Non-monotonic Lyapunov Functions for Stability of Discrete Time Nonlinear and Switched Systems

Amir Ali Ahmadi and Pablo A. Parrilo

***Abstract*— We relax the monotonicity requirement of Lyapunov's theorem to enlarge the class of functions that can provide certificates of stability. To this end, we propose two new sufficient conditions for global asymptotic stability that allow the Lyapunov functions to increase locally, but guarantee an average decrease every few steps. Our first condition is non-convex, but allows an intuitive interpretation. The second condition, which includes the first one as a special case, is convex and can be cast as a semidefinite program. We show that when non-monotonic Lyapunov functions exist, one can construct a more complicated function that decreases monotonically.**

We demonstrate the strength of our methodology over standard Lyapunov theory through examples from three different classes of dynamical systems. First, we consider polynomial dynamics where we utilize techniques from sum-of-squares programming. Second, analysis of piecewise affine systems is performed. Here, connections to the method of piecewise quadratic Lyapunov functions are made. Finally, we examine systems with arbitrary switching between a finite set of matrices. It will be shown that tighter bounds on the joint spectral radius can be obtained using our technique.

I. INTRODUCTION

A. Background

Consider the discrete time dynamical system:

$$x_{k+1} = f(x_k) \tag{1}$$

where the map $f : \mathbb{R}^n \to \mathbb{R}^n$ can be in general nonlinear, non-smooth, or even uncertain. Starting from an initial condition x_0, what can be said about the asymptotic behavior of the state, x_k, as $k \to \infty$? Questions of this flavor play a central role in control theory and engineering, as well as, in sciences such as ecology and economics. More specifically, this paper will focus on the notion of global asymptotic stability (GAS). If we take the unique equilibrium point of (1) to be the origin (i.e., $f(0) = 0$), then we have the following formal definition:

Definition 1: The origin is a globally asymptotically stable equilibrium of (1) if:

- $\forall \epsilon > 0 \ \exists \delta > 0$ such that $\|x_0\| < \delta \Rightarrow \|x_k\| < \epsilon \ \forall k$
- $\lim_{k\to\infty} x_k = 0 \quad \forall x_0 \in \mathbb{R}^n$

In general, the question of determining whether the equilibrium of a nonlinear dynamics is GAS can be extremely difficult. Even for special classes of systems several undecidability and NP-hardness results exist in the literature; see e.g. [1] and [2]. Currently, the primary tool for establishing stability of nonlinear systems is the well-known Lyapunov's direct method, first published in 1892. Lyapunov's theorem comes in many variants. Below, we state the version that establishes GAS:

Theorem 1.1: [3] Consider the dynamical system (1). If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) > 0 \ \forall x \neq 0, V(0) = 0$, and

$$V(x_{k+1}) < V(x_k) \tag{2}$$

then, the origin is a GAS equilibrium of (1).

The significance of this criterion is that it allows stability of the system to be verified without explicitly solving the difference equation. Lyapunov's theorem, in effect, turns the question of determining stability into a search for a so-called *Lyapunov function*, a function of the state that decreases monotonically along trajectories. Unfortunately, the theorem offers no systematic way of performing this search. Although *converse theorems* [3] guarantee the existence of a Lyapunov function for every stable system, the results assume knowledge of the solution of (1) and are therefore useless in practice. Moreover, little is known about the connection of the dynamics f to the Lyapunov function V. Among the few results in this direction, the case of linear systems is well settled since a stable linear system always admits a quadratic Lyapunov function. It is also known that stable and smooth homogeneous systems always have a homogeneous Lyapunov function [4].

In the past few decades, advances in the theory of *convex programming* along with efficient numerical algorithms for solving them have rejuvenated Lyapunov theory. The approach is to parameterize a class of Lyapunov functions with restricted complexity (e.g., quadratics or polynomials) and then pose the search as a convex feasibility problem. For example, it is well-known that the search for a quadratic Lyapunov function for a linear system is a semidefinite program (SDP) (also referred to as a linear matrix inequality (LMI) problem) [5]. In [6], the methodology was extended to piecewise linear systems and piecewise quadratic Lyapunov functions. In 2000, the method of sum-of-squares (SOS) programming was introduced [7], which allowed for computation of SOS polynomial Lyapunov functions for polynomial dynamics [8], [9]. The methodology was extended to handle hybrid systems in [10]. One valuable common feature among the techniques based on convex programming is that if a Lyapunov function of a certain class exists, it will be found. If the problem is infeasible, the variables of the dual program provide a certificate of nonexistence of a Lyapunov function of that class. We will make use of this fact several times in

