A Poset Framework to Model Decentralized Control Problems

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Abstract— **In this paper we use partially ordered sets (posets) to study decentralized control problems arising in different settings. We show that time delayed systems with certain delay structure enables one to endow the problem with poset structure thereby allowing convex reparametrization of the problem. We show how to extend these results to spatially invariant systems. We also study the connection between posets and quadratic invariance and show that in some settings they are equivalent.**

I. Introduction

Traditional control theory deals with the problem of synthesizing centralized controllers, i.e. controllers that assume that all output measurements are available for processing. Many modern control problems however, are large-scale, complex, and decentralized. For such problems, implementing centralized controllers is not feasible, and the study of decentralized decision-making becomes important. Examples of such large-scale systems include flocks of aerial vehicles and the power distribution grid.

It is well-known that in general decentralized control is a hard problem [16]. Blondel and Tsitsiklis [2] have shown that certain instances of such problems are in fact intractable. On the other hand, Voulgaris [14], [15] presented several cases where decentralized control is tractable. In a previous paper [8], the authors were able to generalize these results in an appealing framework using partially ordered sets. This paper includes extensions of that work to other settings. Rotkowitz and Lall [12] have presented a criterion known as quadratic invariance that characterizes a class of problems in decentralized control that have the property that problems become convex in the Youla parameter. Our results are related to this property and we show the connection to their work in our paper. We also show how some preexisting results on quadratic invariance of networked systems with time delays can be interpreted as results about underlying partially ordered sets.

In a previous paper [8], we developed a framework for decentralized control problems using posets. We argued that posets provide the right language and technical tools to talk about a more general notion of *causality* (also referred to as hierarchical control in the literature) among subsystems. Associated to the notion of a poset (which is a combinatorial object) is the notion of an incidence algebra [10], an algebraic object. This algebraic structure allowed us to convexify the problem. We showed that some interesting examples of decentralized control that had been shown to be tractable

in the literature were in fact specific instances of this poset paradigm. In this paper, we extend this poset framework to other cases, for instance systems with time delays.

The main contributions of this paper are the following:
1) We study systems with time delays. It had been shown in a previous paper [6], that subject to certain conditions on the delays between subsytems (namely subadditivity), the resulting problem was quadratically invariant (and thus amenable to convex optimization). We show that there is a natural poset associated with systems with time delays with this subadditivity property, and that the computational tractability is simply an algebraic consequence of this underlying poset.
2) We mention an extension of the preceding to spatially distributed systems. It was shown by Bamieh that spatially invariant systems with a "funnel causal" impulse response was amenable to convex optimization. Using the same poset approach, we are able to generalize the funnel causality condition.
3) We study the relationship between posets and quadratic invariance. We show that quadratic invariance can be naturally interpreted as a transitivity property, and that under certain natural settings, poset structures and quadratic invariance are *exactly equivalent*. We introduce the notion of a quoset, which is a poset modulo an equivalence relation. We show that under similar but somewhat more general conditions, quadratic invariance is equivalent to *quosets*.

Posets are very well-studied objects in combinatorics. The associated notion of incidence algebras and Galois connections were first studied by Rota [10] in a combinatorics setting. Since then, these ideas have been used in engineering and computer science (see [8] and the references therein).

The rest of this paper is organized as follows. In Section II we introduce the order-theoretic and control theoretic preliminaries that will be used throughout the paper. We also review some of our past results in this section. In Section III we study systems with time delays. In Section IV we study an extension of the results of Section III to spatially invariant systems. In Section V, we study the connection between quadratic invariance and posets/incidence algebras. In Section VI we conclude our paper.

II. Preliminaries

A. Order-theoretic Preliminaries

Definition 1: A partially ordered set (or *poset*) $\mathcal{P} = (P, \leq)$ consists of a set P along with a binary relation $\leq$

which has the following properties:

1) $a \preceq a$ (reflexivity),
2) $a \preceq b$ and $b \preceq a$ implies $a = b$ (antisymmetry),
3) $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity).

Posets may be finite or infinite, depending upon the cardinality of the underlying set P.

Example 1: An example of a poset with three elements (i.e., $P = \{a, b, c\}$) with order relations $a \preceq b$ and $a \preceq c$ is shown in Figure 1.
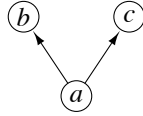

Fig. 1.   A poset on the set $\{1, 2, 3\}$.

Definition 2: Let $\mathcal{P}$ be a poset. Let $\mathbb{Q}$ be a ring. The set of all functions

$$f : \mathcal{P} \times \mathcal{P} \to \mathbb{Q}$$

with the property that $f(x, y) = 0$ if $x \not\preceq y$ is called the *incidence algebra* of $\mathcal{P}$ over $\mathbb{Q}$. It is denoted by $I_{\mathcal{P}}(\mathbb{Q})$. If the ring is clear from the context, we will simply denote this by $I_{\mathcal{P}}$ (we will usually work over the field of rational proper transfer functions, or related extended spaces). When the poset $\mathcal{P}$ is finite, the set of functions in the incidence algebra may be thought of as matrices with a specific sparsity pattern given by the order relations of the poset.

