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Sum of Squares and Polynomial Convexity

Amir Ali Ahmadi and Pablo A. Parrilo

***Abstract*— The notion of *sos-convexity* has recently been proposed as a tractable sufficient condition for convexity of polynomials based on sum of squares decomposition. A multivariate polynomial $p(x) = p(x_1, \ldots, x_n)$ is said to be sos-convex if its Hessian $H(x)$ can be factored as $H(x) = M^T(x)\,M(x)$ with a possibly nonsquare polynomial matrix $M(x)$. It turns out that one can reduce the problem of deciding sos-convexity of a polynomial to the feasibility of a semidefinite program, which can be checked efficiently. Motivated by this computational tractability, it has been speculated whether every convex polynomial must necessarily be sos-convex. In this paper, we answer this question in the negative by presenting an explicit example of a trivariate homogeneous polynomial of degree eight that is convex but not sos-convex.**

I. Introduction

In many problems in applied and computational mathematics, we would like to *decide* whether a multivariate polynomial is convex or to *parameterize* a family of convex polynomials. Perhaps the most obvious instance appears in optimization. It is well known that in the absence of convexity, global minimization of polynomials is generally NP-hard [1], [2], [3]. However, if we somehow know *a priori* that the polynomial is convex, nonexistence of local minima is guaranteed and simple gradient descent methods can find a global minimum. In many other practical settings, we might want to parameterize a family of convex polynomials that have certain properties, e.g., that serve as a convex envelope for a non-convex function, approximate a more complicated function, or fit some data points with minimum error. To address many questions of this type, we need to have an understanding of the algebraic structure of the set of convex polynomials.

Over a decade ago, Pardalos and Vavasis [4] put the following question proposed by Shor on the list of seven most important open problems in complexity theory for numerical optimization: "Given a degree-4 polynomial in n variables, what is the complexity of determining whether this polynomial describes a convex function?" To the best of our knowledge, the question is still open but the general belief is that the problem should be hard (see the related work in [5]). Not surprisingly, if testing membership to the set of convex polynomials is hard, searching and optimizing over them also turns out to be a hard problem.

The notion of *sos-convexity* has recently been proposed as a tractable relaxation for convexity based on semidefinite programming. Broadly speaking, the requirement of positive semidefiniteness of the Hessian matrix is replaced with the existence of an appropriately defined sum of squares decomposition. As we will briefly review in this paper, by drawing some appealing connections between real algebra and numerical optimization, the latter problem can be reduced to the feasibility of a semidefinite program.

Despite the relative recency of the concept of sos-convexity, it has already appeared in a number of theoretical and practical settings. In [6], Helton and Nie use sos-convexity to give sufficient conditions for semidefinite representability of semialgebraic sets. In [7], Lasserre uses sos-convexity to extend Jensen's inequality in convex analysis to linear functionals that are not necessarily probability measures. In a different work [8], Lasserre appeals to sos-convexity to give sufficient conditions for a polynomial to belong to the quadratic module generated by a set of polynomials. More on the practical side, Magnani, Lall, and Boyd [9] use sum of squares programming to find sos-convex polynomials that best fit a set of data points or to find minimum volume convex sets, given by sub-level sets of sos-convex polynomials, that contain a set of points in space.

Even though it is well-known that sum of squares and nonnegativity are not equivalent, because of the special structure of the Hessian matrix, sos-convexity and convexity could potentially turn out to be equivalent. This speculation has been bolstered by the fact that finding a counterexample has shown to be difficult and attempts at giving a non-constructive proof of its existence have seen no success either. Our contribution in this paper is to give the first such counterexample, i.e., the first example of a polynomial that is convex but not sos-convex. This example is presented in Theorem 3.2. Our result further supports the hypothesis that deciding convexity of polynomials should be a difficult problem. We hope that our counterexample, in a similar way to what other celebrated counterexamples [10]–[13] have achieved, will help stimulate further research and clarify the relationships between the geometric and algebraic aspects of positivity and convexity.