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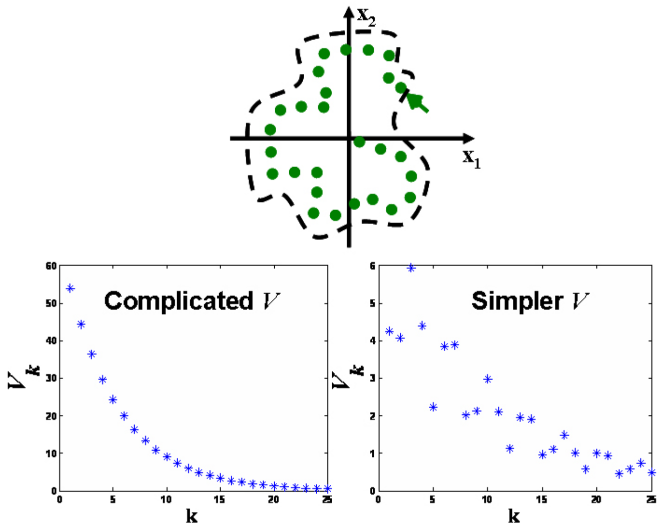

Fig. 1. Motivation for relaxing monotonicity. Level curves of a standard Lyapunov function can be complicated. Simpler functions can decrease on average every few steps.

our examples.

B. Motivation

Despite all the positive progress, it is not too difficult to find stable systems where most of the techniques fail to find a Lyapunov function. Even if one is found, in many situations, the structure of the Lyapunov function can be very complicated. This setback encourages one to think whether the conditions of Lyapunov's theorem are overly conservative.

This paper addresses the following natural question: if it is enough to show $V \to 0$ as $k \to \infty$, why should we require V to decrease monotonically? In fact, there has been earlier work in the literature, mostly in continuous time, on relaxing this condition. Butz in [11] replaces $\dot{V} < 0$ with a condition on $\dot{V}, \ddot{V}$, and $\dddot{V}$ and establishes GAS. However, his condition cannot be verified by a convex program. In a dynamical systems context, Yorke gives a Lyapunov theorem using $\ddot{V}$, but the theorem does not establish GAS [12].

It is perhaps not immediate to see whether relaxing monotonicity would help simplify the structure of Lyapunov functions. Figure 1 explains why we would conceptually expect this to happen. In the top, a hypothetical trajectory is plotted along with a level curve of a candidate Lyapunov function. The problem is that a simple dynamics f (e.g., polynomial of low degree) can produce such trajectory. However, a Lyapunov function V with such level curve must be very complicated (e.g., polynomial of high degree). On the other hand, much simpler functions (maybe even a quadratic) can decrease in a non-monotonic fashion as plotted in the bottom right. Later in the paper, we will verify this intuition with specific examples.

In order to relax monotonicity, two questions need to be answered. (i) Are we able to replace inequality (2) by a condition that allows Lyapunov functions to increase locally but yet guarantee their convergence to zero in the limit? (ii) Can the search for a Lyapunov function with the new condition be cast as a convex program, so that earlier techniques can be readily applied? The contribution of this paper is to give an affirmative answer to both of these questions. Our answer will also illuminate the connection of non-monotonic Lyapunov functions to standard Lyapunov functions.

C. Notation and Organization of the Paper

Our notation is mostly standard. We use superscripts V^1, V^2 to refer to different functions. Some of our Lyapunov functions will decrease monotonically and some will not. Whenever confusion may arise, we refer to a function satisfying Lyapunov's original theorem as a *standard* Lyapunov function. For simplicity, we denote $V(x_k)$ by V_k. Often, we refer to $V_{k+i} - V_k$ as the improvement in i steps, which can either be negative (a decrease in V) or positive (an increase in V). Finally, by f^i, we mean composition of f with itself i times.

The organization of the paper is as follows. In Section II we present our theorems and give some interpretations. In Section III-A, we apply our results to polynomial systems by using SOS programming. Section III-B analyzes stability of piecewise affine systems. In Section III-C, we use non-monotonic Lyapunov functions to find upper bounds on the joint spectral radius of a finite set of matrices. Throughout Section III, we draw comparisons with earlier techniques. Finally, we present our conclusions and some future directions in Section IV.

II. NON-MONOTONIC LYAPUNOV FUNCTIONS

In this section we state our main results which are comprised of two sufficient conditions for global asymptotic stability. Both theorems impose conditions on higher order differences of Lyapunov functions. For clarity, we state our theorems with formulations that only use up to a two-step difference. The generalized versions are presented as corollaries.

A. The Non-Convex Theorem

Our first theorem has a non-convex formulation and it will turn out to be a special case of our second theorem. On the other hand, it allows for an intuitive interpretation of relaxing the monotonicity requirement of (2). For this reason, we present it as a motivation.

Theorem 2.1: Consider the dynamical system (1). If there exists a scalar $\tau \geq 0$, and a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$, such that

$$\begin{aligned} V(x) > 0 \quad \forall x \neq 0 \\ V(0) = 0 \\ \tau(V_{k+2} - V_k) + (V_{k+1} - V_k) < 0 \end{aligned} \tag{3}$$

then the origin is a GAS equilibrium of (1).