Definition 3: Let $\mathcal{P}$ be a poset. The function $\zeta(\mathcal{P}) \in I_{\mathcal{P}}(\mathbb{Q})$ defined by

$$\zeta(\mathcal{P})(x, y) = \begin{cases} 0, & \text{if } x \not\preceq y \\ 1, & \text{otherwise} \end{cases}$$

is called the *zeta-function* of $\mathcal{P}$.

Clearly, the zeta-function of the poset is an element of the incidence algebra.

Example 2: The matrix representation of the zeta function for the poset from Example 1 is as follows:

$$\zeta_{\mathcal{P}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The incidence algebra is the set of all matrices in $\mathbb{Q}^{3 \times 3}$ which have the same sparsity pattern as its zeta function.

Given two functions $f, g \in I_{\mathcal{P}}(\mathbb{Q})$, their sum $f + g$ and scalar multiplication cf are defined as usual. The product $h = f \cdot g$ is defined as follows:

$$h(x, y) = \sum_{z \in \mathcal{P}} f(x, z) g(z, y). \tag{1}$$

As mentioned above, we will frequently think of the functions in the incidence algebra of a poset as square matrices (of appropriate dimensions) inheriting a sparsity pattern dictated by the poset. The above definition of function multiplication is made so that it is consistent with standard matrix multiplication.

Theorem 1: Let $\mathcal{P}$ be a poset. Under the usual definition of addition, and multiplication as defined in (1) the incidence algebra is an associative algebra (i.e. it is closed under addition, scalar multiplication and function multiplication).

Proof: Closure under addition and scalar multiplication is obvious. Let $f, g \in I_{\mathcal{P}}$, and consider elements x, y such that $x \not\preceq y$. If $x \not\preceq y$, there cannot exist a z such that $x \preceq z \preceq y$. Hence, in the above sum, either $f(x, z) = 0$ or $g(z, y) = 0$ for every z, and thus $h(x, y) = 0$. ∎

A standard corollary of this theorem is the following.

Corollary 1: Suppose $A \in I_{\mathcal{P}}$ is invertible. Then $A^{-1} \in I_{\mathcal{P}}$.

In some situations, we will be dealing with two different posets whose order relations are closely related to one another (for example if one poset is a subset of another poset). One natural way of modeling such a situation is using the notion of *Galois connections*.

Definition 4: Let $\mathcal{P} = (P, \preceq)$ and $Q = (Q, \sqsubseteq)$ be finite posets. A pair of maps (ϕ, ψ) where $\phi : P \to Q$ and $\psi : Q \to P$ is said to form a *Galois connection* if it satisfies the following property:

$$q \sqsubseteq \phi(p) \Leftrightarrow \psi(q) \preceq p \text{ for all } p \in \mathcal{P} \text{ and } q \in Q.$$

Figure 2 shows two posets $\mathcal{P}$ and Q related by a Galois connection. Note that poset Q is isomorphic to the subposet of $\mathcal{P}$ with the elements $\{1, 3\}$.
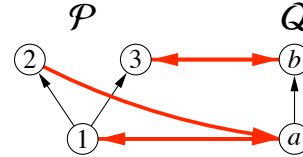

Fig. 2.   Posets $\mathcal{P}$ and Q with a pair of maps that form a Galois connection.

B. Control-theoretic Preliminaries

In this paper we are interested in decentralized structures on linear time invariant systems. We will not particularly emphasize the continuous or discrete time cases as our results apply equally well to both the settings. We will consider systems with the following description: $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the plant output, $w \in \mathbb{R}^{n_w}$ is the exogenous input, $z \in \mathbb{R}^{n_z}$ is the system output. We will be interested in representing our systems via transfer function matrices as

$$P(\omega) = \begin{bmatrix} P_{11}(\omega) & P_{12}(\omega) \\ P_{21}(\omega) & P_{22}(\omega) \end{bmatrix},$$

where $P(\omega) \in \mathbb{C}^{(n_w + n_u) \times (n_z + n_y)}$ is the overall system transfer function. Through the rest of this paper, we abbreviate notation and define $P_{22} = G$. Furthermore, in several cases we will assume that system G has an equal number of inputs and outputs (i.e. $n_u = n_y$). In these cases, we will think of G being composed of several subsystems, each subsystem having one input and one output. While dealing with finite dimensional LTI systems the signal and operator spaces will be the standard ones. In some sections we will be dealing with systems with time-delays, in these cases the systems are no longer finite-dimensional, and the relevant spaces will