The organization of the paper is as follows. Section II is devoted to mathematical preliminaries required for understanding the remainder of this paper. We begin this section by introducing the cone of nonnegative and sum of squares polynomials. We briefly discuss the connection between sum of squares decomposition and semidefinite programming highlighting also the dual problem. Formal definitions of sos-convex polynomials and sos-matrices are also given in this section. In Section III, we present our main result, which is an explicit example of a convex polynomial that is not sos-convex. Some of the properties of this polynomial are also

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discussed at the end of this section.

A more comprehensive version of this paper is presented in [14]. Because our space is limited here, some of the technical details and parts of the proofs have been omitted and can be found in [14]. Furthermore, in [14], we explain how we have found our counterexample using software and by utilizing techniques from sum of squares programming and duality theory of semidefinite optimization. The methodology explained there is of independent interest since it can be employed to search or optimize over a restricted family of nonnegative polynomials that are *not* sums of squares.

II. MATHEMATICAL BACKGROUND

A. Nonnegativity and sum of squares

We denote by $\mathbb{K}[x] := \mathbb{K}[x_1, \ldots, x_n]$ the ring of polynomials in n variables with coefficients in the field $\mathbb{K}$. Throughout the paper, we will have $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$. A polynomial $p(x) \in \mathbb{R}[x]$ is said to be nonnegative or positive semidefinite (psd) if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. Clearly, a necessary condition for a polynomial to be psd is for its total degree to be even. We say that $p(x)$ is a sum of squares (sos), if there exist polynomials $q_1(x), ..., q_m(x)$ such that

$$p(x) = \sum_{i=1}^{m} q_i^2(x). \tag{1}$$

It is clear that $p(x)$ being sos implies that $p(x)$ is psd. In 1888, David Hilbert [15] proved that the converse is true for a polynomial in n variables and of degree d *only* in the following cases:

- $n = 1$ (univariate polynomials of any degree)
- $d = 2$ (quadratic polynomials in any number of variables)
- $n = 2, \ d = 4$ (bivariate quartics)

Hilbert showed that in all other cases there exist polynomials that are psd but not sos. Explicit examples of such polynomials appeared nearly 80 years later, starting with the celebrated example of Motzkin, followed by more examples by Robinson, Choi and Lam, and Lax-Lax and Schmüdgen. See [16] for an outstanding exposition of these counterexamples.

A polynomial $p(x)$ of degree d in n variables has $l = \binom{n+d}{d}$ coefficients and can therefore be associated with the l-tuple of its coefficients, which we denote by $\vec{p} \in \mathbb{R}^l$. A polynomial where all the monomials have the same degree is called a *form*. A form $p(x)$ of degree d is a homogenous function of degree d (since it satisfies $p(\lambda x) = \lambda^d p(x)$), and has $\binom{n+d-1}{d}$ coefficients. The set of forms in n variables of degree d is denoted by $\mathcal{H}_{n,d}$. It is easy to show that if a form of degree d is sos, then the polynomials q_i in the sos decomposition are forms of degree $d/2$. We also denote the set of psd (resp. sos) forms of degree d in n variables by $P_{n,d}$ (resp. $\Sigma_{n,d}$). Both $P_{n,d}$ and $\Sigma_{n,d}$ are closed convex cones [16], and we have the relation $\Sigma_{n,d} \subseteq P_{n,d} \subseteq \mathcal{H}_{n,d}$.

Any form of degree d in n variables can be dehomogenized into a polynomial of degree $\leq d$ in $n-1$ variables by setting $x_n = 1$. Conversely, any polynomial p of degree d in n variables can be homogenized into a form p_h of degree d in $n+1$ variables, by adding a new variable y, and letting

$$p_h(x_1, \ldots, x_n, y) := y^d p(\frac{x_1}{y}, \ldots, \frac{x_n}{y}).$$

The properties of being psd and sos are preserved under homogenization and dehomogenization.

To make the ideas presented so far more concrete, we end this section by discussing the example of Motzkin.

