A Partial Order Approach to Decentralized Control

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Abstract— **In this paper we employ the theory of partially ordered sets to model and analyze a class of decentralized control problems. We show that posets provide a natural way of modeling problems where communication constraints between subsystems have a hierarchical structure. We show that such problems have appealing algebraic properties that can be exploited to parameterize the set of stabilizing controllers. While much of the paper is devoted to problems where the plant and controller have identical communication constraints, we also generalize our theory to case where they may have different communication constraints.**

I. INTRODUCTION

Traditional control theory deals with the problem of synthesizing centralized controllers, i.e., controllers that assume that all output measurements are available for processing. Many important control problems however, are large-scale, complex, and decentralized. For such problems, implementing centralized controllers is often not feasible, and the study of decentralized decision-making becomes important. Examples of such large-scale systems include flocks of aerial vehicles and the power distribution grid.

It is well-known that in general decentralized control is a hard problem, and significant research efforts have been directed towards its many different aspects; see for instance the classical survey [3] for many of the earlier results. More recently, Blondel and Tsitsiklis [4] have shown that in certain instances, decentralized control problems are in fact computationally intractable. On the other hand, Voulgaris [15], [16] presented several cases where decentralized control is in fact tractable. We are able to unify all these examples under an appealing theoretical framework. Rotkowitz and Lall [13] have presented a criterion known as quadratic invariance that characterizes a class of problems in decentralized control that have the property that problems become convex in the Youla parameter. Our results are related to this property and we show the connection to their work in our paper.

In this paper we consider linear time invariant systems with communication constraints, both within the plant and the controller. In most of the paper we are interested in the setting where the system is composed of several interacting subsystems. This subsystem approach enables us to partition the overall transfer function into several local transfer functions. The communication constraints among the subsystems manifest themselves via sparsity constraints in the plant and controller transfer functions. In this paper we consider a specific class of such communication constraints that arise naturally in many decentralized decision-making problems.

The fundamental object of interest in this paper is the notion of a *partially ordered set* or *poset*. We argue that posets provide the right language and technical tools to talk about a more general notion of *causality* (also referred to as hierarchical control in the literature) among subsystems. We show that many interesting examples of decentralized control that have been shown to be tractable in the literature are in fact specific instances of this poset paradigm.

To any given poset (which is a combinatorial object) we can associate its incidence algebra [12], an algebraic object. By exploiting the algebraic structure, we show that several appealing properties follow for this class of problems. These include a nice parameterization of all controllers satisfying the sparsity constraints, convexity guarantees, quadratic invariance, and some deeper theoretical insight relating feedback invariance to certain lattices of invariant subspaces. In most of this paper we consider problems where the communication constraints within the controller are required to mirror the communication constraints within the plant. In the last section we provide a brief outline of a poset-based approach to deal with problems where constraints within the controller may be different from the plant constraints. We show that the natural generalization to this setting involves considering two posets (on the inputs and outputs respectively) along with a *Galois connection*.

The main contributions of this paper are the following:

1) We introduce the notion of a partially ordered set (poset) as a means of modeling causality-like communication constraints between subsystems in a decentralized control setting.
2) We exploit algebraic properties of the problem to show that the set of controllers that satisfy the sparsity constraints can be parameterized explicitly.
3) We present some deeper insight on the feedback invariance properties of such problems in a coordinate-free setting.
4) We generalize the poset based technique from the setting where plant and controller have same communication constraints to the setting where they may have different constraints.

Posets are very well studied objects in combinatorics. The associated notions of incidence algebras and Galois connections were first studied by Rota [12] in a combinatorics setting. Since then, order-theoretic concepts have been used in engineering and computer science; we mention a few

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specific works below. Cousot and Cousot used these ideas to develop tools for formal verification of computer programs in their seminal paper [6]. In control theory too, these ideas have been used by some authors in the past, albeit in somewhat different settings. Ho and Chu used posets to study team theory problems [9]. They were interested in sequential decision making problems where agents must make decisions at different time steps. They study the form of optimal decision-makers when the problems have poset structure. Mullans and Elliot [11] use posets to generalize the notion of time and causality, and study evolution of systems on locally finite posets. Del Vecchio and Murray [14] have used ideas from lattice and order theory to construct estimators for discrete states in hybrid systems.

The rest of this paper is organized as follows. In Section II we introduce the order-theoretic and control theoretic preliminaries that will be used throughout the paper. In Section III we present some examples of communication structures that can be modeled via posets. In Section IV we show how the algebraic properties of systems on posets may be exploited to construct a parameterization of all stabilizing controllers. In Section V we present the notions of structural matrix algebras and their lattices of invariant subspaces. These notions allow us to generalize our theory to a coordinate independent setting. In Section VI, we generalize our results to the case where plant and controller do not necessarily have the same communication constraints. In Section VII we conclude our paper and mention future directions of research.

II. Preliminaries

A. Order-theoretic Preliminaries

Definition 1: A partially ordered set (or *poset*) consists of a set $\mathcal{P}$ along with a binary relation $\preceq$ which is reflexive, antisymmetric and transitive.

In this paper we will be dealing with *finite* posets, i.e. where the set $\mathcal{P}$ is of finite cardinality.

Example 1: An example of a poset with three elements (i.e. $\mathcal{P} = \{a, b, c\}$) with order relations $a \preceq b$ and $a \preceq c$ is shown in Figure 1.

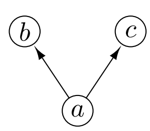

Fig. 1. A poset on the set $\{1, 2, 3\}$.