need to be appropriately extended (see [6], [7], [12]). Given a controller $K \in \mathcal{R}_p^{n_y \times n_u}$, the closed-loop system has transfer function:

$$f(P, K) = P_{11} + P_{12}K(I - GK)^{-1}P_{21}.$$

We are interested in optimal controller-synthesis problems of the form:

$$
\begin{aligned}
\text{minimize} \quad & \| f(P, K) \| \\
\text{subject to} \quad & K \text{ stabilizes } P \\
& K \in S,
\end{aligned}
\tag{2}
$$

where S is some subspace of the space of controllers. In this paper, $\| \cdot \|$ is any norm on $\mathcal{R}_p^{n_z \times n_w}$, chosen to appropriately capture the performance of the closed-loop system. In this paper S will represent different constraints on the controller K (for example sparsity, or delay bounds). It may be noted that for general P and S there is no known technique for solving problem (2).

Problem (2) as presented is a nonconvex problem in K. If the subspace constraint $K \in S$ were not present, then several techniques exist for solving the problem [4]. One approach towards a solution to the problem is to write an explicit parameterization of all stabilizing controllers for the problem. It is desirable to have the closed-loop transfer function be an affine function in the parameter, so that the problem becomes convex. There are different approaches to perform the parametrization, for example the Youla parametrization [12] and the so-called R-parametrization [8].
Rotkowitz and Lall presented a notion called *quadratic invariance* which gives a sufficient condition for reparametrizing a decentralized control problem in a way that is convex in the Youla parameter [12]. On a related note, Shah and Parrilo showed that problems with an underlying poset structure were also amenable to convex reparametrization using the R-parametrization [8], [9]. (It turns out that such problems are in fact quadratically invariant, and thus the Youla parametrization approach may be taken, though there are certain disadvantages to it [8].)

Definition 5: Given a system G and a subspace of constraints for the controller S, it is said to be *quadratically invariant* with respect to G if for all $K \in S$, $KGK \in S$.

Remark Note that if one has an *associative algebra* of matrices $\mathcal{I}$, and if the plant $G \in \mathcal{I}$ and the controller constraint subspace $S = \mathcal{I}$ then due to the algebraic structure, $KGK \in S$ trivially, hence the problem is quadratically invariant.

In the rest of this paper, we will not emphasize this aspect of reparametrization/convexification. The main thrust of the paper is to show how posets, when chosen with insight, give nice classes of problems which are amenable to convex reparametrization.

C. Review of Earlier Work

In this section we briefly review how the authors used posets to study a class of decentralized control problems [8]. Decentralized systems are composed by interconnecting several subsystems. Each subsystem is assumed to be linear and time invariant. Consider an input-output framework where each subsystem is given as a transfer function matrix G. If $G \in \mathcal{R}_p^{n \times n}$, the system is composed of n subsystems. Subsystem i consists of input i and output i (the transfer function between which is G_{ii}). In addition, input i can also affect another subsystem (say subsystem j) in which case $G_{ij}(\omega) \neq 0$ for almost all ω. As in [12], we would like to consider communication constraints between the subsystems being modeled as sparsity constraints on the matrix $G(\omega)$. To this end we define some notation.

Suppose we have a collection of subsystems that are interconnected in a way that is consistent with the partial order structure of a poset $\mathcal{P} = (\{1, \ldots, n\}, \preceq)$. The partial order represents the communication structure in the plant as follows:

Definition 6: The plant $G \in \mathcal{R}_p^{n \times n}$ is said to be *communication-constrained* by poset $\mathcal{P}$ if whenever $j \npreceq i$, subsystem i does not communicate information to subsystem j (i.e. $G_{ji}(\omega) = 0$).

This definition formalizes the notion of information-richness that we mentioned in the previous section, i.e. that $j \preceq i$ implies that j has access to more information that i since $G_{ji} \neq 0$.
In [8] we first studied the case where the controller K must mirror the communication constraints of the plant, i.e. if $i \npreceq j$ then $K_{ij}(\omega) = 0$ (i.e K must also lie in the incidence algebra of $\mathcal{P}$). Since G is given to be in the incidence algebra and the task at hand is to design a K in the incidence algebra, the algebraic structure guarantees that KGK is in the incidence algebra, this giving quadratic invariance and the associated convexity guarantees. Some examples are shown in Figure 3.
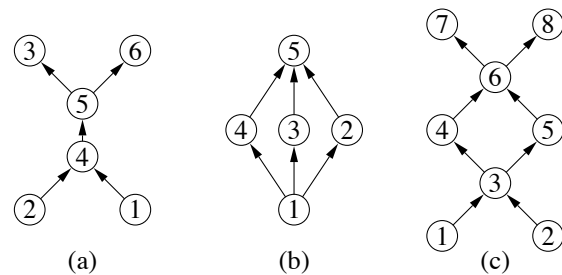
Fig. 3. Examples of other poset communication structures: (a) A multichain (b) A lattice (c) A directed acyclic graph.

