

Constrained Consensus and Optimization in Multi-Agent Networks^{*†}

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Abstract

We present distributed algorithms that can be used by multiple agents to align their estimates with a particular value over a network with time-varying connectivity. Our framework is general in that this value can represent a consensus value among multiple agents or an optimal solution of an optimization problem, where the global objective function is a combination of local agent objective functions. Our main focus is on constrained problems where the estimate of each agent is restricted to lie in a different constraint set.

To highlight the effects of constraints, we first consider a constrained consensus problem and present a distributed “projected consensus algorithm” in which agents combine their local averaging operation with projection on their individual constraint sets. This algorithm can be viewed as a version of an alternating projection method with weights that are varying over time and across agents. We establish convergence and convergence rate results for the projected consensus algorithm. We next study a constrained optimization problem for optimizing the sum of local objective functions of the agents subject to the intersection of their local constraint sets. We present a distributed “projected subgradient algorithm” which involves each agent performing a local averaging operation, taking a subgradient step to minimize its own objective function, and projecting on its constraint set. We show that, with an appropriately selected stepsize rule, the agent estimates generated by this algorithm converge to the same optimal solution for the cases when the weights are constant and equal, and when the weights are time-varying but all agents have the same constraint set.

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1 Introduction

There has been much interest in distributed cooperative control problems, in which several autonomous agents collectively try to achieve a global objective. Most focus has been on the canonical consensus problem, where the goal is to develop distributed algorithms that can be used by a group of agents to reach a common decision or agreement (on a scalar or vector value). Recent work also studied multi-agent optimization problems over networks with time-varying connectivity, where the objective function information is distributed across agents (e.g., the global objective function is the sum of local objective functions of agents). Despite much work in this area, the existing literature does not consider problems where the agent values are constrained to given sets. Such constraints are significant in a number of applications including motion planning and alignment problems, where each agent’s position is limited to a certain region or range, and distributed constrained multi-agent optimization problems.

In this paper, we study cooperative control problems where the values of agents are constrained to lie in closed convex sets. Our main focus is on developing distributed algorithms for problems where the constraint information is distributed across agents, i.e., each agent only knows its own constraint set. To highlight the effects of different local constraints, we first consider a constrained consensus problem and propose a *projected consensus algorithm* that operates on the basis of local information. More specifically, each agent linearly combines its value with those values received from the time-varying neighboring agents and projects the combination on its own constraint set. We show that this update rule can be viewed as a version of the alternating projection method where, at each iteration, the values are combined using weights that are varying in time and across agents, and projected on the respective constraint sets.

We provide convergence and convergence rate analysis for the projected consensus algorithm. Due to the projection operation, the resulting evolution of agent values has nonlinear dynamics, which poses challenges for the analysis of the algorithm’s convergence properties. To deal with the nonlinear dynamics in the evolution of the agent estimates, we decompose the dynamics into two parts: a linear part involving a time-varying averaging operation and a nonlinear part involving the error due to the projection operation. This decomposition allows us to represent the evolution of the estimates using linear dynamics and decouples the analysis of the effects of constraints from the convergence analysis of the local agent averaging. The linear dynamics is analyzed similarly to that of the unconstrained consensus update, which relies on convergence of *transition matrices* defined as the products of the time-varying weight matrices. Using the properties of projection and agent weights, we prove that the projection error diminishes to zero. This shows that the nonlinear parts in the dynamics are vanishing with time and, therefore, the evolution of agent estimates is “almost linear”. We then show that the agents reach consensus on a “common estimate” in the limit and that the common estimate lies in the intersection of the agent individual constraint sets.

We next consider a constrained optimization problem for optimizing a global objective function which is the sum of local agent objective functions, subject to a constraint set given by the intersection of the local agent constraint sets. We focus on distributed algorithms in which agent values are updated based on local information given by the

agent’s objective function and constraint set. In particular, we propose a distributed *projected subgradient algorithm*, which for each agent involves a local averaging operation, a step along the subgradient of the local objective function, and a projection on the local constraint set.