Definition 2: Let $\mathcal{P}$ be a poset. Let $\mathbb{Q}$ be a ring. The set of all functions

$$f : \mathcal{P} \times \mathcal{P} \to \mathbb{Q}$$

with the property that $f(x, y) = 0$ if $x \npreceq y$ is called the *incidence algebra* of P over $\mathbb{Q}$. It is denoted by $I_P(\mathbb{Q})$. If the ring is clear from the context, we will simply denote this by $I_{\mathcal{P}}$ (we will usually work over the field of rational proper transfer functions).

Since the poset P is finite, the set of functions in the incidence algebra may be thought of as matrices with a specific sparsity pattern given by the order relations of the poset.

Definition 3: Let $\mathcal{P}$ be a poset. The function $\zeta(P) \in I_{\mathcal{P}}(\mathbb{Q})$ defined by

$$\zeta(P)(x, y) = \begin{cases} 0, & \text{if } x \npreceq y \\ 1, & \text{otherwise} \end{cases}$$

is called the *zeta-function* of $\mathcal{P}$.

Clearly the zeta-function of the poset is a member of the incidence algebra

Example 2: The matrix representation of the zeta function for the poset from Example 1 is as follows:

$$\zeta_{\mathcal{P}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The incidence algebra is the set of all matrices in $\mathbb{Q}^{3\times 3}$ which have the same sparsity pattern as its zeta function.

Given two functions $f, g \in I_{\mathcal{P}}(\mathbb{Q})$, their sum $f+g$ and scalar multiplication cf are defined as usual. The product $h = f \cdot g$ is defined as follows:

$$h(x, y) = \sum_{z \in \mathcal{P}} f(x, z) g(z, y). \tag{1}$$

As mentioned above, we will frequently think of the functions in the incidence algebra of a poset as square matrices (of appropriate dimension) inheriting a sparsity pattern dictated by the poset. The above definition of function multiplication is made so that it is consistent with matrix multiplication.

Theorem 1: Let $\mathcal{P}$ be a poset. Under the usual definition of addition, and multiplication as defined in (1) the incidence algebra is an associative algebra (i.e. it is closed under addition, scalar multiplication and function multiplication).

Proof: Closure under addition and scalar multiplication is obvious. Let $f, g \in I_{\mathcal{P}}$. Consider elements x, y such that $x \npreceq y$, so that $f(x, y) = g(x, y) = 0$. Indeed if $x \npreceq y$, there cannot exist a z such that $x \preceq z \preceq y$. Hence, in the above sum, either $f(x, z) = 0$ or $g(z, y) = 0$ for every z, and thus $h(x, y) = 0$. ■

A standard corollary of this theorem is the following.

Corollary 1: Suppose $A \in I_{\mathcal{P}}$ is invertible. Then $A^{-1} \in I_{\mathcal{P}}$.

B. Control-theoretic Preliminaries

In paper we are interested in discrete-time linear systems defined by difference equations as follows:

$$\begin{aligned} x[k+1] &= Ax[k] + B_1 w[k] + B_2 u[k] \\ z[k] &= C_1 x[k] + D_{11} w[k] + D_{12} u[k] \\ y[k] &= C_2 x[k] + D_{21} w[k] + D_{22} u[k]. \end{aligned}$$

In this description, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the plant output, $w \in \mathbb{R}^{n_w}$ is the exogenous input, $z \in \mathbb{R}^{n_z}$ is the system output. Rather than working with state-space models, we will throughout be interested in working with their frequency domain representation via the z-transform.

It can be shown [8] that if the above system is stabilizable and detectable (we assume this throughout the paper) then the system (denoted by P) can be described in frequency domain by the following map:

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix},$$

where $P(z) \in \mathbb{C}^{(n_w+n_u)\times(n_z+n_y)}$ is the overall system transfer function. Through the rest of this paper, we abbreviate notation and define $P_{22} = G$. We also assume G to be strictly proper, so that the stabilization of P is equivalent to the stabilization of G (see [8]). Furthermore, in Section III we assume that system G has an equal number of inputs and outputs (i.e. $n_u = n_y$). We will think of G being composed of several subsystems, each subsystem having one input and one output. We denote $\mathcal{R}_p^{m\times n}$ to be the set of rational-proper transfer matrices of dimension $m \times n$. We denote the set of stable proper transfer matrices by $\mathcal{RH}_\infty$. The entries of P can be shown to be rational proper transfer functions, i.e. $P \in \mathcal{R}_p^{(n_w+n_u)\times(n_z+n_y)}$. Given a controller $K \in \mathcal{R}_p^{n_y\times n_u}$, the closed-loop system has transfer function:

$$f(P, K) = P_{11} + P_{12}K(I - GK)^{-1}P_{21}.$$

We are interested in optimal controller-synthesis problems of the form:

$$\begin{array}{ll} \text{minimize} & \| f(P, K) \| \\ \text{subject to} & K \text{ stabilizes } P \\ & K \in S, \end{array} \tag{2}$$

where S is some subspace of the space of controllers. In this paper, $\| \cdot \|$ is any norm on $\mathcal{R}_p^{n_z\times n_w}$, chosen to appropriately capture the performance of the closed-loop system. In this paper S will represent sparsity constraints on the controller K. It may be noted that for general P and S there is no known technique for solving problem (2).

Problem (2) as presented is a nonconvex problem in K. If the subspace constraint $K \in S$ were not present, then several techniques exist for solving the problem [8]. One approach towards a solution to the problem is to write an explicit parameterization of all stabilizing controllers for the problem. It is desirable to have the closed-loop transfer function be an affine function in the parameter, so that the problem becomes convex. We present one such parameterization here [5].