In some cases we may be interested in problems where the number of inputs and outputs are different, and there is no natural subsystem interpretation. In this setup we model the decentralization constraints using Galois connections. We impose a poset structure on the inputs and a different poset structure on the outputs and express communication constraints (i.e. whether input i can communicate with output j in the plant) using the maps between the posets. It turns out that if these maps form a Galois connection then the problem is quadratically invariant [8].

III. Systems with Time Delays

In this section we consider LTI systems with time delays. Given a decentralized plant with communication delays between the different subsystems, we consider the task of designing controllers for the subsystems which interact according to a similar delay structure. It has been known [6], [7] that such communication structures are amenable to convex reparametrization due to their quadratic invariance. In this section we show that posets arise naturally in this setup, that they describe the communication constraints in an intuitive way, and the partial order structure results in convexity.

Consider a system with n subsystems (let $\mathcal{N} = \{1, \ldots, n\}$). Let the system be described by the transfer function matrix G where $G_{ij}(\omega)$ describes the frequency response between input of system j and output of system i. An equivalent way to describe the plant is to specify the impulse responses $g_{ij}(t)$. Define the delay between the subsystems i and j (denoted by D_{ij}) as follows (see Figure 4):

$$D_{ij} = \sup \left\{ \tau : g_{ij}(t) = 0 \text{ for all } t \le \tau \right\}.$$

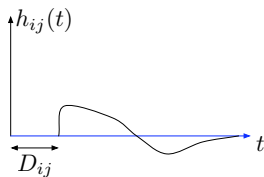Note that since all systems are assumed to be causal, the delays D_{ij} are nonnegative.

Fig. 4. Impulse response $h_{ij}(t)$ along with the associated delay D_{ij}.

We define a relation $\le$ on $\mathcal{N} \times \mathbb{R}$ as follows.

Definition 7: We say that $(j, t_1) \le (i, t_2)$ if

$$t_2 - t_1 \ge D_{ij}.$$

Since the systems we are dealing with are time invariant, what this condition means intuitively is that $(j, t_1) \le (i, t_2)$ if system i at time t_2 is in the cone of influence of an impulse applied at system j at time t_1. We show next that if the delays satisfy a triangle inequality then the relation $\le$ described in Definition 7 is a partial order relation.

Proposition 1: Suppose $D_{ii} = 0$ (i.e. effect of input on output within same subsystems is without delay), $D_{ij} > 0$ (there is nonzero delay between distinct subsystems) and the D_{ij} are subadditive, i.e.

$$D_{ij} + D_{jk} \ge D_{ik}, \qquad (3)$$

for all i, j, k distinct. Then $\le$ is a *partial order* relation.

Proof: Since $D_{ii} = 0$, by definition $(i, t_1) \le (i, t_1)$. If $(i, t_1) \le (j, t_2)$ and $(j, t_2) \le (i, t_1))$ then $t_1 - t_2 \ge 0$ and $t_2 - t_1 \ge 0$ (since delays are nonnegative), thus by definition $t_1 = t_2$. Since $D_{ij} > 0$ for $i \ne j$ it must be the case that $i = j$ giving anti-symmetry. If $(i, t_1) \le (j, t_2)$ and $(j, t_2) \le (k, t_3)$, we have $t_1 \le t_2 \le t_3$. Further, $t_2 - t_1 \ge D_{ji}$ and $t_3 - t_2 \ge D_{kj}$. By (3),

$t_3 - t_1 \ge D_{ji} + D_{kj} \ge D_{ki}$ and hence $(i, t_1) \le (k, t_3)$, verifying transitivity. ∎

Note that this triangle inequality structure on the delays is *exactly* the condition that appears in [6]. What is interesting here is that these delays actually give rise to a natural poset structure, as we have just pointed out (the poset is determined purely by the delays, the actual functional form of the impulse response does not matter). Furthermore, the set of impulse responses $g_{ij}(t)$ which satisfy this delay structure actually forms an algebra of functions under convolution, as the next proposition shows.

Definition 8: Let $\Psi = \left\{ D_{ij} \right\}_{1 \le i, j \le n}$ be a given set of delays. Let $\mathcal{I}_\Psi$ denote the set of (matrix) impulse responses $G(t)$ with the property that $g_{ij}(t) = 0$ if $(j, 0) \not\le (i, t)$. Intuitively $g_{ij}(t) = 0$ means that the effect of an impulse at time $t = 0$ at subsystem j has not reached the output of subsystem i at time t. Thus $\mathcal{I}_\Psi$ is precisely the set of systems which obeys the delay structure prescribed by Ψ. Given a set of impulse responses $F = \left\{ f_{ij}(t) \right\}$ and $G = \left\{ g_{ij}(t) \right\}$ define $F * G$ to be the matrix of impulse responses with $(F * G)_{ij}(t) = \sum_{k=1}^{n} f_{ik} * g_{kj}(t)$.