Note that we have a product of decision variables V and τ in (3). Therefore, this condition cannot be checked via an SDP. We shall overcome this problem in the next subsection.

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But for now, our approach will be to fix τ through a binary search, and then search for V.

Before we provide a proof of the theorem, we shall give an interpretation of condition (3). When $\tau = 0$, we recover Lyapunov's theorem. For $\tau > 0$, condition (3) requires a weighted average of the improvement in one step and the improvement in two steps to be negative. Meaning that V has to decrease on average every two steps. This allows the Lyapunov function to increase in one step (i.e. $V_{k+1} > V_k$), as long as the improvement in two steps is negative enough. Similarly, at some other points in space, we may have $V_{k+2} > V_k$ when there is enough decrease in the first step. The special case of $\tau = 1$ has a nice interpretation. In this case (3) reduces to

$$V_k > \frac{1}{2}(V_{k+1} + V_{k+2}),$$

i.e., at every point in time, the value of the Lyapunov function should be more than the average of the value at the next two future steps. It should intuitively be clear that condition (3) should imply $V_k \to 0$ as $k \to \infty$. The formal proof is as follows.

Proof: (of Theorem 2.1) Consider the sequence $\{V_k\}$. For any given V_k, (3) and the fact that $\tau \geq 0$ imply that either V_{k+1} or V_{k+2} should be strictly less than V_k. Therefore, there exists a subsequence of $\{V_k\}$ that is monotonically decreasing. Since the subsequence is lower bounded by zero, it must converge to some $c \geq 0$. It can be shown (for e.g. by contradiction) that because of continuity of $V(x)$, c must be zero. This part of the proof is similar to the proof of standard Lyapunov theory (see e.g. [3]). Now that we have established a converging subsequence, for any $\varepsilon > 0$, we can find $\bar{k}$ such that $V_{\bar{k}} < \min\{\frac{\varepsilon}{1+\tau}, \frac{\tau\varepsilon}{1+\tau}\}$. Because of positivity of V and condition (3), we have $V_k < \varepsilon \ \forall k > \bar{k}$. Therefore, $V_k \to 0$, which implies $x \to 0$. ∎

We shall provide an alternative proof in Section II-B for the more general theorem. Note that by construction, Theorem 2.1 should work better than requiring $V_{k+1} < V_k$ ($\tau = 0$) and $V_{k+2} < V_k$ (τ large). The following example illustrates that the improvement can be significant.

Example 2.1: (piecewise linear system in one dimension) Consider the piecewise linear dynamical system:

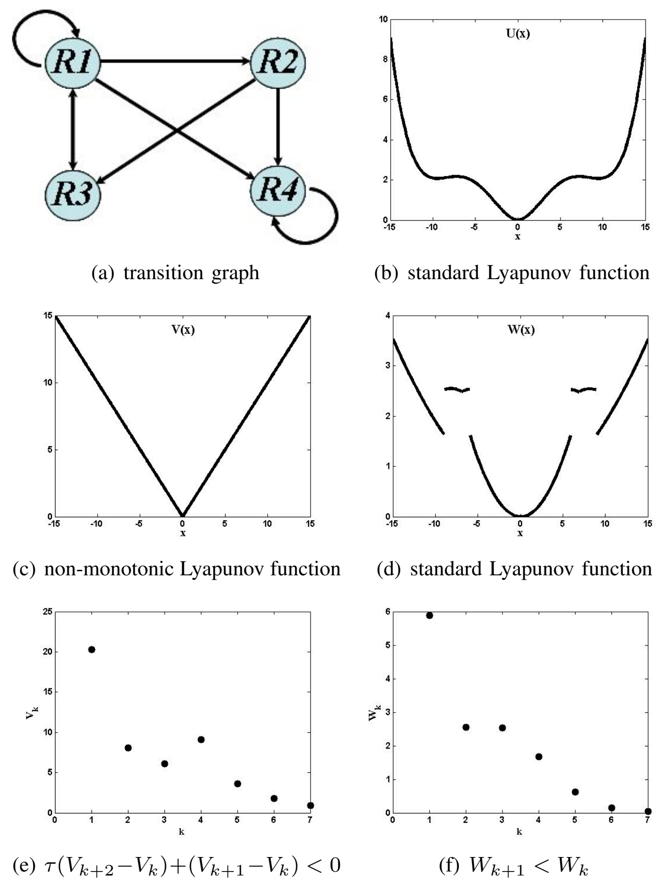
$$x_{k+1} = f(x_k)$$

with

$$f = \begin{cases} A_1 x & |x| \in R_1 = [9, \infty) \\ A_2 x & |x| \in R_2 = [7, 9) \\ A_3 x & |x| \in R_3 = [6, 7) \\ A_4 x & |x| \in R_4 = [0, 6) \end{cases}$$

where $A_1 = \frac{2}{5}$, $A_2 = \frac{3}{4}$, $A_3 = \frac{3}{2}$, and $A_4 = \frac{1}{2}$.