Example 2.1: The Motzkin polynomial

$$M(x_1, x_2) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1 \tag{2}$$

is historically the first known example of a polynomial that is psd but not sos. Positive semidefiniteness follows from the arithmetic-geometric inequality, and the nonexistence of an sos decomposition can be shown by some clever algebraic manipulations (see [16]) or by a duality argument. We can homogenize this polynomial and obtain the Motzkin form

$$M_h(x_1, x_2, x_3) := x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6, \tag{3}$$

which belongs to $P_{3,6} \setminus \Sigma_{3,6}$ as expected.

B. Sum of squares, semidefinite programming, and duality

Deciding nonnegativity of polynomials is an important problem that arises in many areas of systems theory, control, and optimization [17]. Unfortunately, this problem is known to be NP-hard even when the degree of the polynomial is equal to four [2], [3]. On the other hand, deciding whether a given polynomial admits an sos decomposition turns out to be a tractable problem. This tractability stems from the underlying convexity of the problem as first pointed out in [18], [19], [20]. More specifically, it was shown in [20] that one can reduce the problem of deciding whether a polynomial is sos to feasibility of a *semidefinite program* (SDP). Semidefinite programs are a well-studied subclass of convex optimization problems that can be efficiently solved in polynomial time using interior point algorithms. Because our space is limited, we refrain from further discussing SDPs and refer the interested reader to the excellent review paper [21]. The main theorem that establishes the link between sum of squares and semidefinite programming is the following.

Theorem 2.1: ([20], [2]) A multivariate polynomial $p(x)$ in n variables and of degree $2d$ is a sum of squares if and only if there exists a positive semidefinite matrix Q (often called the Gram matrix) such that

$$p(x) = z^T Q z, \tag{4}$$

where z is the vector of monomials of degree up to d

$$z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d]. \tag{5}$$

Given a polynomial $p(x)$, by expanding the right hand side of (4) and matching coefficients of p, we get linear constraints on the entries of Q. We also have the constraint that Q must be a positive semidefinite (PSD[1]) matrix. Therefore, the

[1] To avoid potential confusion, we use the abbreviation psd for positive semidefinite polynomials and PSD for positive semidefinite matrices. We also denote a PSD matrix A with the standard notation $A \succeq 0$.

feasible set of this optimization problem is the intersection of an affine subspace with the cone of PSD matrices. This is exactly the structure of the feasible set of a semidefinite program [21].

Since the entries of the vector of monomials z can generally be algebraically dependent, the matrix Q in the representation (4) is not in general unique. The size of the matrix Q depends on the size of the vector of monomials. When there is no sparsity to be exploited Q will be $\binom{n+d}{d}\times\binom{n+d}{d}$. If the polynomial $p(x)$ is homogeneous of degree $2d$, then it suffices to consider in (4) a vector z of monomials of degree exactly d [2]. This will reduce the size of Q to $\binom{n+d-1}{d}\times\binom{n+d-1}{d}$.

The conversion step of going from an sos decomposition problem to an SDP problem is fully algorithmic and has been implemented in the SOSTOOLS [22] software package. We can input a polynomial $p(x)$ into SOSTOOLS and if the code is feasible, it will return a matrix Q and a vector of monomials z. Since Q is PSD, one can compute its Cholesky factorization $Q = V^T V$, which immediately gives rise to an explicit sos decomposition

$$p(x) = \sum_i (Vz)_i^2. \tag{6}$$

Solutions returned from interior point algorithms are numerical approximations computed via floating point manipulations. In many applications in pure mathematics where the goal is to formally prove a theorem (as is the case in this paper), it is required to get an exact algebraic solution. What we mean by this is that given a polynomial $p(x) \in \mathbb{Q}[x]$, i.e., a polynomial with rational coefficients, we would like to compute a *rational sos decomposition*, i.e., a decomposition only consisting of squares of polynomials in $\mathbb{Q}[x]$. This issue has been studied in detail in [23] where it is shown that the existence of a rational sos decomposition is equivalent to the existence of a Gram matrix with rational entries. SOSTOOLS is endowed with a feature that computes rational decompositions. The work in [23] proposes an efficient mixed symbolic-numerical approach for this purpose and has been separately implemented as an sos package on the computer algebra system Macaulay 2.