Proposition 2: Given a set of delays Ψ which satisfy the conditions of Proposition 1. If $F = \left\{ f_{ij}(t) \right\}, G = \left\{ g_{ij}(t) \right\}$ such that $F, G \in \mathcal{I}_\Psi$ for $1 \le i, j \le n$, then $F * G \in \mathcal{I}_\Psi$.

Proof: Suppose $(j, 0) \not\le (i, t)$. It suffices to show that $(F * G)_{ij}(t) = 0$. Now,

$$(F * G)_{ij}(t) = \sum_{k=1}^{n} \int_{\mathbb{R}_+} f_{ik}(t - \tau) g_{kj}(\tau) d\tau.$$

If $(F * G)_{ij}(t) \ne 0$ then there must be some k, τ such that $g_{kj}(\tau) \ne 0$ and $f_{ik}(t - \tau) \ne 0$. This in turn means that $\tau \ge D_{kj}$ and $t - \tau \ge D_{ik}$. Thus $(j, 0) \le (k, \tau)$ and $(k, \tau) \le (i, t)$. By transitivity, $(j, 0) \le (i, t)$, contrary to our assumption. ∎

Since the impulse responses form a convolutional algebra, the transfer function matrices $F(\omega)$ and $G(\omega)$ form a multiplicative algebra and are thus quadratically invariant. This allows us to conclude the following proposition.

Proposition 3: Given a set of delay constraints Ψ which satisfy the conditions of Proposition 1. Given a plant $G \in \mathcal{I}_\Psi$ with the same delay constraints, the problem of finding a controller $K \in \mathcal{I}_\Psi$ is quadratically invariant.

A natural question to ask is: "What happens when the plant and controller delays are not the same? In particular, what if $G \notin \mathcal{I}_\Psi$?" It turns out that even in the case when (i) the delays in G are longer (or equal to) than those allowed in K and (ii) do not necessarily satisfy the triangle inequality, the problem is quadratically invariant. In other words, a triangle inequality structure is not required on the delays in G. As long as the delays in G are greater than or equal to the delays in K and the delay constraints on the controller K obey a poset structure, the problem remains quadratically invariant [6]. The proof is a straightforward adaptation of the proof of Proposition 2. Let the delays in the the plant be R_{ij}. Assume that $R_{ij} \ge D_{ij}$ (notice that we do not need to assume transitivity of the R_{ij}, though we continue to assume the poset structure on the D_{ij}). We show that if $(j, 0) \not\le (i, t)$ then $(k * g * k)_{ij}(t) = 0$.

$$(k * g * k)_{ij}(t) = \sum_{l=1}^{n} \sum_{m=1}^{n} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} k_{il}(t - \tau_2) g_{lm}(\tau_2 - \tau_1) k_{lj}(\tau_1) d\tau_1 d\tau_2$$

Suppose $(j,0) \not\preceq (i,t)$ but $(k * g * k)_{ij}(t) \neq 0$. Then $k_{il}(t - \tau_2) \neq 0$, $g_{lm}(\tau_2 - \tau_1) \neq 0$, and $k_{mj}(\tau_2) \neq 0$ for some l, m, τ_1, τ_2. Then $(l, \tau_2) \preceq (i, t)$, $(j, 0) \preceq (m, \tau_1)$. Also, since $g_{lm}(\tau_2 - \tau_1) \neq 0$, $\tau_2 - \tau_1 \geq R_{lm}$. But $R_{lm} \geq D_{lm}$. Thus $\tau_2 - \tau_1 \geq D_{lm}$, i.e. $(m, \tau_1) \preceq (l, \tau_2)$. Using transitivity, we get $(j, 0) \preceq (i, t)$, giving a contradiction.

Example 3: Consider the example of a decentralized control system shown in Figure 6. This system has a plant with three subsystems (G_1, G_2 and G_3) and corresponding controllers K_1, K_2 and K_3. The figure shows the delays on the arrows. For example the delay between G_1 and G_2 is 3 time units. Note that the arrow from G_1 to G_3 is one-directional, meaning that G_3 is not allowed to communicate with G_1 (i.e. D_{13} the delay between 3 and 1 is infinite). Notice that the delays on the controller side satisfy the triangle inequality and that for all i, j, the delays between G_i and G_j are larger than those asked between K_i and K_j. Thus the incidence algebra results apply, and the described problem is quadratically invariant.
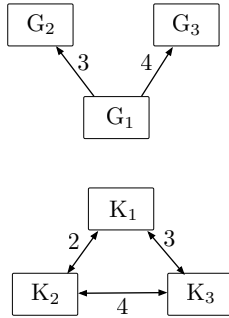

Fig. 5. Decentralized system (3 subsystems) with time delays in the plant and specified delay constraints in the controller.