We would like to establish global asymptotic stability using Lyapunov theory. Since f is odd, it suffices to find a Lyapunov function for half of the space (e.g., $x \geq 0$) and use its mirror image on the other half space. Figure 2(a) illustrates the possible switchings among the four regions. Note that $A_3 > 1$ and $A_3 A_2 > 1$. We claim that no quadratic Lyapunov function exists. Moreover, no quadratic



(a) transition graph (b) standard Lyapunov function

(c) non-monotonic Lyapunov function (d) standard Lyapunov function

(e) $\tau(V_{k+2}-V_k)+(V_{k+1}-V_k) < 0$ (f) $W_{k+1} < W_k$

Fig. 2. Comparison between non-monotonic and standard Lyapunov functions for Example 2.1. The non-monotonic Lyapunov function has a simpler structure and therefore less decision variables.

function can satisfy $V_{k+2} < V_k$. These facts can easily be seen by noting that any positive definite quadratic function will increase if the trajectory moves away from the origin. Therefore, transitions A_3 and $A_3 A_2$ respectively reject the existence of a quadratic Lyapunov function that would decrease monotonically in one or two steps.

In order to satisfy the monotonic decrease of Lyapunov's theorem, we should search for functions that are more complicated than quadratics. Figure 2(b) and 2(d) illustrate two such functions. The first function, U, is a polynomial of degree 4 (on the nonnegative half-space) found through SOS programming. The second function, W, is a piecewise quadratic with four pieces that is obtained by solving an SDP. Figure 2(f) shows the value of W on a trajectory that starts in R_1, visits R_2, R_3, R_1, R_4, and stays in R_4 before it converges to the origin. The corresponding plot for U is omitted to save space.

Next, we apply Theorem 2.1 to prove stability. As shown in Figure 2(c), we can simply take V to be a linear function with slope of 1 on the positive half-space. This V along with any $\tau \in (1.25, 2)$ satisfies (3). Figure 2(e) shows the value of V on the same trajectory described before. Even though from $k = 2$ to $k = 4$ V is increasing, at any point in time condition (3) is satisfied.

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This example clearly demonstrates that relaxing monotonicity can simplify the structure of Lyapunov functions. From a computational point of view, the search for the non-monotonic Lyapunov function only had 2 decision variables: the slope of the line in V and the value of τ. On the other hand, each of the four quadratic pieces of W have three free parameters. If we take advantage of the fact that the piece containing the origin should have no constant or linear terms, we end up with a total of 10 decision variables. As we shall see in Section III-B, both methods will have the same number of constraints. The quartic polynomial U has no constant or linear terms and therefore has 3 decision parameters. However, as the dimension of the space goes up, the difference between the number of free parameters of a quadratic and a quartic grows quadratically in the dimension. We will make many more similar comparisons in Section III for different types of dynamical systems.

We end this section by stating the general version of Theorem 2.1, which requires the Lyapunov function to decrease on average every m steps.

Corollary 2.1: Consider the dynamical system (1). If there exists $m-1$ nonnegative scalars τ_i, and a continuous radially unbounded function $V:\ \mathbb{R}^n \to \mathbb{R}$, such that

$$\begin{array}{rcl} V(x) & > & 0 \quad \forall x \neq 0 \\ V(0) & = & 0 \\ \tau_{m-1}(V_{k+m} - V_k) + \cdots + (V_{k+1} - V_k) & < & 0 \end{array} \tag{4}$$

then the origin is a GAS equilibrium of (1).

Proof: The proof is a straightforward generalization of the proof of Theorem 2.1. ∎

B. The Convex Theorem

In this section we present our main theorem, which will be used throughout Section III.

Theorem 2.2: Consider the dynamical system (1). If there exists two continuous functions $V^1, V^2:\ \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{array}{r} V^2 \text{ and } V^1 + V^2 \text{ are radially unbounded} \\ V^2(x) > 0 \ \forall x \neq 0 \\ V^1(x) + V^2(x) > 0 \ \forall x \neq 0 \\ V^1(0) + 2V^2(0) = 0 \\ (V^2_{k+2} - V^2_k) + (V^1_{k+1} - V^1_k) < 0 \end{array} \tag{5}$$

then the origin is a GAS equilibrium of (1).

