On Equilibria of Distributed Message-Passing Games*

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Abstract. This paper investigates a general class of finite dynamic games where players communicate with each other by message-passing. Messages may be delayed, received out of order, duplicated or lost, but not corrupted. The challenge posed by message-passing games is that the execution of the game is non-deterministic and, thus, the final outcome may not result in an equilibrium. We introduce a notion of termination for message-passing games which is related to self-stabilization. We present theorems relating their termination to equilibria of their corresponding alternating move games. Our results apply to both best-response games in which players always choose the best action in response to other players actions, and better-response games where players may choose any action that improves their current response.

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1 Introduction

Non-cooperative game theory is a natural framework for modeling problems in distributed systems. It has been successfully applied in many contexts, such as congestion and power control in networks [1, 2] and distributed control [3]. These problems may be formulated in a game theoretical framework, where players correspond to the system components which compete for some set of resources. The solution of the problem turns out to be a Nash equilibrium [17], i.e. a configuration of the game from which each player has no incentive to deviate. Research in this area has primarily focused on designing utility functions for the players which guarantee termination of the game at a Nash equilibria ([3, 4, 11]). In many of these formulations, for example in [11], players alternate their actions in a round-robin fashion and the game is repeated infinitely many times. These games are referred in the literature as dynamic games with alternating moves [18].

In some practical contexts, the agents of a system may act asynchronously and communicate via message-passing [6]. Messages may be lost, delayed or delivered out of order. We call these games *dynamic message-passing games*. When a game is played via message-passing, the final outcome may not be a Nash equilibrium, even if the corresponding game with alternating moves has a unique Nash equilibrium which is reached from all states. Notions of games for modeling distributed systems have been recently proposed [14, 16]. These models allow for do not capture an active behavior of the message-passing communication medium.

In this paper, we define message-passing games, where the communication medium is modeled explicitly. The message-passing communication medium used in this paper has been proposed in [10] and investigated further in [20]. The medium may drop, duplicate or delay messages in transit, but cannot block for ever the communication between any two players. We introduce the notion of termination for message-passing games. A game terminates if eventually-always the configuration of the game (including messages in transit) belongs to the set of its Nash equilibria. Our goal is to relate termination of message-passing games to termination of the corresponding games with alternating moves. This is highly relevant, since many works, e.g. [4, 11], have related termination of games with alternating moves to their structure. Therefore, using our results, it would be possible to directly relate termination of message-passing games to the structure of the game.

We focus our attention on finite best-response games, where players always choose the set of actions which maximize their payoffs. This is a very natural strategy for players in finite games since their objective is to terminate in a Nash equilibrium [5, 13]. The main result of the paper shows that the two notions of terminations are equivalent only for games with unique best response. This means that for such class, the game via message-passing terminates if and only if the corresponding game with alternating moves terminates. Differently from the game with alternating moves, in message-passing games players choose their actions based on out-of-date actions of their opponents, possibly executed at different stages. We prove that if each player always best-responds to his opponents, then the game terminates, regardless the actions of the communication medium and the number of equilibria of the game. We show that this equivalence does not hold for more general best-response games.

Termination of asynchronous games, where players take their actions asynchronously, has been widely investigated [3, 4, 19]. However, these works do not model the communication medium as a component of the game. For example, in [19] they do not allow for delays in the transmission and for out-of-order messages. The authors in [3, 4] study termination of games where there is a unique Nash equilibrium and find sufficient conditions on utility functions which ensure convergence to it. Our concept of termination is more flexible because it allows for termination in presence of multiple equilibria of the game.

Our notion of termination relates to self-stabilization [7, 8]. Whatever the initial configuration is, the game eventually-always self-stabilizes in the set of Nash equilibria. Hence, the set of Nash equilibria corresponds to the set of correct states of a self-stabilizing system. Our results prove that self-stabilization is guaranteed only in presence of unique best response games. We refer to [9, 12] for a complete treatment on self-stabilization.

The rest of the paper is organized as follows. Section 2 discusses basic concepts of game theory and introduces finite dynamic best-response games with alternating moves. Section 3 defines dynamic message-passing games along with their communication model and notion of termination. Section 4 relates Nash equilibria of best-response games with alternating moves to termination of message-passing games for the case of unique best response. Section 5 discusses generalizations and applications of our results. Section 6 concludes the paper.