IV. Spatially Invariant Systems

It is possible to extend the results of the preceding subsection to a class of infinite dimensional systems that are spatially distributed [9]. These results were proposed in [9] by these authors. Similar results were independently and simultaneously developed by Rotkowitz et. al. in [3]. These results generalized in multiple directions the previous results of Bamieh and Voulgaris [13]. We briefly review our results in this subsection, since they nicely complement the preceding results.
We consider systems that evolve along spatial coordinates ($x \in \mathcal{X}$) as well as temporal coordinates ($t \in T$). (Though in general we can work with systems which evolve on locally compact abelian groups, to fix ideas, we will think of the underlying spaces as being real valued.) Much like temporal invariance, we say that a system is spatio-temporally invariant if the effect of an impulse at spatial coordinate x_1 at time t_1 at another location x_2 at time t_2 depends only on $x_2 - x_1$ and $t_2 - t_1$. Such systems may be specified by their *spatio-temporal impulse response* $h(x, t)$. This function describes the response of the system at location (x, t) under the influence

of an impulse at $(0, 0)$. Given a system $h(x, t)$ one defines the *support function* $f(x)$ as follows:

$$f(x) = sup \{\tau | h(x, t) = 0 \text{ for all } t < \tau\}. \quad (4)$$

The support function evaluated at x tells one the *delay* involved in the effect of an impulse at the origin to reach x. For example, if the system under consideration were a wave, then the support function would be exactly the light cone centered at the origin.
Bamieh and Voulgaris [13] considered spatially invariant systems with impulse responses whose support functions were *concave* (they called such functions "funnel causal"). They showed that such impulse responses were *convolutionally closed* i.e., if $h(x, t)$ and $g(x, t)$ are two systems with support function $f(x)$, then their convolution

$$(h * g)(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} h(x - \xi, t - \tau)g(\xi, \tau)d\tau d\xi$$

is also supported on $f(x)$. As a consequence of this closure property, the set of spatially invariant controllers with the specified support function $f(x)$ can be reparametrized as a convex set in the Youla domain.
Their result depends on two key assumptions:

- The spatial coordinates have to be one-dimensional,
- The support function must be concave.

Using our poset framework we are able to generalize these results. It turns out that the key property is *subadditivity* (i.e. $f(x_1 + x_2) \leq f(x_1) + f(x_2)$) of the support function. (Note that in the previous section, we assumed that the delays satisfy the triangle inequality. Subadditivity of the support functions is the natural generalization to this case.) If the support functions involved satisfy sub-additivity then it is possible to endow $\mathcal{X} \times T$ with a natural poset structure. Define a relation $\preceq$ on $\mathcal{X} \times T$ as follows. Let $(x_1, t_1) \preceq (x_2, t_2)$ if:

$$t_2 - t_1 \geq f(x_2 - x_1). \quad (5)$$

Notice the similarity between inequality (5) and Definition 7. Both inequalities say that $t_2 - t_1$ should be greater than the delay between the subsystems, thus making (5) the natural extension of the poset definition to the spatial invariance case.

Proposition 4: [9, Proposition 1] Suppose the support function is such that $f(0) = 0$, $f(x) > 0$ for $x \neq 0$ and subadditive. Then $\preceq$ is a partial order relation.

Under these assumptions on the support function, the impulse responses form a convolutional algebra, the usual Youla parametrizations are employed, and convexification follows. Our results hold for multi dimensional spatial coordinates (for example, norms are examples of subadditive support functions in higher dimensions). More interestingly, subadditive support functions are a strictly larger class of functions that contain concave functions as special cases [9]. Figure 6 shows an example of a subadditive support function that is *not* concave. For further details, we encourage the reader to read [9].
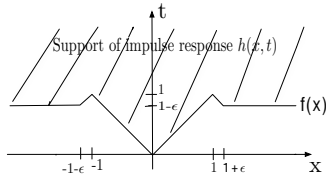
Fig. 6. Support of impulse response.

V. CONNECTION BETWEEN QUADRATIC INVARIANCE AND PARTIAL ORDER STRUCTURE

In this section we want to study the connection between quadratic invariance and posets. We have seen that poset structure implies that the problem is quadratically invariant. We are now interested in understanding the converse, i.e. "does quadratic invariance imply existence of poset-like structure?" Quadratic invariance is really a transitivity property. As argued earlier, posets provide the right language to describe transitive relations. In what follows, we make this connection more concrete. Connections between quadratic invariance and *partially nested structures* as defined in a team-theoretic setting by Ho and Chu [5] have been studied and pointed out by Rotkowitz [11]. The team theoretic problem considers a scenario where there are multiple decision makers who must each make a decision in some order. The paper considers a scenario where the order in which decisions are made satisfy certain *precedence relations*. (Though this terminology is not used in these papers, these precedence relations are in fact closely related to partial order relations.) The paper by Ho [5] shows that problems with this precedence structure (called partially nested problems) are amenable to convex optimization, and moreover, that optimal controllers are linear. Rotkowitz shows that existence of these precedence relations is equivalent to quadratic invariance. Our results are similar in spirit but differ in that they are not restricted to this team-theoretic setting. Moreover, we provide a finer characterization of quadratic invariance in terms of posets and quosets.