We study the convergence behavior of this algorithm for two cases: when the constraint sets are the same, but the agent connectivity is time-varying; and when the constraint sets X_i are different, but the agents use uniform and constant weights in each step, i.e., the communication graph is fully connected. We show that with an appropriately selected stepsize rule, the agent estimates generated by this algorithm converge to the same optimal solution of the constrained optimization problem. Similar to the analysis of the projected consensus algorithm, our convergence analysis relies on showing that the projection errors converge to zero, thus effectively reducing the problem into an unconstrained one. However, in this case, establishing the convergence of the projection error to zero requires understanding the effects of the subgradient steps, which complicates the analysis. In particular, for the case with different constraint sets but uniform weights, the analysis uses an error bound which relates the distances of the iterates to individual constraint sets with the distances of the iterates to the intersection set.

Related literature on parallel and distributed computation is vast. Most literature builds on the seminal work of Tsitsiklis [26] and Tsitsiklis *et al.* [27] (see also [3]), which focused on distributing the computations involved with optimizing a global objective function among different processors (assuming complete information about the global objective function at each processor). More recent literature focused on multi-agent environments and studied consensus algorithms for achieving cooperative behavior in a distributed manner (see [28], [12], [6], [21], [7], and [22, 23]). These works assume that the agent values can be processed arbitrarily and are unconstrained. Another recent approach for distributed cooperative control problems involve using game-theoretic models. In this approach, the agents are endowed with local utility functions that lead to a game form with a Nash equilibrium which is the same as or close to a global optimum. Various learning algorithms can then be used as distributed control schemes that will reach the equilibrium. In a recent paper, Marden *et al.* [14] used this approach for the consensus problem where agents have constraints on their values. Our projected consensus algorithm provides an alternative approach for this problem.

Most closely related to our work are the recent papers [18, 17], which proposed distributed subgradient methods for solving unconstrained multi-agent optimization problems. These methods use consensus algorithms as a mechanism for distributing computations among the agents. The presence of different local constraints significantly changes the operation and the analysis of the algorithms, which is our main focus in this paper. Our work is also related to incremental subgradient algorithms implemented over a network, where agents sequentially update an iterate sequence in a cyclic or a random order [4, 15, 24, 13]. In an incremental algorithm, there is a single iterate sequence and only one agent updates the iterate at a given time. Thus, while operating on the basis of local information, incremental algorithms differ fundamentally from the algorithm studied in this paper (where all agents update simultaneously). Furthermore, the work in [4, 15, 24, 13] assumes that the constraint set is known by all agents in the system, which is in a sharp contrast with the algorithms studied in this paper (our primary interest

is in the case where the information about the constraint set is distributed across the agents).

The paper is organized as follows. In Section 2, we introduce our notation and terminology, and establish some basic results related to projection on closed convex sets that will be used in the subsequent analysis. In Section 3, we present the constrained consensus problem and the projected consensus algorithm. We describe our multi-agent model and provide a basic result on the convergence behavior of the transition matrices that govern the evolution of agent estimates generated by the algorithms. We study the convergence of the agent estimates and establish convergence rate results for constant uniform weights. Section 4 introduces the constrained multi-agent optimization problem and presents the projected subgradient algorithm. We provide convergence analysis for the estimates generated by this algorithm. Section 5 contains concluding remarks and some future directions.

2 Notation, Terminology, and Basics

A vector is viewed as a column, unless clearly stated otherwise. We denote by x_i or $[x]_i$ the i -th component of a vector x . When $x_i \geq 0$ for all components i of a vector x , we write $x \geq 0$. We write x' to denote the transpose of a vector x . The scalar product of two vectors x and y is denoted by $x'y$. We use $\|x\|$ to denote the standard Euclidean norm, $\|x\| = \sqrt{x'x}$.

A vector $a \in \mathbb{R}^m$ is said to be a *stochastic vector* when its components a_j are non-negative and their sum is equal to 1, i.e., $\sum_{j=1}^m a_j = 1$. A set of m vectors $\{a^1, \dots, a^m\}$, with $a^i \in \mathbb{R}^m$ for all i , is said to be *doubly stochastic* when each a^i is a stochastic vector and $\sum_{i=1}^m a_j^i = 1$ for all $j = 1, \dots, m$. A square matrix A is said to be doubly stochastic when its rows are stochastic vectors, and its columns are also stochastic vectors.