2 Games with Alternating Moves

In this section, we define finite and dynamic games with alternating moves with special attention on best-response games. In all these games, the actions sets of the players are finite and the game is repeated infinitely many times.

2.1 Basic Game Theoretical Concepts

We review some basic concepts of game theory and refer the reader to [18] for a more detailed treatment. We next define the concept of game in normal form.

Definition 1. *A game in normal form G is a triple (N, A, u) consisting of (1) $N := \{1, 2, \ldots n\}$ the set of players; (2) $A := \prod_{i \in N} A_i$ the set of global actions, where A_i is the set of player i's actions; (3) $u := (u_1, u_2, \ldots, u_n)$ the set of players payoffs where $u_i : A \to \mathbb{R}$ is the payoff (or utility) function of player i.*

Elements of A are also called *action profiles*. We denote by i, j, k arbitrary players. Given $a \in A$, a_i denotes the action of player i in a and a_{-i} the actions of his opponents, i.e. $(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. Under this notation, we may write $a = (a_i, a_{-i})$. We denote $A_{-i} = \prod_{j \neq i} A_j$. The power set of A_i is denoted by

$\mathcal{P}(A_i)$. We abbreviate $G = (N, A, u)$ with G throughout the paper. An example of a normal form game is shown in Figure 1(a).

Definition 2. *A game G is (a) finite, if N is finite and A_i is finite $\forall i$, (b) dynamic, if players repeat the game infinitely many times, (c) with alternating moves if players alternates in taking their moves.*

The steady-state configurations of a game G, i.e. configurations in which no player has the incentive to improve his payoff, are called Nash equilibria.

Definition 3. *$a^* \in A$ is a Nash equilibrium of G if $\forall i : u_i(a^*) \geq u_i(b, a^*_{-i}) \forall b \in A_i$*

We denote by A^* the set of Nash equilibria of G and by A^*_i the set of the actions of player i which occur in a Nash equilibria, i.e. $A^*_i = \{b \in A_i : \exists a^* \in A^*, a^*_i = b\}$.

2.2 Best-Response Games

Best-Response Games. Each player can take only actions which maximize his payoff. Each player i has a function $\beta_i : A_{-i} \to \mathcal{P}(A_i)$. For each $a_{-i} \in A_{-i}$, the function β_i returns the set of actions which maximize the utility of player i.

Definition 4. *A best-response game $G_\beta = (N, A, u, \beta)$ is a normal form game where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and $\forall i$, $\beta_i : A_{-i} \to \mathcal{P}(A_i)$, such that $\forall a_{-i}$ $\beta_i(a_{-i}) = \{b \in A_i \ : \ u_i(b, a_{-i}) \geq u_i(b', a_{-i}) \ \forall b' \in A_i\}$*

The function β_i is called the *best response function* of player i. If $\forall i$, $\forall a_{-i} \in A_{-i}$, $|\beta_i(a_{-i})| = 1$, the game is a best-response game with *unique best response*. We refer to it as *unique best-response game* throughout the paper. We denote it by $G_{\beta U}$; if $N = 2$, we denote it by $G_{\beta U2}$. An example of best response function for the game in Figure 1(a) is shown in Figure 1(b).

Graphical Representation. Best-response games may be pictorially represented as directed graphs. Given G_β, the corresponding best-response graph $Gr_\beta = (V, E)$ is a directed graph where the vertices are the actions of G and the set of edges E is constructed using the set of best response actions.

Definition 5. *$Gr_\beta = (V, E)$ is a best-response graph of G_β if (1) $V = A$ and (2) $\forall i, a_{-i} \in A_{-i}, b, b' \in A_i$, $((b', a_{-i}), (b, a_{-i})) \in E$ if $b' \notin \beta_i(a_{-i})$, $b \in \beta_i(a_{-i})$.*

We refer to the pair (b, a_{-i}) as a *best-response vertex.* Notice that a best-response vertex for player i contains only incoming edges from non best-response vertices associated to player i. Examples of best-response graphs are shown in Figure 2(a) and 2(b). The graphs corresponding to $G_{\beta U2}, G_{\beta U}$ are denoted by $Gr_{\beta U2}, Gr_{\beta U}$.