Let $\mathcal{H}_{stab}$ denote the set of stable closed loop transfer matrices achieved by some controller that internally stabilizes the plant, i.e.

$$\mathcal{H}_{stab} = \{H | H = P_{11} + P_{12}K(I - GK)^{-1}P_{21}, \\ K \text{ stabilizes } P\}.$$

Let $R = K(I - GK)^{-1}$. Then it is well-known [5] that $\mathcal{H}_{stab}$ can be parameterized in terms of R via

$$\mathcal{H}_{stab} = \{P_{11} + P_{12}RP_{21} | RG \in \mathcal{RH}_\infty, R \in \mathcal{RH}_\infty, \\ I + GR \in \mathcal{RH}_\infty, (I + GR)G \in \mathcal{RH}_\infty\}. \tag{3}$$

Furthermore, given R, the controller K may be recovered by

$$K := h_G(R) = (I + RG)^{-1}R. \tag{4}$$

Lemma 1: Assume $G \in I_{\mathcal{P}}$. Then $R \in I_{\mathcal{P}}$ if and only if $K \in I_{\mathcal{P}}$.

Proof: Follows from the definition of R, (4) and Corollary 1. ■

Lastly, if G is stable, then $\mathcal{H}_{stab}$ has a simple parameterization:

$$\mathcal{H}_{stab} = \{P_{11} + P_{12}RP_{23} | R \in \mathcal{RH}_\infty\}. \tag{5}$$

III. Examples of Communication Structures Arising from Posets

In this section we study some examples of posets. Some communication structures that have been studied in the past (and arise in practice) can be described using posets [15], [16]. The intuition behind modeling communication among subsystems via posets is as follows. We say that subsystems i and j satisfy $i \preceq j$ if all the information that is available to subsystem j is also available to subsystem i. It means that subsystem i is more *information-rich*. We will formalize this notion in the next section.

A. Independent subsystems

For the trivial poset on the set $\{1, 2, \ldots, n\}$ where there are no partial order relations between any of the elements (i.e. all the elements are independent of each other) corresponds to the case where the subsystems exchange no communication whatsoever (all subsystems have access to only their own information, thus K and G are diagonal). It is readily seen that this is just the case where one is required to stabilize n independent plants using independent controllers, a problem that reduces to a classical control problem. At the other extreme is the case where the poset is *totally ordered*. This is the case of nested control [15], which we study below.

B. Nested systems

This is a class of systems where the transfer functions have a block-triangular structure. Nested systems have been analyzed by Voulgaris [16], [15]. Such structures arise in cases where there are several subsystems, with each subsystem contained within the subsequent subsystem so that the arrangement forms a nest. There is one-way communication among the subsystems (say from the inside to the outside, or vice-versa).

For simplicity, consider a system with just two subsystems P_1 and P_2. The internal subsystem is P_1 which can communicate information to the outer subsystem P_2 (but not vice-versa). The task at hand is to design a controller that obeys this same nested-communication architecture. The following is the set of plant outputs, control inputs, exogenous outputs and exogenous inputs respectively:

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad z := \begin{bmatrix} z_1 \\ z_{12} \\ z_2 \end{bmatrix} \quad w := \begin{bmatrix} w_1 \\ w_{12} \\ w_2 \end{bmatrix}$$

The sparsity pattern generated by the communication constraints for the controller and the plant are as follows:

$$G = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \quad K = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Figure 2 depicts such a nested system in a block diagram. It is easy to see that G and K are matrices in the incidence

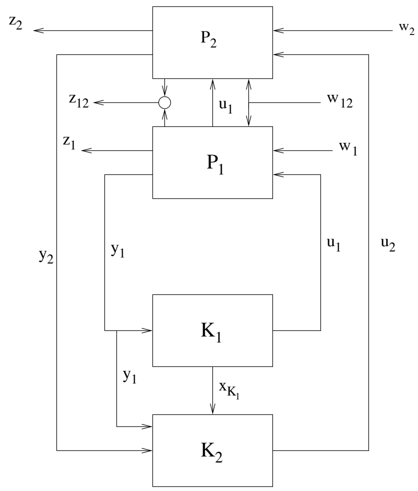

Fig. 2. A system with nested communication constraints.

algebra generated by the poset over $\{1,2\}$ with $2 \preceq 1$. This is consistent with the intuition that since subsystem 2 has access to information from subsystem 1 to make decisions (K_{21} is not required to be 0), subsystem 2 is more information-rich. Voulgaris [15] showed that for such nested systems, the optimal control problem can be reduced to a convex problem in the Youla parameter. In the next section we will see that this result follows as a special case of a more general result that is true for all poset-systems.

C. Other examples

The example cited in the above subsection shows that nested systems are just special cases of those arising from posets. Clearly, many other communication structures can be modeled as posets. Some such examples include multi-chains, lattices and closures of directed acyclic graphs. A few such examples are shown in Fig. 4.

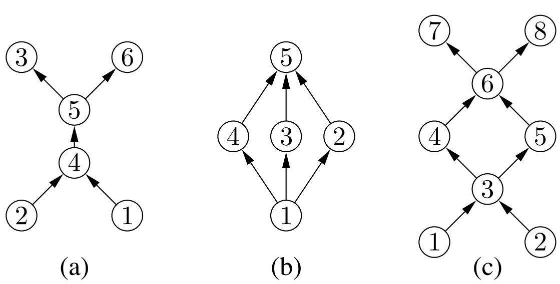

Fig. 3. Examples of other poset communication structures: (a) A multichain (b) A lattice (c) A directed acyclic graph.