The inequality (5) is linear in the decision variables V^1 and V^2. This will alow us to check condition (5) via a semidefinite program. Note that Theorem 2.1 is a special case of Theorem 2.2, when $V^1 = V$ and $V^2 = \tau V$. Unlike Theorem 2.1, Theorem 2.2 maps the state into two Lyapunov functions instead of one. In this fashion, the improvement in one and two steps are measured using two different metrics. The theorem states that as long as the sum of the two improvements is negative at any point in time, stability is guaranteed and both V^1 and V^2 will converge to zero. Figure 3 illustrates the trajectory of a hypothetical dynamical system at three consecutive instances of time. Here, V^1 and V^2 ar taken to be quadratics and therefore have ellipsoidal level sets. Since the decrease in the horizontal ellipsoid in two steps is larger than the increase of the vertical ellipsoid in the first step, inequality (5) is satisfied.

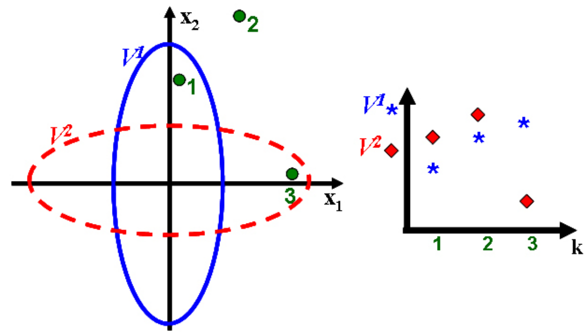

Fig. 3. Interpretation of Theorem 2.2. On the left, three consecutive instances of the trajectory are plotted along with level sets of V^1 and V^2. V^1 measures the improvement in one step, and V^2 measures the improvement in two steps. The plot on the right shows that inequality (5) is satisfied.

The following proof will use the conditions of Theorem 2.2 to explicitly construct a standard Lyapunov function.

Proof: (of Theorem 2.2) We start by rewriting (5) in the form

$$V^2_{k+2} + V^1_{k+1} < V^2_k + V^1_k.$$

Adding V^2_{k+1} to both sides and rearranging terms we get

$$V^1_{k+1} + V^2_{k+1} + V^2_{k+2} < V^1_k + V^2_k + V^2_{k+1}.$$

If we define $W(x) = V^1(x) + V^2(x) + V^2(f(x))$, the last inequality implies that $W_{k+1} < W_k$. It is easy to check from the assumptions of the theorem that W will be continuous, radially unbounded, and will satisfy $W(x) > 0\ \forall x \neq 0$, and $W(0) = 0$. Therefore, W is a *standard* Lyapunov function for (1). ∎

The explicit construction of a standard Lyapunov function in this proof suggests that non-monotonic Lyapunov functions are equivalent to standard Lyapunov functions of a very specific structure. The function $W(x)$ is parameterized not only with the value of the current state x, but also with the future value of the state $f(x)$. We will demonstrate in Section III that parameterizing W in this fashion and searching for V^1 and V^2 can often be advantageous over a direct search for a standard Lyapunov function of similar complexity. The reason is that depending on f itself, $W(x)$ will have a more complicated structure than $V^1(x)$ and $V^2(x)$. For example, if f is a polynomial of degree d and V^1 and V^2 are polynomials of degree q, then W will be of higher degree dq. As a second example, suppose f is piecewise linear with $\mathcal{R}$ pieces. If two smooth quadratic functions V^1 and V^2 satisfy the conditions of Theorem 2.2, then there will be a standard Lyapunov function W which is piecewise quadratic with $\mathcal{R}$ pieces. From a computational point of view, this additional complexity directly translates into more decision variables. These facts will become more clear in Section III, where we compare standard Lyapunov techniques to our methodology for specific examples.

Next, we generalize Theorem 2.2 to m-step differences.

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Corollary 2.2: Consider the dynamical system (1). If there exists continuous functions $V^1, \cdots, V^m : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} \sum_{i=j}^{m} V^i \text{ radially unbounded for } j = 1, \cdots, m \\ \sum_{i=j}^{m} V^i(x) > 0 \ \forall \ x \neq 0 \text{ for } j = 1, \cdots, m \\ \sum_{i=1}^{m} iV^i(0) = 0 \\ (V_{k+m}^m - V_k^m) + \cdots + (V_{k+1}^1 - V_k^1) < 0 \end{aligned} \tag{6}$$

then the origin is a GAS equilibrium of (1).

Proof: Similar to the proof of Theorem 2.2, it can be shown that $\sum_{j=1}^{m} \sum_{i=j}^{m} V^i(f^{j-1})$ is a standard Lyapunov function. ■

III. APPLICATIONS AND EXAMPLES

In this section, we apply our results to polynomial systems, piecewise affine systems, and linear systems with arbitrary switching. In all of the examples, our approach will be as follows. We fix a certain class of Lyapunov functions (e.g., quadratics) and we show that no function within that class satisfies $V_{k+1} < V_k$ or $V_{k+2} < V_k$. Then, we find functions V^1 and V^2 of the same class that prove stability based on Theorem 2.2. In most cases, we will write out the LMIs explicitly to provide guidelines for the users. Throughout, the reader should keep in mind that Corollary 2.2 with $m > 2$ can lead to better results than Theorem 2.2 at the expense of computing higher order differences.

