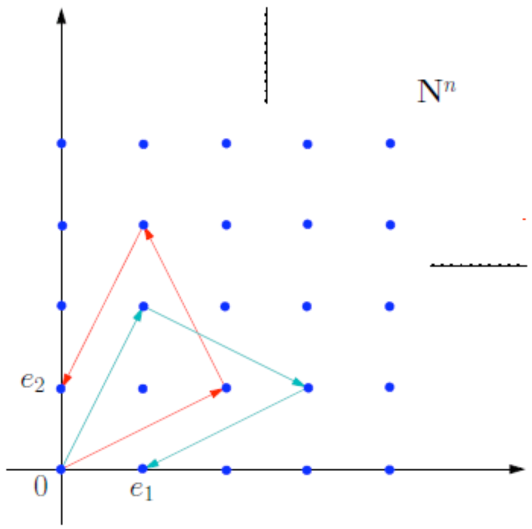


  - Direct applications to graph grammars, infinite graph reachability, Petri nets, etc.
  - What are the obstructions to reachability?

D. Tarraf and P.A. Parrilo "Commutative relaxations of word problems," CDC2007

Reachability and word problems

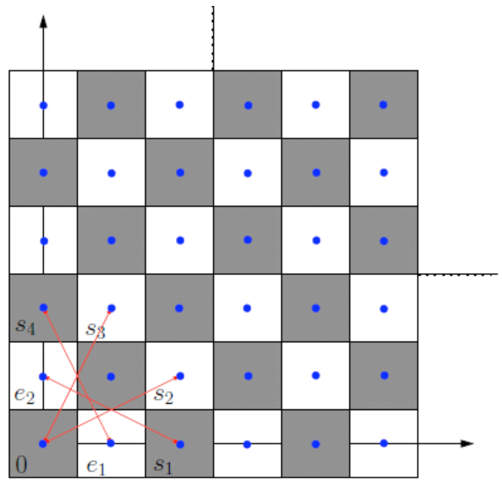
- String grammars: finite alphabet and production rules
- Relaxations: commutative and/or symmetric versions
- Algebraic reformulation in terms of ideal membership and nonnegativity (cf. Mayr-Meyer)
- Convexity enables duality-based considerations



D. Tarraf and P.A. Parrilo "Commutative relaxations of word problems," CDC2007

Reachability and word problems

- Results:
  - Characterization in terms of polynomial identities and nonnegativity constraints
  - Yields a hierarchy of linear programming (LP) conditions
  - Zero-to-all reachability equivalent to finitely many point-to-point problems
  - Progress towards higher-order relaxations, that do not rely on commutativity assumptions



Related resources

- Papers, tutorials, etc.
  - www.mit.edu/~parrilo
  - www.hot.caltech.edu/math.html
- Software: SOSTOOLS
  - www.mit.edu/~parrilo/sostools

Relaxations for reachability and word problems

Parrilo, Tarraf (MIT)

Goal: efficient tests

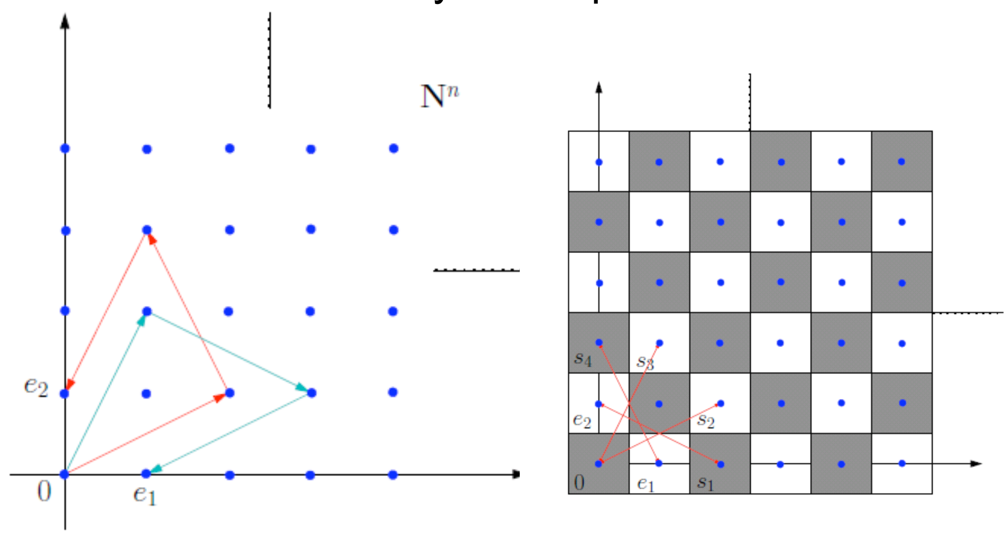
- Can we transition between two states, using only moves from a given finite set? (word problem for finite semi-Thue systems, generally undecidable)
- Direct applications to graph grammars, infinite graph reachability, Petri nets, etc.
- What are the obstructions to reachability?