By construction, Nash equilibria of G_β are sink vertices of Gr_β. Given $v \in V$, we denote by v_i its i-th component and by v_{-i} all of its components except for component i. We next define the concept of projection of an action.

Definition 6. *The projection of $a_i \in A_i$ in Gr is $V_{a_i} = \{(a_i, v_{-i}) : v_{-i} \in A_{-i}\}$*

This is the set of vertices of V where player i plays a_i while his opponents can choose any feasible action in their action set. Similarly, we can define the *projection* of $a \in A$, as $V_a = \cup_{i \in N} V_{a_i}$.

We next recall the definition of vertex degree in a directed graph. Given a vertex v, we denote by $d(v)$ the in-degree of v, i.e. the number of edges incoming into v. We define the in-degree of a set of vertices V', denoted by $d(V')$, as the total number of edges directed from vertices outside V' to vertices inside V', i.e. $d(V') = |\{(v,w) \in E : v \in V - V', w \in V'\}|$.

2.3 Global states, and Terminating Executions

In this subsection, we define the notion of terminating executions (or trajectories) of a finite dynamic game G with alternating moves. We first introduce the *state* s of G as an array of length N whose i-th component, denoted by s_i, is the last action of player i. Initially, s_i is arbitrary. We denote by s_{-i} the array of length $N-1$ consisting of all components of s expect for i.

An *execution* (or *trajectory*) of G is an infinite sequence of states and actions $(s^0, b_1, s^1, b_2 \ldots)$, where (1) s^0 is an arbitrary initial state, (2) $\forall i$, $b_i \in A_j$ for some player j, (3) $s^i_k = s^{i-1}_k$ for all $k \neq j$, $s^i_j = b_i$. In case when $G = G_\beta$ then $b_i \in \beta_j(s^i_{-j})$.

An execution *terminates* if it reaches (and remain) in a Nash equilibrium. A game terminates if all its executions terminate.

Definition 7. *G terminates if* $\diamond \Box\, s \in A^*$.

Let G_β be a best-response game. We next give a termination condition in term of its corresponding graph Gr_β.

Lemma 1. *G_β terminates if and only if Gr_β is acyclic.*

Proof. The proof follows by contradiction. $\Box$

3 Message-Passing Games

In this section, we introduce message-passing games where players communicate by exchanging messages. Messages may be potentially lost, duplicated or arrive out-of-order.

Definition 8. *A message-passing game $G' = (N, A, u, \mathcal{C})$ is a normal form game with communication medium $\mathcal{C}$.*

We denote by G the corresponding game with alternating moves, i.e. the game having the same triple (N, A, u), but without the communication medium. Similarly, given the best-response games $G_{\beta U}, G_{\beta U2}, G_\beta$ we denote by $G'_{\beta U}$, $G'_{\beta U2}$, G'_β the corresponding message-passing games having the same tuple (N, A, u, β). Since Nash equilibria are related to the structure of the game, rather than to the communication mechanism used in playing the game, we would have that G

and G' have the same Nash equilibrium action profiles and the same graphical representation.

We next describe $\mathcal{C}$ and the notion of terminating executions of G'.

Communication Medium. $\mathcal{C}$ is a message-passing broadcast channel. It has four primitives: *send*, *receive*, *drop* and *duplicate*. The *send* and *receive* primitives are executed by players, while the *drop* and *duplicate* primitives are executed by $\mathcal{C}$. When player i executes a $send(m)$, where m is a message, m is stored in $\mathcal{C}$ and broadcasted to all his opponents. When i executes a $receive(m)$, m is removed from $\mathcal{C}$. The primitive $drop(m)$ removes m from $\mathcal{C}$, while the primitive $duplicate(m)$ duplicates m on $\mathcal{C}$. Denote by $\sharp(m)$ the total number of copies of m. We make the following assumptions on the behavior of $\mathcal{C}$.

Assumption 1. *$\mathcal{C}$ has the following properties (A1.) $\diamond\ m \notin \mathcal{C}$; (A2.) $\sharp(m)$ is bounded; (A3.) $\forall i, j$, i receives a message from j infinitely often.*

Assumption (A1.) means that messages are eventually dropped or received; Assumption (A2.) means that the total number of copies of a message is finite; Assumption (A3.) excludes the possibility that all messages between two players are dropped, i.e. it is always the case that eventually some message from i is delivered to j.