Consider the problem of designing an optimal controller $K \in S$ as described in problem 2. In this section we revisit the model where decentralization constraints are viewed as sparsity constraints in the controller. Let $K \in \mathcal{R}^{n_y} \times \mathcal{R}^{n_u}$. Define a subset if indices of K via $\mathcal{J} \subseteq \{1, \ldots, n_y\} \times \{1, \ldots, n_u\}$. Then the subspace constraint is defined as $K_{ij} = 0$ for all $(i, j) \in \mathcal{J}$. Quadratic invariance reduces to the following transitive property in this model [12, Theorem 26]:

Theorem 2: The subspace S is quadratically invariant with respect to a specified plant G if and only if for all $K \in S$ and all i, j, k, l,

$$K_{ij} G_{jk} K_{kl} (1 - K_{il}) = 0. \qquad (6)$$

Remark Let us interpret equation (6) in an intuitive way. Let us denote the constraint $K_{ij} \neq 0$ by $i \rightarrow_K j$ (which denotes that there is a path from i to j in the controller) and $G_{jk} \neq 0$ by $j \rightarrow_G k$ (i.e. that there is a path from j to k in the plant).

Then the equation (6) states that:

$$i \rightarrow_K j, j \rightarrow_G k, k \rightarrow_K l \text{ implies } i \rightarrow_K l. \qquad (7)$$

The transitive structure becomes more apparent now. What quadratic invariance is saying is that the overall graph of the closed loop (which is comprised of a combination of subgraphs of the plant and the controller) is transitively closed. The condition means that if l is not allowed to communicate to i in the controller then there must exist no path from l to i around the closed loop (because such a path would produce a way for l to communicate to i by going once around the closed loop).

When the graph inside the plant and the controller is identical, quadratic invariance reduces to transitive closure of this (identical) graph. We next show that in this scenario quadratic invariance corresponds to existence of poset structure.

A. Existence of Posets

Consider a plant G and a decentralized control problem with sparsity constraints $K \in S$ such that $K_{ij} = 0$ for all $(i, j) \in \mathcal{J}$ for some index set $\mathcal{J}$. Consider the square case i.e. $n_y = n_u$. Let $\mathcal{N} = \{1, \ldots, n_y\}$. We say that a given decentralized control problem is *plant-controller symmetric* if the given plant also satisfies the sparsity constraints of the controller (i.e. $G \in S$).

Proposition 5: Consider a plant-controller symmetric control problem. Suppose the following assumptions are true of the index set:

1) $(i, i) \notin \mathcal{J}$
2) $(i, j) \notin \mathcal{J} \Leftrightarrow (j, i) \in \mathcal{J}$ for distinct i, j.
3) The problem is quadratically invariant.

Then there exists a poset $\mathcal{P}$ over n_y elements such that S is the incidence algebra of $\mathcal{P}$.

Proof: Since both G and K are $n_y \times n_y$ matrices, it is enough to construct a poset on n_y elements and show that the sparsity pattern of S exactly corresponds to the incidence algebra of this poset. Let us define our candidate for the partial order $\preceq$ as follows: $i \preceq j$ if $(i, j) \notin \mathcal{J}$. We need to verify that this is indeed a partial order relation. Since $(i, i) \notin \mathcal{J}$, we clearly have $i \preceq i$ thus verifying reflexivity. If $i \preceq j$ and $j \preceq i$ then it must be the case that $(i, j) \notin \mathcal{J}$ and $(j, i) \notin \mathcal{J}$. However the second assumption in the statement of the proposition excludes the possibility of such i, j being distinct, thus $i = j$ and we have anti-symmetry. Finally, suppose we have $i \preceq j$ and $j \preceq l$ (i.e. $(i, j) \notin \mathcal{J}$ and $(j, l) \notin \mathcal{J}$). Choose index k such that $k = j$ and use quadratic invariance to conclude from equation (7) that $(i, l) \notin \mathcal{J}$. Thus $i \preceq l$, verifying transitivity. The incidence algebra of this poset is the set of elements such that $K_{ij} = 0$ if $i \not\preceq j$, (i.e. $(i, j) \in \mathcal{J}$) which is exactly the definition of S. $\blacksquare$

B. Existence of Quosets

It is natural to ask: "to what extend does the preceding theorem generalize?" It turns out that one can in fact relax

the second assumption (anti-symmetry). It is possible to have a more general notion of a partial order in the absence of anti-symmetry. In that setting, distinct elements can be equivalent, and the partial order is defined on the quotient set modulo the equivalence. The resulting object is very like a poset (called a quotient poset or *quoset*, sometimes it is called a *preorder* in the literature). There is a corresponding algebraic object, analogous to the incidence algebra, called the *structural matrix algebra* [1].