A very useful feature of sos-programming is that when the semidefinite program deduced from Theorem 2.1 is infeasible, we get a *certificate* that the polynomial is *not* sos (though it might still be psd). This certificate is readily given to us by a feasible solution of the dual semidefinite program. By definition, the dual cone $\Sigma_{n,d}^*$ of the sum of squares cone $\Sigma_{n,d}$ is the set of all linear functionals μ that take nonnegative values on it, i.e.,

$$\Sigma_{n,d}^* := \{\mu \in \mathcal{H}_{n,d}^*, \ \ \langle \mu, p\rangle \geq 0 \ \forall p \in \Sigma_{n,d}\}. \tag{7}$$

Here, the dual space $\mathcal{H}_{n,d}^*$ denotes the space of all linear functionals on $\mathcal{H}_{n,d}$, and $\langle .,. \rangle$ represents a pairing between the elements of the primal and the dual space. If a polynomial is not sos, we can find a dual functional $\mu \in \Sigma_{n,d}^*$ that separates it from the closed convex cone $\Sigma_{n,d}$. The basic idea behind this is the well known separating hyperplane theorem in convex analysis; see e.g. [24]. In Section III, we will see a concrete example of the use of duality when we prove that our polynomial is not sos-convex. For a more thorough treatment of the duality theory in semidefinite and sum of squares programming, we refer to reader to [21] and [2] respectively.

C. Sum of squares matrices and sos-convexity

The notions of positive semidefiniteness and sum of squares of scalar polynomials can be naturally extended to polynomial matrices, i.e., matrices with entries in $\mathbb{R}[x]$. We say that a symmetric polynomial matrix $P(x) \in \mathbb{R}[x]^{m\times m}$ is PSD if $P(x)$ is PSD for all $x \in \mathbb{R}^n$. It is straightforward to see that this condition holds if and only if the polynomial $y^T H(x) y$ in $m+n$ variables $[x; y]$ is psd. The definition of an sos-matrix is as follows [25], [26], [27].

Definition 1: A symmetric polynomial matrix $P(x) \in \mathbb{R}[x]^{m\times m}$, $x \in \mathbb{R}^n$ is an sos-matrix if there exists a polynomial matrix $M(x) \in \mathbb{R}[x]^{s\times m}$ for some $s \in \mathbb{N}$, such that $P(x) = M^T(x)M(x)$.

Lemma 2.2: A polynomial matrix $P(x) \in \mathbb{R}[x]^{m\times m}$, $x \in \mathbb{R}^n$ is an sos-matrix if and only if the scalar polynomial $y^T P(x) y$ is a sum of squares in $\mathbb{R}[x; y]$.

Proof: One direction is trivial: if $P(x)$ admits the factorization $M^T(x)M(x)$, then the scalar polynomial $y^T M^T(x) M(x) y = (M(x)y)^T (M(x)y)$ is sos. For the reverse direction see [25]. ∎

Lemma 2.2 enables us to easily check whether a given polynomial matrix is an sos-matrix with the machinery explained in Section II-B. Remarkably, in the univariate case ($x \in \mathbb{R}$), any PSD polynomial matrix $P(x) \in \mathbb{R}[x]^{m\times m}$ is an sos-matrix; see e.g. [28]. For more details about univariate sos-matrices, their interesting connection to the famous Kalman-Yakubovich-Popov lemma, as well as an efficient eigenvalue-based method for finding their sos decomposition, we refer the reader to [29].