In the initial set-up of the game, $\mathcal{C}$ may be not empty, i.e. there can be an arbitrary, but finite, number of messages in transit.

States. The *state* s of G' is the composition of the state ς of its players and the state of $\mathcal{C}$. The state of $\mathcal{C}$ is the set of messages in transit on $\mathcal{C}$. The state ς_i of player i is an action profile in A, whose i-th component equals to the current action of i and whose k-th component, $k \neq i$, is equal to the last received action for player k. Initially, ς_i is arbitrary.

Communication Protocol. Players play the message-passing game G' via $\mathcal{C}$. They cannot see each other. Player i executes the *send* primitive infinitely often; however, the number of messages sent within a finite time interval is finite. A message m is a pair (i, ς_i) where i is the sender of m and ς_i is the state of player i. Player i chooses a move based on the last actions received from his opponents. We assume that i takes an action in reply to any received message. When player i receives a message (j, ς_j) from player j, he updates his state as follows. For all $k \neq i$, he sets the k-th component of ς_i to the k-th component of ς_j. The i-th component of ς_i remains unchanged.

Terminating Executions. The set of actions associated with player i in s, denoted by $s_{(i)}$, contains: (1) the action stored in the i-th component of ς_j, $\forall j$ (2) any action b contained in a message (k, a) in transit on $\mathcal{C}$ having $a_i = b$ for any player k. An *execution* (or trajectory) of G' is an infinite sequence $(s^0, c_1, s^1, c_2, \ldots)$ where s^0 is an arbitrary initial state of G' and $c_i \in \{send, receive, drop, duplicate\}$. We define an execution of G' terminating if it loops among its Nash equilibrium points. Hence,

Definition 9. *G' terminates if $\diamond \square\ \forall i\ s_{(i)} \subseteq A_i^*$.*

The game G' terminates if for all its executions (starting in any arbitrary initial state and satisfying Assumption 1), eventually all actions of player i ($\forall i$) belong to the set of Nash equilibria. This notion is equivalent to the notion of termination of games with alternating moves. It is also equivalent to the notion of self-stabilization.

4 Terminating Message-Passing Games

In this section, we relate termination in best-response games with alternating moves to termination in the corresponding message-passing game. This is true only for games with unique best response.

Theorem 1. *$G'_{\beta U}$ terminates if and only if $G_{\beta U}$ terminates.*

Although the theorem holds for any number of players, we report the proof only for the case of two players in order to convey the main ideas without obscuring them with many details. The proof for a number $N > 2$ players follow using the same idea.

Theorem 2. *$G'_{\beta U2}$ terminates if and only if $G_{\beta U2}$ terminates.*

Before proving the theorem, we show a key property of $Gr_{\beta U2}$. To this purpose, we define the following two predicates:

$$\mathcal{N}(Gr_{\beta U2}) = \langle\ \exists i,\ f \in A_i\ :\ V_f \cap A^* = \emptyset\ \rangle$$
$$\mathcal{Z}(Gr_{\beta U2}) = \langle\ \exists j,\ b \in A_j\ :\ d(V_b) = 0\ \rangle$$

The first predicate is true if there exists some action f whose projection V_f does not contain Nash equilibria. The second predicate is true if there exists some action b whose projection V_b has no incoming edges from vertices outside the projection. By construction, V_b cannot contain Nash equilibria. The following property holds.

Lemma 2. *If $Gr_{\beta U2}$ is acyclic and $\mathcal{N}(Gr_{\beta U2})$ holds. Then $\mathcal{Z}(Gr_{\beta U2})$ holds.*

Proof. The proof follows by contradiction. Assume that $\mathcal{Z}(Gr_{\beta U2})$ does not hold, i.e. $\forall i, \forall b \in A_i$, $d(V_b) > 0$.

The key idea is to construct a sequence of projections $S_P = (P_1, P_2 \ldots P_l)$ such that $\forall j \in \{1, \ldots l\}$ (i.) there is a directed edge from a vertex in P_j to a vertex in P_{j-1}, (ii.) $P_j \cap A^* = \emptyset$ (iii.) $P_j \neq P_r$, for all $r \neq j$. We will show that P_l, i.e. the last projection of the sequence, has no incoming edges, which contradicts the hypothesis.