IV. Systems with Same Plant and Controller Communication Constraints

In this section we consider systems that are composed by interconnecting several subsystems. Each subsystem is assumed to be linear and time invariant. We consider an input-output framework where each subsystem is given as a transfer function matrix G. If $G \in \mathcal{R}_p^{n\times n}$, the system is composed of n subsystems. Subsystem i consists of input i and output i (the transfer function between which is G_{ii}). In addition, input i can also affect another subsystem (say subsystem j) in which case $G_{ij}(z) \neq 0$ for almost all z. As in [13], we would like to consider communication constraints between the subsystems being modeled as sparsity constraints on the matrix $G(z)$. To this end we define some notation.

Suppose we have a collection of subsystems that are interconnected in a way that is consistent with the partial order structure of a poset $\mathcal{P} = (\{1,\ldots,n\},\preceq)$. The partial order represents the communication structure in the plant as follows:

Definition 4: The plant $G \in \mathcal{R}_p^{n\times n}$ is said to be *communication-constrained* by poset $\mathcal{P}$ if whenever $j \npreceq i$, subsystem i does not communicate information to subsystem j (i.e. $G_{ji}(z) = 0$).

This definition formalizes the notion of information-richness that we mentioned in the previous section, i.e. that $j \preceq i$ implies that j has access to more information that i since $G_{ji} \neq 0$.

In this section we are interested in the case where the controller K must mirror the communication constraints of the plant, i.e. if $i \npreceq j$ then $K_{ij}(z) = 0$ (i.e K must also lie in the incidence algebra of $\mathcal{P}$). We denote the set of all stabilizing controllers that lie in the incidence algebra by $C_{stab}(\mathcal{P})$. Let the set of all closed loop transfer functions that are stabilized by $K \in C_{stab}(\mathcal{P})$ be denoted by $\mathcal{H}_{stab}(\mathcal{P})$. We are interested in parameterizing $C_{stab}(\mathcal{P})$ and $\mathcal{H}_{stab}(\mathcal{P})$. Our approach will be as follows. First we will construct an explicit stabilizing controller in $C_{stab}(\mathcal{P})$. Using this controller in the feedback loop, we reduce the problem to the case where the plant is stable and then use equation (5) to parameterize the set of all closed loop maps. Before we do so, we must make an important remark and a related assumption.

Remark Suppose we have a plant $G \in I_{\mathcal{P}}(\mathcal{R}_p)$ with G_{ij} unstable for some $i \neq j$. The task of internally stabilizing the plant G with a controller $K \in I_{\mathcal{P}}(\mathcal{R}_p)$ is *impossible*. This is because G_{ij} does not have a feedback path. To illustrate this consider an example with two subsystems forming a nest (i.e. the block-triangular case we saw Section III-B). Suppose

$$G = \begin{bmatrix} 0 & 0 \\ G_{21} & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix},$$

where G_{21} is unstable and K is some stabilizing controller in the incidence algebra. By Theorem 1 and Corollary 1 it is easy to see that $R = K(I - GK)^{-1}$ is also in the incidence algebra (i.e. it is lower triangular). However, if this were the case, one can readily check that

$$(I + GR)G = \begin{bmatrix} 0 & 0 \\ G_{21} & 0 \end{bmatrix}.$$

It is impossible for this to be stable, thus the controller cannot be internally stabilizing, yielding a contradiction.

In addition, for the problem to make sense we also need that the G_{ii} be internally stabilizable.

Assumptions

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(a) For $i \neq j, G_{ij}(z)$ is stable
(b) The G_{ii} are stabilizable.

The argument in the remark above shows that if we are interested in plants with poset constraints, and if we want to stabilize the plant with a controller that also satisfies the same poset constraints, then Assumption (a) is necessary. This assumption is reasonable because it encompasses the case where subsystems pass static signals among each other (i.e. $G_{ij}(z)$ are constants). In fact, there may even be dynamics between subsystems as long as these dynamics are stable.

A. *Explicit construction of a stabilizing controller*

Armed with Assumption (a), we can now explicitly construct a stabilizing controller as follows. Suppose we are given $G \in I_{\mathcal{P}}(\mathcal{R})_p)$. Without loss of generality, we assume that G is lower triangular. (It is always possible to put it in triangular form by constructing a linear extension of the poset i.e. extending the partial order to a total order which is consistent [1, Prop 1.4].) We choose an R which is diagonal and such that $R_{ii}, R_{ii}G_{ii}, (I+G_{ii}R_{ii}), (I+G_{ii}R_{ii})G_{ii}$ are stable. We can always choose such an R because the existence of such R_{ii} are equivalent to the G_{ii} being internally stabilizable. Moreover, since R is diagonal, it is also in the incidence algebra. The diagonal entries of $RG, R, I+GR$, and $(I+GR)G$ are stable by construction. It can be easily verified that the off-diagonal entries of these matrices are stable because they are sums and products of stable entries (recall that by assumption (a) the G_{ij} are stable for $i \neq j$). Hence all four of the transfer functions are stable, and by (3) we have a stable closed loop. We choose the corresponding stabilizing controller to be simply $K_{nom} = (I+RG)^{-1}R$.