Approach: symbolic-numeric

- Relaxations: commutative and/or symmetric versions
- Algebraic reformulation in terms of ideal membership and nonnegativity
- Convexity enables duality-based considerations

Results to date

- Characterization in terms of polynomial identities and nonnegativity constraints
- Yields a hierarchy of linear programming (LP) conditions
- Zero-to-all reachability equivalent to finitely many point-to-point problems
- Progress towards higher-order relaxations, that do not rely on commutativity assumptions



D. Tarraf and P.A. Parrilo "Commutative relaxations of word problems," submitted to CDC2007.

Caltech-Murray, FA9550-06-1-0303

Analysis via Non-monotonic Lyapunov Functions

Ahmadi, Parrilo (MIT)

Goal: stability and performance

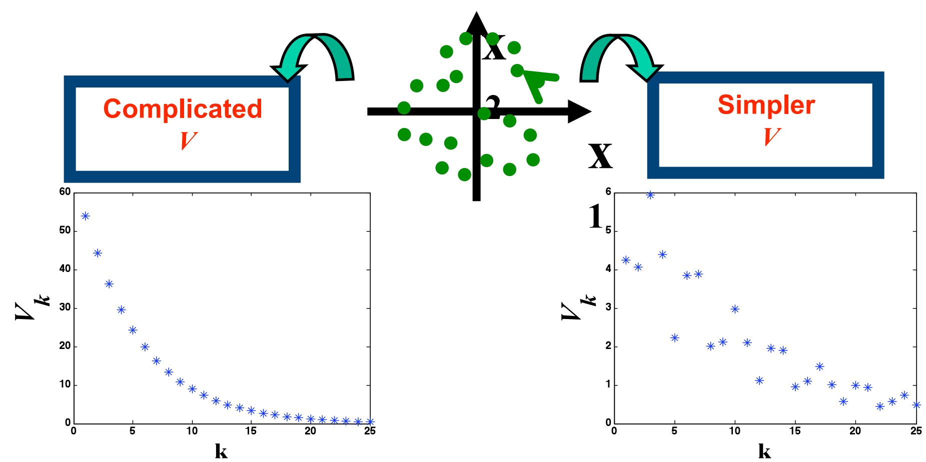
- Traditional Lyapunov-based analysis relies on monotone invariants (e.g., energy)
- This often forces descriptions requiring high algebraic complexity
- Is it possible to relax the monotonicity assumption?

Approach: convexity-based

- Require nonnegativity of linear combinations of time derivatives
- Algebraic reformulation in terms of polynomial nonnegativity
- Yields tractable conditions, verifiable by convex optimization

Results to date

- Convexity-based conditions, checkable by SOS/semidefinite programming
- Easy to apply, more powerful than standard conditions
- Connections with other techniques (e.g., vector Lyapunov functions)
- Many extensions to discrete /continuous/hybrid/switched, etc.



A. A. Ahmadi and P.A. Parrilo "Non-monotonic Lyapunov Functions for Stability of Discrete Time Nonlinear and Switched Systems," to appear, CDC2008.

Caltech-Murray, FA9550-06-1-0303

Partial orders and decentralized control

Shah, Parrilo, (MIT)

Goal: understand information flow

- A new framework to reason about information flow in terms of partially ordered sets (posets).
- What are the structures amenable to decentralized control design?

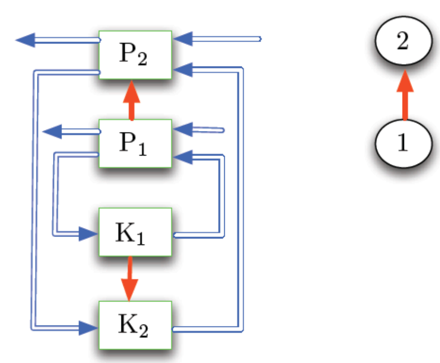
Approach: incidence algebras

- Posets and incidence algebras
- Abstract flow of information, generalize notions of causality
- Yields convexity of the underlying control problems. Relations with quadratic invariance.

$$\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Results to date

- Generalizes sequential and partially nested structures (e.g., leader -follower)
- Convex characterization of poset -preserving controllers, via Youla
- Captures the right level of abstraction, rich algebraic and combinatorial tools
- Extensions to more complicated situations, via Galois connections



P. Shah and P.A. Parrilo "A partial order approach to decentralized control," to appear, CDC2008.

Caltech-Murray, FA9550-06-1-0303