We write $\text{dist}(\bar{x}, X)$ to denote the standard Euclidean distance of a vector \bar{x} from a set X , i.e.,

$$\text{dist}(\bar{x}, X) = \inf_{x \in X} \|\bar{x} - x\|.$$

We use $P_X[\bar{x}]$ to denote the projection of a vector \bar{x} on a closed convex set X , i.e.,

$$P_X[\bar{x}] = \arg \min_{x \in X} \|\bar{x} - x\|.$$

In the subsequent development, the properties of the projection operation on a closed convex set play an important role. In particular, we use the projection inequality, i.e., for any vector x ,

$$(P_X[x] - x)'(y - P_X[x]) \geq 0 \quad \text{for all } y \in X. \quad (1)$$

We also use the standard non-expansiveness property, i.e.,

$$\|P_X[x] - P_X[y]\| \leq \|x - y\| \quad \text{for any } x \text{ and } y. \quad (2)$$

In addition, we use the properties given in the following lemma.

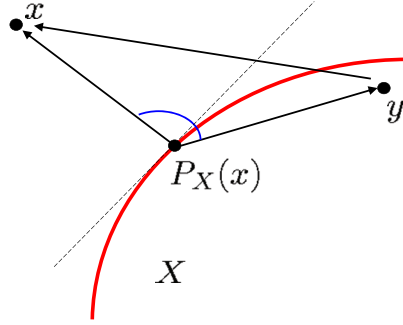


Figure 1: Illustration of the relation between the projection error and feasible directions of a convex set at the projection vector.

Lemma 1 Let X be a nonempty closed convex set in \mathbb{R}^n . Then, we have for any $x \in \mathbb{R}^n$,

- (a) $(P_X[x] - x)'(x - y) \leq -\|P_X[x] - x\|^2$ for all $y \in X$.
- (b) $\|P_X[x] - y\|^2 \leq \|x - y\|^2 - \|P_X[x] - x\|^2$ for all $y \in X$.

Proof. (a) Let $x \in \mathbb{R}^n$ be arbitrary. Then, for any $y \in X$, we have

$$(P_X[x] - x)'(x - y) = (P_X[x] - x)'(x - P_X[x]) + (P_X[x] - x)'(P_X[x] - y).$$

By the projection inequality [cf. Eq. (1)], it follows that $(P_X[x] - x)'(P_X[x] - y) \leq 0$, implying

$$(P_X[x] - x)'(x - y) \leq -\|P_X[x] - x\|^2 \quad \text{for all } y \in X.$$

(b) For an arbitrary $x \in \mathbb{R}^n$ and for all $y \in X$, we have

$$\begin{aligned} \|P_X[x] - y\|^2 &= \|P_X[x] - x + x - y\|^2 \\ &= \|P_X[x] - x\|^2 + \|x - y\|^2 + 2(P_X[x] - x)'(x - y). \end{aligned}$$

By using the inequality of part (a), we obtain

$$\|P_X[x] - y\|^2 \leq \|x - y\|^2 - \|P_X[x] - x\|^2 \quad \text{for all } y \in X.$$

■

Part (b) of the preceding Lemma establishes a relation between the projection error vector and the feasible directions of the convex set X at the projection vector, as illustrated in Figure 2.

We next consider nonempty closed convex sets $X_i \subseteq \mathbb{R}^n$, for $i = 1, \dots, m$, and an averaged-vector \hat{x} obtained by taking an average of vectors $x_i \in X_i$, i.e., $\hat{x} = \frac{1}{m} \sum_{i=1}^m x_i$ for some $x_i \in X_i$. We provide an “error bound” that relates the distance of the averaged-vector \hat{x} from the intersection set $X = \bigcap_{i=1}^m X_i$ to the distance of \hat{x} from the individual

sets X_i . This relation, which is also of independent interest, will play a key role in our analysis of the convergence of projection errors associated with various distributed algorithms introduced in this paper. We establish the relation under an interior point assumption on the intersection set $X = \cap_{i=1}^m X_i$ stated in the following:

Assumption 1 Given sets $X_i \subseteq \mathbb{R}^n$, $i = 1, \dots, m$, let $X = \cap_{i=1}^m X_i$ denote their intersection. There is a vector $\bar{x} \in \text{int}(X)$, i.e., there exists a scalar $\delta > 0$ such that

$$\{z \mid \|z - \bar{x}\| \leq \delta\} \subset X.$$

We provide an error bound relation in the following lemma.