B. *Parametrization of all stabilizing controllers in the incidence algebra*

Note that since R is diagonal (and hence trivially in the incidence algebra), by Lemma 1, K_{nom} is in the incidence algebra and also stabilizing. We use K_{nom} in the closed loop to stabilize the plant, so that the problem is reduced to the case of a stable plant. Now we treat the system with K_{nom} in the closed loop as the "new plant", which is already stable. Let

$$\tilde{P}(z) = \begin{bmatrix} \tilde{P}_{11}(z) & \tilde{P}_{12}(z) \\ \tilde{P}_{21}(z) & \tilde{G}(z) \end{bmatrix},$$

where $\tilde{P}$ is the closed loop map obtained via K_{nom}. Since this is stable, by (5) we have that the set of all stable closed loop maps achievable by a controller in $I_{\mathcal{P}}$ is

$$\mathcal{H}_{stab}(\mathcal{P}) = \left\{ \tilde{P}_{11} + \tilde{P}_{12}R\tilde{P}_{21} | R \in \mathcal{RH}_\infty \cap I_{\mathcal{P}} \right\}.$$

Finally, the set of all stabilizing controllers in the incidence algebra is

$$\mathcal{C}_{stab}(\mathcal{P}) = \left\{ (I+R\tilde{G})^{-1}R | R \in \mathcal{RH}_\infty \cap I_{\mathcal{P}} \right\}.$$

Using this parameterization, one can reduce the optimal control problem (2) to the convex problem:

$$\begin{array}{ll} \text{minimize} & \| \tilde{P}_{11} + \tilde{P}_{12}R\tilde{P}_{21} \| \\ \text{subject to} & R \in \mathcal{RH}_\infty \cap I_{\mathcal{P}}. \end{array}$$

Remark Rotkowitz and Lall present a property known as *quadratic invariance* in their paper [13]. A plant P and a subspace (of controllers) S is defined to be quadratically invariant if for every $K \in S$, $KGK \in S$. Note that our problem is quadratically invariant. We have $G \in I_{\mathcal{P}}(\mathcal{R}_p^{n\times n})$ and the subspace constraint is also $K \in I_{\mathcal{P}}(\mathcal{R}_p^{n\times n})$. By Theorem 1, $I_{\mathcal{P}}(\mathcal{R}_p^{n\times n})$ is an algebra of matrices, hence $KGK \in I_{\mathcal{P}}(\mathcal{R}_p^{n\times n})$. One of the advantages of our approach is that the problem has more structure, and hence (unlike their work) we do not need to assume the existence of a nominal stable controller (we explicitly construct a controller, and it need not be stable for our parameterization to work). Also our parameterization is free, i.e. it does not have any constraints apart from $R \in \mathcal{RH}_\infty \cap I_{\mathcal{P}}$.

V. Structural Matrix Algebras

In this section, we generalize the results of the previous section from incidence algebras of posets to a more coordinate independent setting. We defined posets and their corresponding incidence algebras in a such a way that the matrices involved were sparse (the sparsity being inherited from the poset). However, sparsity is a coordinate-dependant concept (because a change of basis would cause a sparse matrix to become dense). It is natural to question wether the results of the previous sections were really a consequence of the sparsity of the problem. In this section we show that on the contrary, the theory presented above is really a consequence of some deeper, coordinate-independent properties. We wish to restate the results and intuition of the previous section in a way that depends purely on properties of certain subspaces of the underlying problem. We will show that posets and incidence algebras are closely related to lattices of these subspaces.

Another reason for studying this abstraction is that it allows us to extend the notion of causality among subsystems to the notion of causality in signal spaces. We will see that certain subspaces of signals have an invariance property that ensures that any signals in those spaces remain invariant under feedback. In this abstract point of view, there are four different objects which are equivalent: (a) posets (b) incidence algebras (c) lattices of ideals of the poset (d) lattices of invariant subspaces.

Much of this section is devoted to explaining the equivalence between these four objects (Fig. 4 summarizes this equivalence). Posets and lattices of ideals are purely combinatorial objects. Incidence algebras are algebraic objects, however they are coordinate dependent. We will show in this section, there is an associated object that is both combinatorial and algebraic, namely the lattice of invariant subspaces, which is coordinate independent. We show that these invariant subspaces have a nice control-theoretic interpretation.

We begin with some definitions related to lattices.

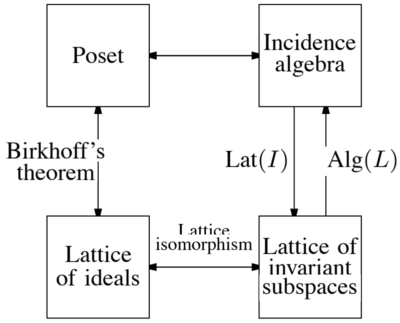

Fig. 4. Figure showing the relationship between posets, incidence algebras, lattices of ideals and lattices of invariant subspaces.

Definition 5: A *lattice* is a partially ordered set $(L, \preceq)$ with the following properties:

1) For all $a, b \in L$ there exists a unique least upper bound called the *join* of a and b and denoted by $a \wedge b$ (i.e. $a \preceq g, b \preceq g \Rightarrow a, b \preceq a \wedge b \preceq g$).

2) For all $a, b \in L$ there exists a unique greatest lower bound called the *meet* of a and b and denoted by $a \vee b$ (i.e. $l \preceq a, l \preceq b \Rightarrow l \preceq a \vee b \preceq a, b$).

A lattice which has an element $0 \in L$ such that $0 \preceq a$ for all $a \in L$ is called a *lattice with 0*. Similarly, A lattice which has an element $1 \in L$ such that $a \preceq 1$ for all $a \in L$ is called a *lattice with 1*.