Construction of S_P. Set $P_1 = P_o$, where $o \in A_i$ for some player i satisfying $V_o \cap A^* = \emptyset$. Such an action o exists since $\mathcal{N}(Gr_{\beta U2})$ holds.
Consider an arbitrary P_{j-1}, and assume that $P_{j-1} \cap A^* = \emptyset$. Then $P_{j-1} = V_f$ for some action $f \in A_i$, for some i. Denote by w the unique best-response vertex in P_{j-1}. By assumption (of contradiction), there exists $z \in P_{j-1}$ with an incoming

edge from some vertex outside P_{j-1}; by construction $z \neq w$, since $P_{j-1} \cap A^* = \emptyset$. The vertex z has $z_i = f$ and $z_k = g$, $k \neq i$, for some $g \in A_k$. Set $P_j = P_g$, where (z, w) is the edge directed from P_j to P_{j-1}. P_j cannot contain Nash equilibrium vertices. This is because z, which is the unique best-response vertex of P_j, has an outgoing edge to w.

Claim 1. $P_j \cap A^* = \emptyset$ for all j.
Proof. It follows from the construction of the projection sequence. It is initially true from the hypothesis, and it is maintained by each newly constructed projection.

Claim 2. $P_j \neq P_r$ for all $r \neq j$.
Proof. By contradiction, assume that $\exists r$ with $P_j = P_r$. Wlog, assume that P_j precedes P_r in S_P. By construction, there exists a path from a vertex $v \in V_r$ to a vertex $t \in V_j$ where each edge along the path connects two projections in the sequence $P_{r-1} \dots P_{j+1}$. If $t = v$, then the path is a cycle, thus contradicting the assumption of the lemma. If instead $t \neq v$, then v must be the unique best-response vertex in P_r; hence, there exists the edge $(t, v) \in Gr_{\beta U2}$. In that case, there would again be a cycle from t to v in the graph $Gr_{\beta U2}$, which contradicts the assumption of the lemma.

Claim 3. S_P is finite.
Proof. It follows from Claim 2 which establishes that all projections are different and from the fact that the graph is finite.

Claim 4. $d(P_l) = 0$.
Proof. From Claim 3, it follows that P_l exists. Moreover, from Claim 1, $P_l \cap A^* = \emptyset$. Therefore, all vertices in P_l (1) either intersect projections which are already in the sequence and thus, by Claim 2, they cannot have incoming edges otherwise a cycle would appear (2) or intersect projections which are not in the sequence. In this case they cannot have incoming edges because otherwise P_l would not have been the last projection of the sequence, so they only have outgoing edges.

Main proof. It follows from Claim 4 that $d(P_l) = 0$, which leads to a contradiction because it violates the predicate $\mathcal{Z}$. □

We briefly discuss the implications of this lemma. An action $b \in A_i$ having $d(V_b) = 0$ cannot be played by i because it is not a best-response to any action of his opponent. We next prove that eventually any message containing this action disappears from the system.

Lemma 3. *If $d(V_b) = 0$ in $Gr_{\beta U2}$, $b \in A_i$, for some i then $\diamond \Box\, m \notin \mathcal{C}$, where $m = (k, a)$ with $k \in N$, $a \in A$ and $a_i = b$.*

Proof. By assumption $d(V_b) = 0$, thus the action b is not a best response for player i to any action of his opponent. Therefore, player i can never send a message (i, a) with $a \in A$ and $a_i = b$. By assumption, in any initial state, the number of messages in transit on $\mathcal{C}$ storing b in the i-th component is finite. Moreover, the number of their copies is finite by Assumption (A2.). By Assumption (A1.)

all these messages (and their copies) will be eventually received, or lost. Hence, all messages storing b in the i-th component will eventually disappear. □

As in Lemma 1, we can relate termination conditions of best-response message-passing games to acyclic graphs.

Lemma 4. *$G'_{\beta U2}$ terminates if and only if $Gr_{\beta U2}$ is acyclic.*

Proof. ($\Rightarrow$). By setting message delays appropriately, it is possible to simulate games with alternating moves using message-passing games; hence, this direction follows from Lemma 1.