A. Polynomial Systems

Conditions of Lyapunov's theorem for polynomial systems reduce to checking nonnegativity of certain polynomials on the whole space. This problem is known to be NP-hard. A tractable sufficient condition for global nonnegativity of a polynomial function is the existence of a sum of squares (SOS) decomposition.

A multivariate polynomial $p(x_1, ..., x_n) := p(x)$ is a sum of squares, if there exist polynomials $q_1(x), ..., q_m(x)$ such that

$$p(x) = \sum_{i=1}^{m} q_i^2(x).$$

Notice that $p(x)$ being SOS implies $p(x) \geq 0$. The converse is not true, except for special cases [7]. However, unlike nonnegativity, the search for an SOS decomposition of a polynomial can be cast as an SDP, which can be solved efficiently in polynomial time. The conversion step is fully algorithmic, and has been implemented in the SOSTOOLS [13] software package. Over the past few years, SOS programming has shown to be a powerful technique for construction of Lyapunov functions [8], [9], [10]. Luckily, we can readily apply the same methodology to find non-monotonic Lyapunov functions. More specifically, we will search for V^1 and V^2 that satisfy

$$\begin{aligned} V^2(x) \quad \text{SOS} \\ V^1(x) + V^2(x) \quad \text{SOS} \\ -\{V^2(f(f(x))) - V^2(x) + V^1(f(x)) - V^1(x)\} \quad \text{SOS.} \end{aligned} \tag{7}$$

Example 3.1: Consider the discrete time polynomial dynamics in dimension two:

$$f = \begin{pmatrix} \frac{3}{10}x_1 \\ x_1 + \frac{1}{2}x_2 + \frac{7}{18}x_2^2 \end{pmatrix}.$$

One can check that no quadratic SOS function V can satisfy

$$-\{V(f(x)) - V(x)\} \quad \text{SOS.}$$

Since there is no gap between SOS and nonnegativity in dimension two and degree up to four [7], we can be certain that in fact no quadratic Lyapunov function exists for this system. We can also check that no quadratic SOS function will satisfy

$$-\{V(f(f(x))) - V(x)\} \quad \text{SOS.}$$

On the other hand, from SOSTOOLS and the SDP solver SeDuMi [14] we get that condition (7) is satisfied with

$$V^1 = 0.063x_1^2 - 0.123x_1x_2 - 1.027x_2^2$$

$$V^2 = 0.731x_1^2 + 0.095x_1x_2 + 1.756x_2^2.$$

Stability follows from Theorem 2.2. It is easy to check that $W(x) = V^1(x) + V^2(x) + V^2(f(x))$ will be a standard Lyapunov function of degree four. Alternatively, we could have directly searched for a standard Lyapunov function of degree four. However, a polynomial of degree d in n variables has $\binom{n+d}{d}$ coefficients. Therefore, as the dimension goes up, one quartic will have significantly more decision parameters than two quadratics.

B. Piecewise Affine Systems

Piecewise affine (PWA) systems are systems of the form

$$x_{k+1} = A_i x_k + a_i, \qquad \text{for } x_k \in R_i \tag{8}$$

where R_i's are polyhedral partitions of the state space. There has been much recent interest in systems of this type because, among other reasons, they provide a practical framework for modeling and approximation of hybrid and nonlinear systems. In [6], the method of piecewise quadratic (PWQ) Lyapunov functions was introduced to analyze stability of continuous time PWA systems. Discrete time analogs of this technique have also been studied (see e.g. [15], [16]). A detailed comparison of different stability techniques for discrete time PWA systems is presented in [17]. In this section, we compare the strength of non-monotonic Lyapunov functions to some of the other techniques through an example. It will be shown that, in some cases, instead of a standard piecewise quadratic Lyapunov function, smooth non-monotonic Lyapunov functions can prove stability.

Example 3.2: (Discretized flower dynamics) Consider the the following PWA system

$$x_{k+1} = \begin{cases} A_1 x_k, & x_k^T H x_k > 0 \\ A_2 x_k, & x_k^T H x_k \leq 0 \end{cases}$$

where $A_1 = \lambda e^{2A_1^{CT}}$, and $A_2 = \frac{1}{\lambda} e^{2A_2^{CT}}$.
The matrices A_1^{CT}, A_2^{CT}, and H are as in [6] (with a minor correction)

$$A_1^{CT} = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}, \quad A_2^{CT} = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix},$$
$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and $\lambda \geq 1$ will be a scaling parameter in this problem. We will compare different techniques based on the range of λ for which they can prove stability.