Lemma 2 Let $X_i \subseteq \mathbb{R}^n$, $i = 1, \dots, m$, be nonempty closed convex sets that satisfy Assumption 1. Let $x^i \in X_i$, $i = 1, \dots, m$, be arbitrary vectors and define their average as $\hat{x} = \frac{1}{m} \sum_{i=1}^m x^i$. Consider the vector $s \in \mathbb{R}^n$ defined by

$$s = \frac{\epsilon}{\epsilon + \delta} \bar{x} + \frac{\delta}{\epsilon + \delta} \hat{x},$$

where

$$\epsilon = \sum_{j=1}^m \text{dist}(\hat{x}, X_j),$$

and δ is the scalar given in Assumption 1.

- (a) The vector s belongs to the intersection set $X = \cap_{i=1}^m X_i$.
- (b) We have the following relation

$$\|\hat{x} - s\| \leq \frac{1}{\delta m} \left(\sum_{j=1}^m \|x^j - \bar{x}\| \right) \left(\sum_{j=1}^m \text{dist}(\hat{x}, X_j) \right).$$

As a particular consequence, we have

$$\text{dist}(\hat{x}, X) \leq \frac{1}{\delta m} \left(\sum_{j=1}^m \|x^j - \bar{x}\| \right) \left(\sum_{j=1}^m \text{dist}(\hat{x}, X_j) \right).$$

Proof.

(a) We first show that the vector s belongs to the intersection $X = \cap_{i=1}^m X_i$. To see this, let $i \in \{1, \dots, m\}$ be arbitrary and note that we can write s as

$$s = \frac{\epsilon}{\epsilon + \delta} \left(\bar{x} + \frac{\delta}{\epsilon} (\hat{x} - P_{X_i}[\hat{x}]) \right) + \frac{\delta}{\epsilon + \delta} P_{X_i}[\hat{x}].$$

By the definition of ϵ , it follows that $\|\hat{x} - P_{X_i}[\hat{x}]\| \leq \epsilon$, implying by the interior point assumption (cf. Assumption 1) that the vector $\bar{x} + \frac{\delta}{\epsilon}(\hat{x} - P_{X_i}[\hat{x}])$ belongs to the set X , and therefore to the set X_i . Since the vector s is the convex combination of two vectors in the set X_i , it follows by the convexity of X_i that $s \in X_i$. The preceding argument is valid for an arbitrary i , thus implying that $s \in X$.

(b) Using the definition of the vector s and the vector \hat{x} , we have

$$\|\hat{x} - s\| = \frac{\epsilon}{\epsilon + \delta} \left\| \frac{1}{m} \sum_{j=1}^m x^j - \bar{x} \right\| \leq \frac{\epsilon}{\delta m} \sum_{j=1}^m \|x^j - \bar{x}\|.$$

Substituting the definition of ϵ yields the desired relation. ■

3 Constrained Consensus

In this section, we describe the constrained consensus problem. In particular, we introduce our multi-agent model and the projected consensus algorithm that is locally executed by each agent. We provide some insights about the algorithm and we discuss its connection to the alternating projections method. We also introduce the assumptions on the multi-agent model and present key elementary results that we use in our subsequent analysis of the projected consensus algorithm. In particular, we define the transition matrices governing the linear dynamics of the agent estimate evolution and give a basic convergence result for these matrices. The model assumptions and the transition matrix convergence properties will also be used for studying the constrained optimization problem and the projected subgradient algorithm that we introduce in Section 4.

3.1 Multi-Agent Model and Algorithm

We consider a set of agents denoted by $V = \{1, \dots, m\}$. We assume a slotted-time system, and we denote by $x^i(k)$ the estimate generated and stored by agent i at time slot k . The agent estimate $x^i(k)$ is a vector in \mathbb{R}^n that is constrained to lie in a nonempty closed convex set $X_i \subseteq \mathbb{R}^n$ known only to agent i . The agents' objective is to cooperatively reach a consensus on a common vector through a sequence of local estimate updates (subject to the local constraint set) and local information exchanges (with neighboring agents only).

We study a model where the agents exchange and update their estimates as follows: To generate the estimate at time $k+1$, agent i forms a convex combination of its estimate $x^i(k)$ with the estimates received from other agents at time k , and takes the projection of this vector on its constraint set X_i . More specifically, agent i at time $k+1$ generates its new estimate according to the following relation:

$$x^i(k+1) = P_{X_i} \left[\sum_{j=1}^m a_j^i(k) x^j(k) \right], \quad (3)$$

where $a^i = (a_1^i, \dots, a_m^i)'$ is a vector of nonnegative weights.