($\Leftarrow$). We can construct a sequence of games $S_G = (G_1, G_2, \ldots G_r)$ (and a sequence of corresponding graphs $S_{Gr} = (Gr_1, Gr_2, \ldots Gr_r)$) where $G_1 = G'_{\beta U2}$ and $\forall k \neq r$, $\mathcal{N}(Gr_k)$ holds, while $\mathcal{N}(Gr_r)$ does not hold. Each game G_{k+1} is obtained from G_k by removing an action $b_k \in A_i$ for some player i, such that $d(V_{b_k}) = 0$. Such an action is guaranteed to exist by Lemma 2 if $\mathcal{N}(Gr_k)$ holds. Using Lemma 3, the action b_k eventually disappears from the game. Hence, eventually always, G_k and G_{k+1} are equivalent. This construction continues until the assumptions of Lemma 2 are violated ($\neg\mathcal{N}(G_r)$), which occurs at index r. In the last game G_r, we have that $\forall i, \forall f \in A_i$, $V_f \cap A^* \neq \emptyset$; or equivalently, $\forall i, \forall f \in A_i$, $\exists a^* \in A^*$ such that $a^*_i = f$. Hence, in G_r, $s_{(i)} \subseteq A^*_i$. □

Proof of Theorem 2. Combining Lemma 1 and 4, the theorem follows. □

5 Discussion

In this section we discuss some generalizations and applications of the results derived earlier in the paper.

Non-Unique Best-Response Games. Theorem 1 holds only for unique best-response games. When the best response is not unique, we can construct message-passing games which do not terminate. This result holds for $N \geq 2$. Consider the two-player game in Figure 2(a). This game has two Nash equilibria, $A^* = \{(A, 1), (B, 3)\}$. Since the graph is acyclic, the game with alternating moves terminates in A^* by Lemma 1. As illustrated in Figure 2(d), it is not true that $\diamond \Box\, (s_{(1)} \subseteq \{A, B\} \wedge s_{(2)} \subseteq \{1, 3\})$. We have that the actions $C, 2$ are executed infinitely often.

Amount of information. The communication protocol defined in this paper assume that players send their complete state. Consider the following protocol, where player i sends the pair (i, b), where $b \in A_i$ is the i-th component of ς_i. The action b is the player best-response to the actions of his opponents stored in his state. When a player j receives the message (i, b), he updates only the i-th component of his state ς_j with b. Notice that for two-player best-response games, this communication protocol is equivalent to the one assumed in this paper. Under this new model of communication, Theorem 1 does not hold for any number of players. When $N \geq 3$, there exist unique best-response message-passing games which do not terminate, while their corresponding games with alternating moves terminate.An example is provided in Figure 2(b). This is a three-player game with unique Nash equilibrium $(1, 1, 1)$. It terminates if played with alternating moves since the graph is acyclic. However, Figure 2(e) shows an execution of the corresponding message-passing game which does not terminate. In this execution the action 2 is executed infinitely often by all players.

Better-Response Games. In better-response games, each player may take any action which improves his current payoff. Each player i stores a *better response function* $B_i : A \to \mathcal{P}(A_i)$ such that $\forall a \in A$, $B_i(a) = \{b \in A_i \ : \ u_i(b, a_{-i}) > u_i(a)\}$. An example of better response function for the game in Figure 1(a) is shown in Figure 1(c). Better-response graphs may be defined similarly. As an example, the graph corresponding to the game in Figure 1(a)- 1(c) is shown in Figure 2(c).

Lemma 1 holds for better-response games as well. However, termination of better-response message-passing games cannot be related to termination of their corresponding games with alternating moves. This result holds for $N \geq 2$. Figure 2(c) shows a two-player better-response game with unique equilibrium given by the pair $(A, 3)$, which terminates if played with alternating moves. The game does not terminate if played via message-passing, as it can be seen from the trajectory in Figure 2(f) which executes actions $B, 1, 2$ infinitely often.