Definition 6: An element p of a lattice L is called *irreducible* if for all $x, y \in L$,

$$p = x \vee y \quad \text{implies} \quad p = x \quad \text{or} \quad p = y.$$

Definition 7: A lattice $(L, \preceq)$ is said to be distributive if it has the following identities hold for all $x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Note: Either property implies the other. These properties also generalize in the obvious way to finitely many variables.

Figure 6 shows some examples of lattices. Lattices arise

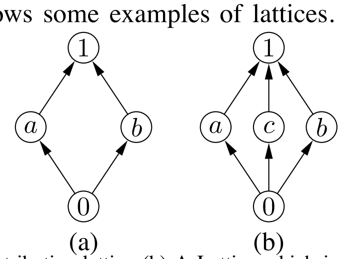

Fig. 5. (a) A distributive lattice (b) A Lattice which is not distributive.

naturally in the study of combinatorics [1], [10], [7]. Given a set S, the set 2^S of all subsets with partial order taken to be set inclusion forms a distributive lattice (the join and meet are respectively set unions and intersections). Sublattices of distributive lattices are also distributive (this follows immediately from their definition).

Lattices also arise in linear algebra [2] because the set of subspaces of a vector space is a distributive lattice under subspace inclusion. Let V be a finite dimensional vector space, and denote the set of all subspaces of V by $\mathcal{P}(V)$. Let $U, W \in \mathcal{P}(V)$. The meet is simply $U \wedge W = U \cap W$. The join of two subspaces is given by $U \vee W = \text{span}(U, W)$. Sublattices of $\mathcal{P}(V)$ are also obviously distributive. We seek to exploit these connections between the combinatorial and linear-algebraic aspects of the problem to arrive at the required abstraction.

Definition 8: A subset I of a poset $\mathcal{P}$ is an ideal of $\mathcal{P}$ if for all $x, y \in \mathcal{P}$

$$x \in I \quad \text{and} \quad y \preceq x \quad \text{imply} \quad y \in I.$$

For $A \subseteq \mathcal{P}$ the set $\{x \in L | x \preceq a \text{ for some } a \in A\}$ is an ideal, called the *ideal generated by* A, and denoted by $I(A)$. Ideals generated by single elements are called principal ideals.

It is easy to show that the set theoretic union and intersection of ideals are also ideals. Hence the family $\mathcal{I}(\mathcal{P})$ of ideals of a poset form a lattice under inclusion. Since it is a sublattice of $2^{\mathcal{P}}$ the lattice of ideals is a distributive lattice. Hence starting with a poset, one can easily construct the associated lattice of ideals $\mathcal{I}(\mathcal{P})$. Conversely, starting with a lattice one can construct the associated *poset of irreducible elements* by simply restricting to the irreducible elements, with the ordering being induced by the original lattice. This gives a natural way of associating posets with lattices. The following theorem by Birkhoff tells us posets and the associated lattices of ideals are equivalent in this sense.

Theorem 2: (Birkhoff) [1] Let $(L, \preceq)$ be a finite distributive lattice with 0 and $P \subseteq L$ the sub-poset of its irreducible elements different from zero. Then $L \cong \mathcal{I}(\mathcal{P})$ by means of the lattice isomorphism

$$\phi : a \longrightarrow I(\{a\}).$$

Conversely, for a poset $\mathcal{P}$, the lattice of ideals $\mathcal{I}(\mathcal{P})$ is distributive, with its poset of irreducible elements $\neq 0$ isomorphic to $\mathcal{P}$.

Proof: See [1] pages 33-34. ∎

Having established the connection between posets and lattices of ideals, let us explore the connection between incidence algebras and lattices of subspaces. Let V be a finite dimensional vector space over a field F. Let $M_n(F)$ be the set of all $n \times n$ matrices with entries in F. Let $\mathcal{A} \subseteq 2^{M_n(F)}$ be a family of matrices. Let $\mathcal{B} \subseteq \mathcal{P}(V)$ be a set of subspaces of V. We define the following two objects:

$$\begin{aligned} Lat(\mathcal{A}) &= \{W \subseteq V | A(W) \subseteq W \quad \text{for all} \quad A \in \mathcal{A}\} \\ Alg(\mathcal{B}) &= \{A \subseteq M_n(F) | A(W) \subseteq W \quad \text{for all} \quad W \in \mathcal{B}\}. \end{aligned} \tag{6}$$

Given a family of matrices $\mathcal{A}$, $Lat(\mathcal{A})$ is the set of subspaces that are invariant with respect to every member in the matrix family. Similarly, given a set of subspaces $\mathcal{B}$, $Alg(\mathcal{B})$ is the set of matrices for which all the subspaces in the family are invariant.

Theorem 3 (Akkurt et al. [2]): 1) Let I be an incidence algebra of a poset $\mathcal{P}$. Then the set of invariant subspaces $Lat(I)$ is a distributive lattice.

2) Conversely if L is some distributive lattice of sub-

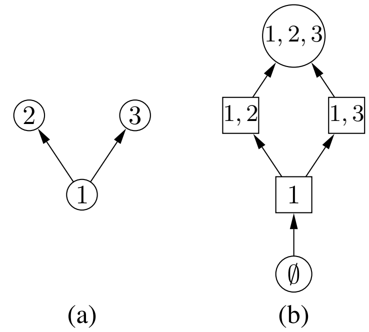

Fig. 6. Example: (a)A poset (b)The corresponding lattice of ideals (square nodes indicate the irreducible elements).

spaces, the associated matrix algebra $M(L)$ is the incidence algebra on the poset of the irreducible elements of L.