In the multivariate case, however, not every PSD polynomial matrix must be an sos-matrix. The first counterexample is due to Choi [13]. Even though Choi did not have polynomial matrices in mind, in [13] he showed that not every psd biquadratic form is a sum of squares of bilinear forms. His counterexample can be rewritten as the following polynomial matrix

$$C(x) = \begin{bmatrix} x_1^2 + 2x_2^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_2^2 + 2x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_3^2 + 2x_1^2 \end{bmatrix}, \tag{8}$$

which is PSD for all $x \in \mathbb{R}^3$ but is not an sos-matrix.

We will now specialize polynomial matrices to the Hessian matrix and discuss convexity of polynomials. It is well known that a polynomial $p(x) := p(x_1, \ldots, x_n)$ is convex if

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and only if its Hessian

$$H(x) = \begin{bmatrix} \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 p}{\partial x_1 \partial x_n} \\ \frac{\partial^2 p}{\partial x_2 \partial x_1} & \frac{\partial^2 p}{\partial x_2^2} & \cdots & \frac{\partial^2 p}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 p}{\partial x_n \partial x_1} & \frac{\partial^2 p}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 p}{\partial x_n^2} \end{bmatrix} \tag{9}$$

is PSD for all $x \in \mathbb{R}^n$, i.e., is a PSD polynomial matrix.

Definition 2: ([6]) A polynomial $p(x)$ is sos-convex if its Hessian $H(x)$ is an sos-matrix.

With this definition, it must be clear that sos-convexity is a sufficient condition for convexity of polynomials. In the univariate case, sos-convexity is in fact equivalent to convexity. The reason is that the Hessian of a univariate polynomial is simply a scalar univariate polynomial. As explained in Section II-A, every psd univariate polynomial is sos.

In the multivariate case, even though we know that not every PSD polynomial matrix is an sos-matrix, it has been speculated that because of the special structure of the Hessian as the matrix of the second derivatives, convexity and sos-convexity of polynomials could perhaps be equivalent. We will show in the next section that this is not the case. Note that the example of Choi in (8) does not serve as a counterexample. The polynomial matrix $C(x)$ in (8) is not a valid Hessian, i.e., it cannot be the matrix of the second derivatives of any polynomial. If this was the case, the third partial derivatives would commute. However, we have in particular

$$\frac{\partial C_{1,1}(x)}{\partial x_3} = 0 \neq -x_3 = \frac{\partial C_{1,3}(x)}{\partial x_1}.$$

In [28], Choi, Lam, and Reznick generalize the earlier results of Choi [13] and provide more examples of psd multiforms that are not sos. Some of their examples can be rewritten as PSD polynomial matrices that are not sos-matrices. In a similar fashion, we can show that their matrices also fail to be valid Hessians.

III. A POLYNOMIAL THAT IS CONVEX BUT NOT SOS-CONVEX

We start this section with a lemma that will appear in the proof of our main result.

Lemma 3.1: If $P(x) \in \mathbb{R}[x]^{m \times m}$ is an sos-matrix, then all its $2^m - 1$ principal minors[2] are sos polynomials. In particular, $\det(P)$ and the diagonal elements of P must be sos polynomials.

Proof: The proof follows from the Cauchy-Binet formula and is presented in [14]. ∎

[2]We remind the reader that the principal minors of an $m \times m$ matrix A are the determinants of all $k \times k$ $(1 \leq k \leq m)$ sub-blocks whose rows and columns come from the same index set $S \subset \{1, \ldots, m\}$.

Remark 3.1: The converse of Lemma 3.1 does not hold. The polynomial matrix of Choi given in (8) serves as a counterexample. It is easy to check that all 7 principal minors of $C(x)$ are sos polynomials and yet it is not an sos-matrix. This is in contrast with the fact that a polynomial matrix is PSD if and only if all its principal minors are psd polynomials. The latter statement follows almost immediately from the well-known fact that a constant matrix is PSD if and only if all its principal minors are nonnegative.

We are now ready to state our main result.

Theorem 3.2: There exists a polynomial that is convex but not sos-convex.