If we search for a smooth[1] quadratic Lyapunov function satisfying $V_{k+1} < V_k$, the problem will be infeasible even for $\lambda = 1$. As a second attempt, we search for a smooth quadratic function that satisfies $V_{k+2} < V_k$. Stability is proven for $\lambda \in [1, 1.114]$. Our next purpose is to show that by combining the improvement in one step and the improvement in two steps using quadratic non-monotonic Lyapunov functions, better results will be obtained. By taking V^i to be $x^T P_i x$, the conditions of Theorem 2.2 reduce to the following set of LMIs

$$\begin{array}{rcl} P_2 & \succ & 0 \\ P_1 + P_2 & \succ & 0 \\ (A_i^T A_j^T P_2 A_j A_i - P_2) + (A_i^T P_1 A_i - P_1) & \prec & 0 \\ \text{when } x \in R_i \text{ and } A_i x \in R_j, \quad \forall i, j \in \{1, 2\}. & & \end{array} \tag{9}$$

In order to impose the last inequality in (9) only on regions of space where $A_j A_i$ is a possible future transition, we use the $\mathcal{S}$-procedure technique [5]. The LMIs in (9) will prove stability for $\lambda \in [1, 1.221)$, which is a strictly larger range than what was obtained before.
Next, we shall comment on the connection of this approach to piecewise quadratic Lyapunov functions, which we denote by $x^T Q_i x$.

A search for a PWQ Lyapunov function can be posed by the following set of LMIs [17]

$$\begin{array}{rcl} Q_1 & \succ & 0 \\ Q_2 & \succ & 0 \\ (A_i^T Q_j A_i - Q_i) & \prec & 0 \\ \text{when } x \in R_i \text{ and } A_i x \in R_j, \quad \forall i, j \in \{1, 2\}. & & \end{array} \tag{10}$$

If we ignore the positivity conditions, (9) and (10) show that the two methods have the same number of constraints. It is relatively straightforward to check that whenever P_1 and P_2 satisfy (9),

$$Q_i = P_1 + P_2 + A_i^T P_2 A_i \quad i = 1, 2 \tag{11}$$

[1]Reference [17] refers to this as a common quadratic Lyapunov function. This is not to be confused with common quadratic in the context of arbitrary switching. We avoid using this terminology to emphasize that the $\mathcal{S}$-procedure relaxation is used on the regions.

will satisfy the LMIs in (10). This is in agreement with the standard Lyapunov function that we constructed in the proof of Theorem 2.2. On the other hand, existence of PWQ Lyapunov functions does not in general imply feasibility of the LMIs in (9). However, for the example discussed above, piecewise quadratic Lyapunov functions also prove stability for $\lambda \in [1, 1.221)$.
We should point out that the method of smooth non-monotonic Lyapunov functions is searching only for two functions P_1 and P_2 independent of the number of regions. On the other hand, PWQ Lyapunov functions have to find as many quadratic functions as the number of regions. This in turn results in more decision variables and more positivity constraints.

To obtain a method that works at least as well as (and most likely strictly better than) standard PWQ Lyapunov functions, one can take V^1, V^2, or both in Theorem 2.2 to be piecewise quadratic.

C. Approximation of the Joint Spectral Radius

In this section, we consider a dynamical system of the type

$$x_{k+1} = A_{\sigma(k)} x_k \tag{12}$$

where σ is a mapping from the integers to a finite set of indices $\{1, ..., m\}$. The question of interest is to determine whether the discrete inclusion (12) is *absolutely asymptotically stable* (AAS), i.e., asymptotically stable for all switching sequences.

It turns out [18] that (12) is AAS if and only if the *joint spectral radius* (JSR) of the matrices $A_1, ..., A_m$ is strictly less than one. The joint spectral radius represents the maximum growth rate obtained by taking arbitrary products of the matrices A_i. It is formally defined as [19]:

$$\rho(A_1, \cdots, A_m) := \lim_{k \to \infty} \max_{\sigma \in \{1, \cdots, m\}^k} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{\frac{1}{k}} \tag{13}$$

where the value of ρ is independent of the norm used in (13). For a given set of matrices, testing whether $\rho < 1$ is undecidable [20]. Moreover, computation and even approximation of the JSR is difficult [21]. Here, we will be interested in providing bounds on the JSR. Clearly, the spectral radius of any finite product of matrices gives a lower bound on ρ. Computing upper bounds is a much more challenging task. We explain our technique for a pair of matrices A_1, A_2. The generalization to a finite set of matrices is straightforward.