Proof: See [2] and the references therein. ■

Theorem 3 justifies the equivalence of distributive lattices of invariant subspaces with the other three objects depicted in Fig. 4. As we have seen already, the poset structure on the subsystems allows us to interpret the plant and controller G and K via the incidence algebra and have a notion of *causality among subsystems*. On the other hand, the invariant subspaces of the incidence algebra are subspaces of signal spaces, and a lattice structure allows us to have a similar causality notion in the signal space. Suppose $u \in \mathbb{C}^{n_u}$ is some plant input and $u \in V$ for some invariant subspace V of the invariant subspace of the incidence algebra. Since $G(V) \subseteq V$, we know that $Gu \in V$ and similarly $KGu \in V$. Thus, problems with such causality structure have the property that *signals remain invariant under feedback*. If the input signal belongs to some subspace in the lattice, after feedback, the new signal will lie in the same subspace (possibly a smaller subspace).

Example 3: Consider the poset on $\mathcal{P} = \{1, 2, 3\}$ shown in Fig. 6(a). The family of ideals of this poset is $\{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. This family can be arranged into a distributive lattice as shown in Fig. 6(b). The irreducible elements ($\neq 0$) of this lattice are the square nodes. As is readily seen, restricting the partial order to the irreducible elements recovers the original poset.

The incidence algebra of this poset is the set of all matrices that have the following sparsity pattern as shown in Example 2 earlier. The subspaces that are invariant with respect every matrix in the incidence algebra are the subspaces

$$0, \quad \text{span}(e_1), \quad \text{span}(e_1, e_2), \quad \text{span}(e_1, e_3), \quad \text{span}(e_1, e_2, e_3),$$

where e_1, e_2, e_3 are the standard unit vectors of the vector space. (The observant reader may notice the correspondence between the indices of the basis vectors of these subspaces and the elements of the lattice of ideals shown in Fig. 6(b)). For example, if $v \in \text{span}(e_1, e_3)$ then v would have the sparsity pattern $v = \begin{bmatrix} * & 0 & * \end{bmatrix}^T$. Then the matrix-vector product Av would be of the form

$$\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \begin{bmatrix} * \\ 0 \\ * \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}.$$

The sparsity pattern is preserved, thus $A(\text{span}(e_1, e_3)) \subseteq \text{span}(e_1, e_3)$. One can arrange the above subspaces into a lattice (with partial order given by subspace inclusion). Upon doing so, one would obtain *the same lattice as the lattice of ideals of* $\mathcal{P}$.

VI. Systems with different plant and controller communication constraints

A. Computational Tractability

In this section we examine a more general setting where the controller is not necessarily required to mirror the communication constraints of the plant. In the previous section, we viewed the system as a collection of interconnected subsystems. We move away from this view here. However, we still wish to study such systems from a partial-order point of view. Instead of having a partial order on subsystems, we now have a partial order on the set of inputs and a (possibly different) partial order on the set of outputs. The communication constraints between the inputs and outputs in the plant and the controller are given by a pair of maps between the two posets. We show that if the pair of maps have the special property of being a *Galois connection* [12],[10] the controller synthesis problem (subject to the communication constraints) is amenable to convex optimization.

Definition 9: Let $\mathcal{P}$ and $\mathcal{Q}$ be finite posets. A pair of maps (ϕ, ψ) where $\phi : P \to Q$ and $\psi : Q \to P$ is said to form a *Galois connection* if it satisfies the following property:

$$p \preceq \phi(q) \Leftrightarrow \psi(p) \preceq q \text{ for all } p \in \mathcal{P} \text{ and } q \in \mathcal{Q}.$$

Let $G(z) \in \mathcal{R}_p^{n_y \times n_u}$ be a system with n_u inputs and n_y outputs. Let $\mathcal{P} = (\{1, \ldots, n_u\}, \preceq_{\mathcal{P}})$ and $\mathcal{Q} = (\{1, \ldots, n_y\}, \preceq_{\mathcal{Q}}$) be posets on the index sets of the inputs and outputs respectively. (We will often hide the subscript on the inequality symbol when it is clear from the context).

Definition 10:

(1) We say that the plant is *communication-constrained by* ϕ if whenever $q \not\preceq \phi(p)$ input p cannot communicate with output q (i.e. $G_{qp} = 0$ for almost all z).

(2) We say that the controller is *communication-constrained by* ψ if whenever $p \not\preceq \psi(q)$ output q cannot communicate with input p (i.e. $K_{pq} = 0$ for almost all z).

The set of all controllers that are communication-constrained by ψ is a subspace. This subspace is denoted by $S(\psi)$. Similarly, the set of all plants that are constrained by ϕ are denoted by $S(\phi)$.

Remark These definitions generalize the notion of an incidence algebra to the case when we have two posets. For instance if the two posets are the same (i.e. $\mathcal{P} = \mathcal{Q}$) and we choose the Galois connection to be the identity map between them, then it can be easily verified that $S(\phi) = S(\psi) = I_{\mathcal{P}}$.

Theorem 4: Let $\mathcal{P}$ and $\mathcal{Q}$ be posets on the index sets of the inputs and outputs respectively. Let $G \in \mathcal{R}_p^{n_y \times n_u} \in S(\phi)$ and $G \in \mathcal{R}_p^{n_u \times n_y} \in S(\psi)$. Then $KGK \in S(\psi)$ and $GKG \in S(\phi)$.