The relation in Eq. (3) defines the *projected consensus algorithm*. The method can be interpreted as a multi-agent algorithm for finding a point in common to the given closed convex sets X_1, \dots, X_m . Note that the problem of finding a common point can be formulated as an unconstrained convex optimization problem of the following form:

$$\begin{aligned} & \text{minimize} && \frac{1}{2} \sum_{i=1}^m \|x - P_{X_i}[x]\|^2 \\ & \text{subject to} && x \in \mathbb{R}^n. \end{aligned} \quad (4)$$

In view of this optimization problem, the method can be interpreted as a *distributed gradient algorithm* where each agent is assigned an objective function $f_i(x) = \frac{1}{2} \|x - P_{X_i}[x]\|^2$. At each time $k + 1$, an agent incorporates new information $x^j(k)$ received from some of the other agents and generates a weighted sum $\sum_{j=1}^m a_j^i(k)x^j(k)$. Then, the agent updates its estimate by taking a step (with stepsize equal to 1) along the negative gradient of its own objective function $f_i = \frac{1}{2} \|x - P_{X_i}\|^2$ at $x = \sum_{j=1}^m a_j^i(k)x^j(k)$. In particular, since the gradient of f_i is $\nabla f_i(x) = x - P_{X_i}[x]$ (see Theorem 1.5.5 in Facchinei and Pang [10]), the update rule in Eq. (3) is equivalent to the following gradient descent method for minimizing f_i :

$$x^i(k+1) = \sum_{j=1}^m a_j^i(k)x^j(k) - \left(\sum_{j=1}^m a_j^i(k)x^j(k) - P_{X_i} \left[\sum_{j=1}^m a_j^i(k)x^j(k) \right] \right).$$

This view of the update rule motivates our line of analysis of the projected consensus method. In particular, motivated by the objective function of problem (4), we use $\sum_{i=1}^m \|x^i(k) - x\|^2$ with $x \in \cap_{i=1}^m X_i$ as a Lyapunov function measuring the progress of the algorithm (see Section 3.6).¹

3.2 Relation to Alternating Projections Method

The method of Eq. (3) is related to the classical *alternating or cyclic projection method*. Given a finite collection of closed convex sets $\{X_i\}_{i \in \mathcal{I}}$ with a nonempty intersection (i.e., $\cap_{i \in \mathcal{I}} X_i \neq \emptyset$), the alternating projection method finds a vector in the intersection $\cap_{i \in \mathcal{I}} X_i$. In other words, the algorithm solves the unconstrained problem (4). Alternating projection methods generate a sequence of vectors by projecting iteratively on the sets (either cyclically or with some given order), see Figure 2(a). The convergence behavior of these methods has been established by Von Neumann [20] and Aronszajn [1] for the case when the sets X_i are affine; and by Gubin *et al.* [11] when the sets X_i are closed and convex. Gubin *et al.* [11] also have provided convergence rate results for a particular form of alternating projection method. Similar rate results under different assumptions have also been provided by Deutsch [8], and Deutsch and Hundal [9].

The constrained consensus algorithm [cf. Eq. (3)] generates a sequence of iterates for each agent as follows: at iteration k , each agent i first forms a linear combination

¹ We focus throughout the paper on the case when the intersection set $\cap_{i=1}^m X_i$ is nonempty. If the intersection set is empty, it follows from the definition of the algorithm that the agent estimates will not reach a consensus. In this case, the estimate sequences $\{x^i(k)\}$ may exhibit oscillatory behavior or may all be unbounded.