Because of the scaling property of the JSR, for any $\lambda \in (0, \infty)$, if we can prove AAS of (12) for the scaled pair of matrices λA_1 and λA_2, then $\frac{1}{\lambda}$ is an upper bound on $\rho(A_1, A_2)$. References [22] and [23] have respectively used common quadratic and common SOS polynomial Lyapunov functions to prove upper bounds on ρ. Here, we will use common non-monotonic Lyapunov functions for this purpose. For the special case where V^1 and V^2 are quadratics (i.e. $V^i = x^T P_i x$), Theorem 2.2 suggests that the following LMIs have to be satisfied to get an upper bound of $\frac{1}{\lambda}$ on $\rho(A_1, A_2)$.

$$\begin{array}{rcl} P_2 & \succ & 0 \\ P_1 + P_2 & \succ & 0 \\ (\lambda^4 A_i^T A_j^T P_2 A_j A_i - P_2) + (\lambda^2 A_i^T P_1 A_i - P_1) & \prec & 0 \\ \forall i, j \in \{1, 2\}. & & \end{array} \tag{14}$$

When P_2 is set to zero, the method of common quadratics is recovered. Similarly, when P_1 is set to zero, the LMIs will find a common quadratic that satisfies $V_{k+2} < V_k$. It is easy to see that the existence of a common quadratic in one step implies the existence of a common quadratic in two steps, but the converse is not true. Therefore, setting $P_1 = 0$ will produce upper bounds that are at least as tight as those obtained from setting $P_2 = 0$. Below, we show with two examples that when we use both P_1 and P_2 in (14) to combine the improvement in one and two steps, we can provide strictly tighter bounds on the JSR.

Example 3.3: ([23], Example 2) We consider the problem of finding an upper bound for the JSR of the following pair of matrices:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

It is not difficult to show that $\rho(A_1, A_2) = 1$. Using common quadratic standard Lyapunov functions, one would obtain an upper bound of $\sqrt{2} \approx 1.41$. A common quadratic standard Lyapunov function for A_1A_1, A_2A_1, A_1A_2, and A_2A_2 would produce an upper bound of $\sqrt[4]{2} \approx 1.19$. On the other hand, common quadratic non-monotonic Lyapunov functions can achieve an upper bound of $1 + \varepsilon$ for any $\varepsilon > 0$. Given ε, the LMIs (14) will be feasible with

$$P_1 = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix}, \quad P_2 = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}$$

with any $\beta > 0, \quad 1 - \frac{4\varepsilon}{1+\varepsilon} < \frac{\alpha}{\beta} < 1$.

We should mention that in [23], it is shown that a common SOS quartic Lyapunov function also achieves an upper bound of $1 + \varepsilon, \quad \forall \varepsilon > 0$.

Example 3.4: ([23], Example 4) We consider the following three randomly generated 4×4 matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 7 & 4 \\ 1 & 6 & -2 & -3 \\ -1 & -1 & -2 & -6 \\ 3 & 0 & 9 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 3 & 0 & -2 \\ -2 & 1 & 4 & 9 \\ 4 & -3 & 1 & 1 \\ 1 & -5 & -1 & -2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 4 & 5 & 10 \\ 0 & 5 & 1 & -4 \\ 0 & -1 & 4 & 6 \\ -1 & 5 & 0 & 1 \end{bmatrix}$$

A lower bound on the JSR is $\rho(A_1A_3)^{\frac{1}{2}} \approx 8.91$ [23]. Method of common quadratic satisfying $V_{k+1} < V_k$, common quadratic satisfying $V_{k+2} < V_k$, and common non-monotonic quadratic satisfying Theorem 2.2 respectively produce upper bounds equal to $9.77, 9.19$, and 8.98. A common SOS quartic satisfying $V_{k+1} < V_k$ produces an upper bound of 8.92 [23]. This bound is tighter than what we obtained from quadratic non-monotonic functions. However, the latter technique will have 20 decision parameters for this example in contrast with 35 needed to find a homogeneous quartic function.

Even though, throughout Section III we have used quadratic non-monotonic Lyapunov functions, the reader should keep in mind that better results can be obtained by taking V^1 and V^2 of Theorem 2.2 to be SOS polynomials.

IV. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper we addressed the following natural question: why should we require a Lyapunov function to decrease monotonically at every step? We gave a sufficient condition for GAS that allows the Lyapunov functions to increase locally while guaranteeing their convergence to zero in the limit. The conditions of our main theorem were convex. Therefore, all the techniques developed for finding Lyapunov functions based on convex programming can readily be applied. We showed that whenever a non-monotonic Lyapunov function is found, one can construct a standard Lyapunov function from it. However, the standard Lyapunov function will have a more complicated structure. The nature of this additional complexity depends on the dynamics itself. We demonstrated the advantages of our methodology over standard Lyapunov theory through examples from polynomial systems, and linear systems with constrained and arbitrary switching.

Our work leaves three future directions to be explored. First, it would be interesting to see if continuous time analogs of non-monotonic Lyapunov functions can be established by imposing a convex condition on higher derivatives of Lyapunov functions. Second, the connection of our methodology to vector Lyapunov functions (e.g. [24]) needs to be clarified. Since our main theorem measures the improvement in different steps with different Lyapunov functions, we suspect that our non-monotonic functions may be a special type of vector Lyapunov functions. Finally, other control applications such as synthesis, or robustness and performance analysis can be explored using non-monotonic Lyapunov functions.

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