Proof: We prove $KGK \in S(\psi)$ (the proof for $GKG \in S(\phi)$ is similar). Suppose $G \in S(\phi)$ and $K \in S(\psi)$.

$$KGK_{ij} = \sum_k \sum_l K_{ik} G_{kl} K_{lj}.$$

Suppose $i \npreceq \psi(j)$ so that $K_{ij} = 0$. If $i \npreceq \psi(k)$ then $K_{ik} = 0$. Similarly if $l \npreceq \psi(j)$ then $K_{lj} = 0$ and if $k \npreceq \phi(l)$ then $G_{kl} = 0$. Hence, the only nonzero terms in the above summation occur for indices that satisfy $i \preceq \psi(k)$, $k \preceq \phi(l)$, and $l \preceq \psi(j)$. Using Definition 5, since (ϕ, ψ) form a Galois connection $k \preceq \phi(l) \Leftrightarrow \psi(k) \preceq l$. Hence, $i \preceq \psi(k) \preceq l \preceq \psi(j)$. However, these three conditions imply that $i \preceq \psi(j)$ which contradicts our initial assumption. ■

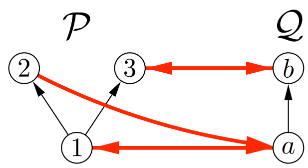

Fig. 7. Posets $\mathcal{P}$ and $\mathcal{Q}$ with a pair of maps that form a Galois connection.

Remark A consequence of Theorem (4) is that problems with such communication constraints are quadratically invariant [13], and can hence be recast as convex problems in the Youla parameter.

B. Example

Consider a system with two inputs ($\mathcal{P} = \{1, 2, 3\}$) and two outputs ($\mathcal{Q} = \{a, b\}$). The input and output spaces are endowed with a partial order structure and a pair of maps (ϕ, ψ) that form a Galois connection (see Fig. 7).

$$\phi(1) = a \qquad \phi(2) = a \qquad \phi(3) = b$$
$$\psi(a) = 1 \qquad \psi(b) = 3.$$

This order-theoretic structure results in communication constraints on the plant G and controller K as shown in (7) below. In the plant, output a can be influenced by inputs 1,2, and 3, and output b can only be influenced by input 1. On the controller side, input 1 can be affected by outputs a and b, input 3 by output b, while input 2 can be affected by none of the outputs. The corresponding G and K would be sparse with the following sparsity structure:

$$G = \begin{bmatrix} * & * & * \\ 0 & 0 & * \end{bmatrix} \quad K = \begin{bmatrix} * & * \\ 0 & 0 \\ 0 & * \end{bmatrix}. \qquad (7)$$

It can be easily verified that the problem with these sparsity constraints is quadratically invariant. The quadratic invariance of the problem depends only on the interconnection between the inputs and outputs in the controller and plant. The emphasis here is that when such constraints are modeled using order-theoretic considerations as explained above, quadratic invariance and the attendant convexity guarantees follow.

VII. Conclusion

We presented a new framework for modeling and analyzing a class of decentralized control problems based on posets. We showed that problems of this type have several appealing properties: (a) They unify many examples studied in the literature under a common theoretical framework, (b) the set of stabilizing controller has a nice parametrization (c) one can generalize the theory to a coordinate independent setting (d) one can also generalize the theory to the setting where plant and controller may have different constraints.

References

[1] M. Aigner. *Combinatorial theory*. Springer-Verlag, 1979.
[2] M. Akkurt, G. P. Barker, and M. Wild. Structural matrix algebras and their lattices of invariant subspaces. *Linear Algebra and its Applications*, 394:25–38, 2005.
[3] N. Sandell, P. Varaiya, M. Athans, and M. Safonov. Survey of decentralized control methods for large scale systems. *IEEE Transactions on Automatic Control*, 23(2):108–128, 1978.
[4] V. D. Blondel and J. N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36(9):1249–1274, 2000.
[5] S. P. Boyd and C. H. Barratt. *Linear Controller Design: Limits of Performance*. Prentice Hall, 1991.
[6] P. Cousot and R. Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *Conference Record of the Fourth Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pages 238–252, Los Angeles, California, 1977. ACM Press, New York, NY.
[7] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, 1990.
[8] B. A. Francis. *A course in $\mathcal{H}_\infty$ control theory*. Springer-Verlag New York, Inc., New York, NY, USA, 1986.
[9] Y.-C. Ho and K.-C. Chu. Team decision theory and information structures in optimal control problems-part I. *IEEE Transactions on Automatic Control*, 17(1):15–22, 1972.
[10] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott. *Continuous Lattices and Domains, In Encyclopedia of Mathematics and its Applications*, volume 93. Cambridge University Press, 2003.
[11] R. E. Mullans and D.L. Elliott. Linear systems on partially ordered time sets. In *Proceedings of the IEEE Conference on Decision and Control including the 12th Symposium on Adaptive Processes*, volume 12, pages 334–337, 1973.
[12] G.-C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Probability theory and related fields*, 2(4):340–368, 1964.
[13] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Transactions on Automatic Control*, 51(2):274–286, 2006.
[14] D. Del Vecchio and R. M. Murray. Discrete state estimators for a class of hybrid systems on a lattice. In *HSCC*, pages 311–325, 2004.
[15] P. Voulgaris. Control of nested systems. In *Proceedings of the American Control Conference*, 2000.
[16] P. Voulgaris. A convex characterization of classes of problems in control with specific interaction and communication structures. In *Proceedings of the American Control Conference*, 2001.

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