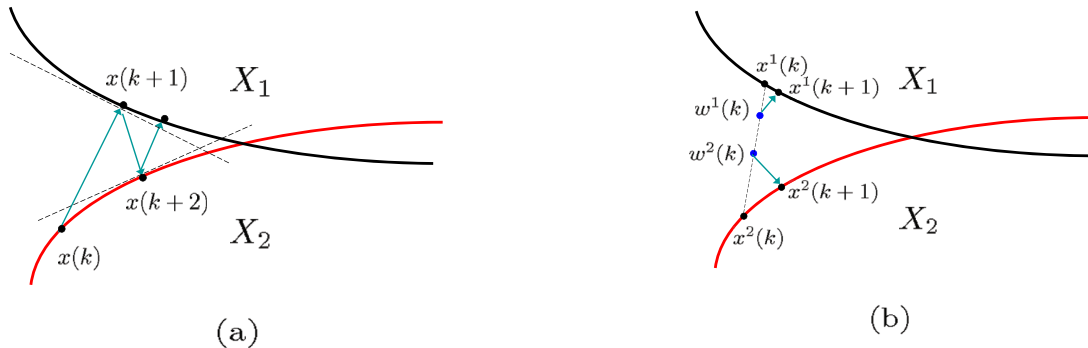


Figure 2: Illustration of the connection between the alternating/cyclic projection method and the constrained consensus algorithm for two closed convex sets X_1 and X_2 . In plot (a), the alternating projection algorithm generates a sequence $\{x(k)\}$ by iteratively projecting onto sets X_1 and X_2 , i.e., $x(k+1) = P_{X_1}[x(k)]$, $x(k+2) = P_{X_2}[x(k+1)]$. In plot (b), the projected consensus algorithm generates sequences $\{x^i(k)\}$ for agents $i = 1, 2$ by first combining the iterates with different weights and then projecting on respective sets X_i , i.e., $w^i(k) = \sum_{j=1}^m a_j^i(k)x^j(k)$ and $x^i(k+1) = P_{X_i}[w^i(k)]$ for $i = 1, 2$.

of the other agent values $x^j(k)$ using its own weight vector $a^i(k)$ and then projects this combination on its constraint set X_i . Therefore, the projected consensus algorithm can be viewed as a version of the alternating projection algorithm, where the iterates are combined with the weights varying over time and across agents, and then projected on the individual constraint sets, see Figure 2(b).

We conclude this section by noting that the alternate projection method has much more structured weights than the weights we consider in this paper. As seen from the assumptions on the agent weights in the following section, the analysis of our projected consensus algorithm (and the projected subgradient algorithm introduced in Section 4) is complicated by the general time variability of the weights $a_j^i(k)$.

3.3 Assumptions

Following Tsitsiklis [26] (see also Blondel *et al.* [5]), we adopt the following assumptions on the weight vectors $a^i(k)$, $i \in \{1, \dots, m\}$ and on information exchange.

Assumption 2 (*Weights Rule*) There exists a scalar η with $0 < \eta < 1$ such that for all $i \in \{1, \dots, m\}$,

- (a) $a_i^i(k) \geq \eta$ for all $k \geq 0$.
- (b) If $a_j^i(k) > 0$, then $a_j^i(k) \geq \eta$.

Assumption 3 (*Doubly Stochasticity*) The vectors $a^i(k) = (a_1^i(k), \dots, a_m^i(k))'$ satisfy:

- (a) $a^i(k) \geq 0$ and $\sum_{j=1}^m a_j^i(k) = 1$ for all i and k , i.e., the vectors $a^i(k)$ are stochastic.
- (b) $\sum_{i=1}^m a_j^i(k) = 1$ for all j and k .

Informally speaking, Assumption 2 says that every agent assigns a substantial weight to the information received from its neighbors. This guarantees that the information from each agent influences the information of every other agent persistently in time. In other words, this assumption guarantees that the agent information is mixing at a nondiminishing rate in time. Without this assumption, information from some of the agents may become less influential in time, and in the limit, resulting in loss of information from these agents.

Assumption 3(a) establishes that each agent takes a convex combination of its estimate and the estimates of its neighbors. Assumption 3(b), together with Assumption 2, ensures that the estimate of every agent is influenced by the estimates of every other agent with the same frequency in the limit, i.e., all agents are equally influential in the long run.

We now impose some rules on the agent information exchange. At each update time t_k , the information exchange among the agents may be represented by a directed graph (V, E_k) with the set E_k of directed edges given by

$$E_k = \{(j, i) \mid a_j^i(k) > 0\}.$$

Note that, by Assumption 2(a), we have $(i, i) \in E_k$ for each agent i and all k . Also, we have $(j, i) \in E_k$ if and only if agent i receives the information x^j from agent j in the time interval (t_k, t_{k+1}) .

We next formally state the connectivity assumption on the multi-agent system. This assumption ensures that the information of any agent i influences the information state of any other agent infinitely often in time.