Potential Games. Best-response and better-response games are special examples of *potential* games. Potential games, introduced in [15], define a global utility of the system called potential function. The potential function is such that any action taken by the player gives the same improvement on the global utility as it does on the player's utility. Formally, G is a potential game if $\exists P : A \to \mathbb{R}$ such that $\forall i, \forall a_{-i} \in A_{-i}, \forall b, c \in A_i : u_i(b, a_{-i}) - u_i(c, a_{-i}) = P(b, a_{-i}) - P(c, a_{-i})$. The function P is called potential function. In [15, 21] the authors characterize sufficient and necessary conditions of terminating potential games with alternating moves. Under the message-passing communication model, it is possible to exhibit examples of potential games which do not terminate, while they reach a Nash equilibrium (and therefore terminate) if played with alternating moves.

6 Conclusions

In this paper, we have introduced finite dynamic message-passing games. These games relax the perfect communication assumption of finite dynamic games with alternating moves, and allow players to (1) take turns in a non-deterministic fashion (2) choose their actions based on out-of-date actions of their opponents, possibly executed at different stages. Messages are sent over a non-reliable communication medium, thus they may be lost, delayed, reordered or duplicated. We have introduced a notion of termination for these games, which generalizes the notion of termination in games with alternating moves. We have related this notion to self-stabilization. We have shown that the two notions of terminations are equivalent only in the case of unique best-response games. Using the graph structure of the game, we have proved that the graph must be acyclic to guarantee termination. In all other cases, we have provided examples of games which terminate if played with alternating moves, but do not terminate via message-passing. We have finally discussed the applicability of our results to the class of better-response and potential games.

In the future, we would like to further investigate conditions which guarantee termination of message passing games in more general contexts. We would also like to consider non-finite games, where the action sets of the player may be infinite.

		Player 2		
		1	2	3
	A	0, 0	3, 3	7, 7
Player 1	B	2, 2	0, 0	6, 6
	C	3, 3	4, 4	5, 5

(a)

$\beta_1(1) = \beta_1(2) = \{C\}$
$\beta_1(3) = \{A\}$
$\beta_2(A) = \beta_2(B) = \{3\}$
$\beta_2(C) = \{3\}$

(b)

$B_1(B,1) = B_1(A,2) = \{C\}$
$B_1(B,3) = \{A\}$
$B_1(A,3) = \emptyset$
$B_1(C,1) = B_1(C,2) = \emptyset$
$B_1(A,1) = \{B,C\}$
$B_1(B,2) = \{A,C\}$
$B_1(C,3) = \{A,B\}$

(c)

Fig. 1. Payoff matrix of a normal form game (*left*), best response functions for the game (*center*), better response function of Player 1 for the game (*right*).

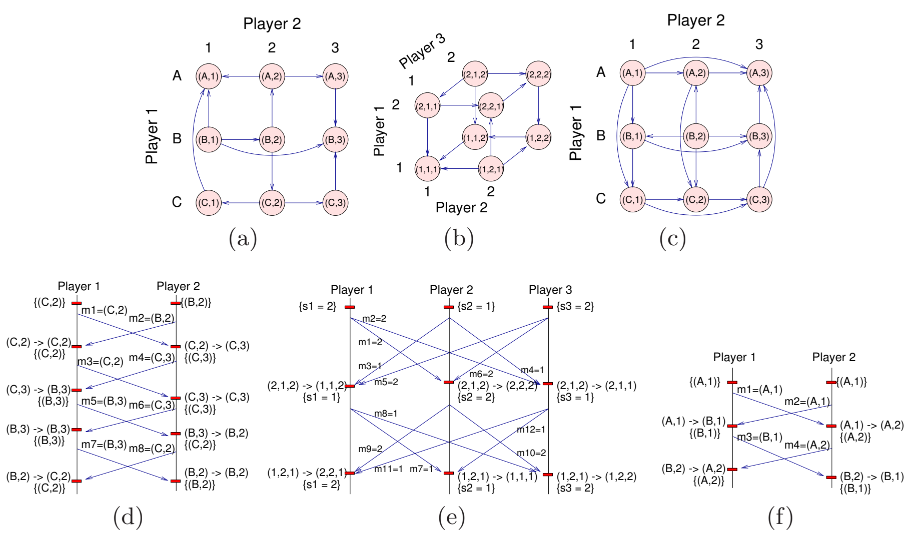

Fig. 2. The graph and an execution of a non-unique best-response game (*left*), unique best-response game (*center*), better-response game (*right*).

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