Definition 9: A *quoset* $Q = (Q, \precsim)$ is a set Q with a binary relation $\precsim$ such that $\precsim$ is reflexive and transitive. Thus it is possible for distinct elements i, j to satisfy $i \precsim j$ and $j \precsim i$ (we will call such elements equivalent and denote this by $i \simeq j$). This notion of a quoset captures the intuition that if i and j can communicate to each other and if they have the same level of information richness then they are equivalent. One defines the analogue of an incidence algebra as follows.

Definition 10: Let $\mathbb{F}$ be a ring. Let $\mathcal{M}$ be the set of functions $f : Q \times Q \to \mathbb{F}$ with the property that $f(i, j) = 0$ if $i \not\precsim j$ for all i, j.

We leave it as an easy exercise to the reader to verify that $\mathcal{M}$ is an associative algebra. Figure 7 shows an example of a quoset and the sparsity pattern of the associated structural matrix algebra. The analogue of Proposition 5 to quosets is
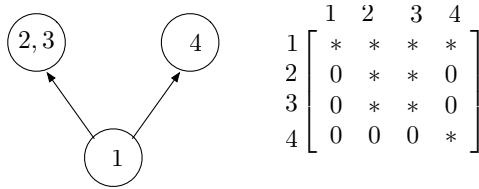

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{cccc}
* & * & * & * \\
0 & * & * & 0 \\
0 & * & * & 0 \\
0 & 0 & 0 & *
\end{array} \right]
\end{array}
$$

Fig. 7.   A quoset and the sparsity pattern of its associated structural matrix algebra. Elements with a '$*$' indicate possible nonzero elements.

the following.

Proposition 6: Consider a plant-controller symmetric control problem. Suppose the following assumptions are true of the index set:

1) $(i, i) \notin \mathcal{J}$
2) The problem is quadratically invariant.

Then there exists a quoset Q over n_y elements such that S is the structural matrix algebra of Q.

Proof: Again we construct a candidate quoset and verify the associated properties. We say that $i \precsim j$ if $(i, j) \notin \mathcal{J}$. The verification of the properties are very similar to that of Proposition 5, we leave this routine step to the reader. ∎
We have thus seen that condition (2) from Proposition 5 can be relaxed, and that in the relaxed setting quadratic invariance is equivalent to existence of quoset structure in the problem. What happens when condition (1) is relaxed (i.e. we allow some elements $(i, i) \in \mathcal{J}$)? It turns out that it is possible to add the reflexive relation without adding or affecting any of the transitivity relations. (This operation of adding reflexive relations is called the *reflexive closure*).

Proposition 7: Suppose we have a plant-controller symmetric control problem with a specified index set (of sparsity constraints) $\mathcal{J}$. (The sparsity constraints are thus $K_{ij} = 0$ for $(i, j) \in \mathcal{J}$.) The problem is quadratically invariant if and only if the reflexive closure of $\mathcal{J}^c$ is a quoset over the elements of $\mathcal{N}$. The set S is the structural matrix algebra corresponding to this quoset.

Proof: We first note that taking reflexive closure of a transitively closed set does not affect any of the relations between distinct elements. Define $\mathcal{I} = \mathcal{J}^c$. Define $i \precsim j$ if $(i, j) \in \mathcal{I}$. Suppose we add the reflexive relations so that $\mathcal{I}' = \mathcal{I} \cup (\bigcup_{i \in \mathcal{N}} \{(i, i)\})$. Consider the transitive closure of $\mathcal{I}'$. The only way new relations can be added is by combining transitive relations with the newly added reflexive relations. Thus if $i \precsim j$ and $j \precsim k$, we know that for distinct i, j, k we already have $i \precsim k$. If $j = i$ or $j = k$ we get no new relations. Hence $\mathcal{I}'$ is its own transitive closure.
Suppose the reflexive closure is a quoset. We know that in the closure operation, no new relations between distinct elements were introduced, hence transitivity is unaffected. By (6) the problem is quadratically invariant. Conversely, if the problem is quadratically invariant, we know from (6) that $\mathcal{I}$ is transitively closed. Thus if we take the reflexive closure, by Proposition 6 the resulting set is a quoset. ∎

VI. Conclusion

We presented a poset based framework to study decentralized control problems. We were able to extend our previous work on decentralized control using posets to other settings. We showed a way to interpret some preexisting results on systems with time delays using posets. We were able to extend these results to the case of spatially distributed systems, We also studied the connection between quadratic invariance and posets and showed that they were equivalent in certain settings. Under somewhat more general conditions, we showed that quadratic invariance is equivalent to the existence of quoset strucure.

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