Proof: We claim that the following trivariate form of degree 8

$$\begin{aligned} p(x) \;=\; & 32x_1^8 + 118x_1^6x_2^2 + 40x_1^6x_3^2 + 25x_1^4x_2^4 \\ & -43x_1^4x_2^2x_3^2 - 35x_1^4x_3^4 + 3x_1^2x_2^4x_3^2 \\ & -16x_1^2x_2^2x_3^4 + 24x_1^2x_3^6 + 16x_2^8 \\ & +44x_2^6x_3^2 + 70x_2^4x_3^4 + 60x_2^2x_3^6 + 30x_3^8 \end{aligned} \tag{10}$$

has these properties. Let $H(x)$ denote the Hessian of $p(x)$. Convexity follows from the fact that

$$(x_1^2 + x_2^2 + x_3^2)H(x) = M^T(x)M(x), \tag{11}$$

for some polynomial matrix $M(x)$. Equivalently,

$$(x_1^2 + x_2^2 + x_3^2)y^T H(x) y \tag{12}$$

is a sum of squares in $\mathbb{R}[x; y]$, which shows that $H(x)$ is a PSD polynomial matrix. We have used SOSTOOLS along with the SDP solver SeDuMi [30] to get an explicit sos decomposition of (12) with Gram matrices that have rational entries. This decomposition is given in [14], where a more complete version of this work is presented. As explained in Section II-B, rational Gram matrices lead to a *rational* sos decomposition which constitutes a formal proof.

To prove that $p(x)$ is not sos-convex, by Lemma 3.1 it suffices to show that

$$\begin{aligned} H_{1,1}(x) \;=\; & \frac{\partial^2 p}{\partial x_1 \partial x_1} \\ =\; & 1792x_1^6 + 3540x_1^4x_2^2 + 1200x_1^4x_3^2 \\ & +300x_1^2x_2^4 - 516x_1^2x_2^2x_3^2 - 420x_1^2x_3^4 \\ & +6x_2^4x_3^2 - 32x_2^2x_3^4 + 48x_3^6 \end{aligned} \tag{13}$$

is not sos (thought it must be psd because of convexity). Let

$$\begin{aligned} \mathcal{S} := \; & \text{span}\{x_1^6, x_1^4x_2^2, x_1^4x_3^2, x_1^2x_2^4, x_1^2x_2^2x_3^2, \\ & x_1^2x_3^4, x_2^4x_3^2, x_2^2x_3^4, x_3^6\} \end{aligned} \tag{14}$$

be the set of all trivariate sextic forms which only consist of the monomials in (14). Note that $H_{1,1}$ belongs to $\mathcal{S}$. We will prove that $H_{1,1}$ is not sos by presenting a dual functional ξ that separates $H_{1,1}$ from $\Sigma_{3,6} \cap \mathcal{S}$.

Consider the vector of coefficients of $H_{1,1}$[3] with the ordering as written in (13):

$$\vec{H}_{1,1}^T = [1792, 3540, 1200, 300, -516, -420, 6, -32, 48]. \tag{15}$$

Using the same ordering, we can represent our dual functional ξ with the vector

$$\begin{aligned} c^T = \quad & [0.039, 0.051, 0.155, 0.839, 0.990, \\ & 1.451, 35.488, 20.014, 17.723], \end{aligned} \tag{16}$$

which will serve as a separating hyperplane. We have

$$\langle \xi, H_{1,1} \rangle = c^T \vec{H}_{1,1} = -8.948 < 0. \tag{17}$$

On the other hand, we claim that for any form $w \in \Sigma_{3,6} \cap \mathcal{S}$, we will have

$$\langle \xi, w \rangle = c^T \vec{w} \geq 0. \tag{18}$$

Indeed, if w is sos, by Theorem 2.1 it can be written in the form

$$w(x) = z^T Q z = \mathrm{Tr}\ Q \cdot zz^T, \tag{19}$$

for some $Q \succeq 0$, and a vector of monomials

$$z^T = [x_1^3, x_1x_2^2, x_1x_3^2, x_1^2x_2, x_2x_3^2, x_1x_2x_3, x_3x_1^2, x_3x_2^2, x_3^3] \tag{20}$$

that includes all monomials of degree 3 except for x_2^3, which is not required. It is not difficult to see that

$$c^T \vec{w} = \mathrm{Tr}\, Q \cdot (zz^T)|_c, \tag{21}$$

where by $(zz^T)|_c$ we mean a matrix where each monomial in zz^T is replaced with the corresponding element of the vector c (or zero, if the monomial is not in $\mathcal{S}$). This yields the matrix

$$(zz^T)|_c = \tag{22}$$

$$\begin{bmatrix} 0.039 & 0.051 & 0.155 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.051 & 0.839 & 0.990 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.155 & 0.990 & 1.451 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.051 & 0.990 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.990 & 20.014 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.990 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.155 & 0.990 & 1.451 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.990 & 35.488 & 20.014 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.451 & 20.014 & 17.723 \end{bmatrix}$$

We can check that this matrix is positive definite. Therefore, equation (21) along with the fact that Q is PSD implies that (18) holds. This completes the proof. ■

We end our discussion with a few remarks on some of the properties of the polynomial $p(x)$ in (10).

Remark 3.2: The Gram matrices in the sos decomposition of (12) (presented in [14]) turn out to be positive definite. This shows that for all $x \neq 0$, $H(x)$ is a positive definite matrix and hence $p(x)$ is in fact *strictly convex*; i.e.,

$$\forall x, \bar{x} \in \mathbb{R}^3 \quad \text{and} \quad \lambda \in (0,1)$$

$$p(\lambda x + (1-\lambda)\bar{x}) < \lambda p(x) + (1-\lambda)p(\bar{x}).$$

[3]As a trivariate form of degree 6, $H_{1,1}$ should have 28 coefficients. We refrain from showing the coefficients that are zero since our analysis is done in the lower dimensional subspace of the nonzero coefficients.

Remark 3.3: Because of strict convexity and the fact that $H_{1,1}$ is *strictly* separated from $\Sigma_{3,6}$ (see (17)), it follows that $p(x)$ is strictly in the interior of the set of trivariate forms of degree 8 that are convex but not sos-convex. In other words, there exists an ε-neighborhood of polynomials around $p(x)$, such that every polynomial in this neighborhood is also convex but not sos-convex.

Remark 3.4: As explained in Section II-A, we can dehomogenize the form in (10) into a polynomial in two variables by letting

$$p_{dh}(x_1, x_2) := p(x_1, x_2, 1). \tag{23}$$

The bivariate polynomial p_{dh} has degree 8 and we can check that it is still convex but not sos-convex. It is interesting to note that p_{dh} is an example with the minimum possible number of variables since we know that all convex univariate polynomials are sos-convex. As for minimality in the degree, we do not know if an example with lower degree exists. However, we should note that a bivariate form of degree 4 cannot be convex but not sos-convex. The reason is that the entries of the Hessian of such polynomial would be bivariate quadratic forms. It is known that a matrix with such entries is PSD if and only if it is an sos-matrix [13].

Remark 3.5: Unlike nonnegativity and sum of squares, sos-convexity may not be preserved under homogenization. To give a concrete example, one can check that $\bar{p}_{dh}(x_2, x_3) := p(1, x_2, x_3)$ is sos-convex, i.e., the 2×2 Hessian of $\bar{p}_{dh}(x_2, x_3)$ is an sos-matrix.

Remark 3.6: It is easy to argue that the polynomial p in (10) must itself be nonnegative. Since p is strictly convex, it has a unique global minimum. Clearly, the gradient of p has no constant terms and hence vanishes at the origin. Therefore, $x = 0$ must be the unique global minimum of p. Because we have $p(0) = 0$, it follows that p is in fact positive definite.

Remark 3.7: In [6], Helton and Nie prove that if a nonnegative polynomial is sos-convex, then it must be sos. Since p is not sos-convex, we cannot directly use their result to claim that p is sos. However, we have independently checked that this is the case simply by getting an explicit sos decomposition of p using SOSTOOLS.

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