Assumption 4 (*Connectivity*) The graph (V, E_∞) is strongly connected, where E_∞ is the set of edges (j, i) representing agent pairs communicating directly infinitely many times, i.e.,

$$E_\infty = \{(j, i) \mid (j, i) \in E_k \text{ for infinitely many indices } k\}.$$

We also adopt an additional assumption that the intercommunication intervals are bounded for those agents that communicate directly. In particular, this is stated in the following.

Assumption 5 (*Bounded Intercommunication Interval*) There exists an integer $B \geq 1$ such that for every $(j, i) \in E_\infty$, agent j sends its information to a neighboring agent i at least once every B consecutive time slots, i.e., at time t_k or at time t_{k+1} or \dots or (at latest) at time t_{k+B-1} for any $k \geq 0$.

In other words, the preceding assumption guarantees that every pair of agents that communicate directly infinitely many times exchange information at least once every B time slots.²

²It is possible to adopt weaker connectivity assumptions for the multi-agent model as those used in the recent work [16].

3.4 Transition Matrices

We introduce matrices $A(s)$, whose i -th column is the weight vector $a^i(s)$, and the matrices

$$\Phi(k, s) = A(s)A(s+1) \cdots A(k-1)A(k) \quad \text{for all } s \text{ and } k \text{ with } k \geq s,$$

where

$$\Phi(k, k) = A(k) \quad \text{for all } k.$$

We use these matrices to describe the evolution of the agent estimates associated with the algorithms introduced in Sections 3 and 4. The convergence properties of these matrices as $k \rightarrow \infty$ have been extensively studied and well-established (see [26], [12], [29]). Under the assumptions of Section 3.3, the matrices $\Phi(k, s)$ converge as $k \rightarrow \infty$ to a *uniform steady state distribution for each s at a geometric rate*, i.e., $\lim_{k \rightarrow \infty} \Phi(k, s) = \frac{1}{m} ee'$ for all s . The fact that transition matrices converge at a geometric rate plays a crucial role in our analysis of the algorithms. Recent work has established explicit convergence rate results for the transition matrices [18, 17]. These results are given in the following proposition without a proof.

Proposition 1 Let Assumptions 2, 3, 4 and 5 hold. Then, we have the following:

- (a) The entries $[\Phi(k, s)]_j^i$ of the transition matrices converge to $\frac{1}{m}$ as $k \rightarrow \infty$ with a geometric rate uniformly with respect to i and j , i.e., for all $i, j \in \{1, \dots, m\}$,

$$\left| [\Phi(k, s)]_j^i - \frac{1}{m} \right| \leq 2 \frac{1 + \eta^{-B_0}}{1 - \eta^{B_0}} (1 - \eta^{B_0})^{\frac{k-s}{B_0}} \quad \text{for all } s \text{ and } k \text{ with } k \geq s.$$

- (b) In the absence of Assumption 3(b) [i.e., the weights $a^i(k)$ are stochastic but not doubly stochastic], the columns $[\Phi(k, s)]^i$ of the transition matrices converge to a stochastic vector $\phi(s)$ as $k \rightarrow \infty$ with a geometric rate uniformly with respect to i and j , i.e., for all $i, j \in \{1, \dots, m\}$,

$$|[\Phi(k, s)]_j^i - \phi_j(s)| \leq 2 \frac{1 + \eta^{-B_0}}{1 - \eta^{B_0}} (1 - \eta^{B_0})^{\frac{k-s}{B_0}} \quad \text{for all } s \text{ and } k \text{ with } k \geq s.$$

Here, η is the lower bound of Assumption 2, $B_0 = (m-1)B$, m is the number of agents, and B is the intercommunication interval bound of Assumption 5.

3.5 Convergence

In this section, we study the convergence behavior of the agent estimates $\{x^i(k)\}$ generated by the projected consensus algorithm (3) under Assumptions 2–5. We write the update rule in Eq. (3) as

$$x^i(k+1) = \sum_{j=1}^m a_j^i(k) x^j(k) + e^i(k), \quad (5)$$

where $e^i(k)$ represents the error due to projection given by

$$e^i(k) = P_{X_i} \left[\sum_{j=1}^m a_j^i(k) x^j(k) \right] - \sum_{j=1}^m a_j^i(k) x^j(k). \quad (6)$$

As indicated by the preceding two relations, the evolution dynamics of the estimates $x^i(k)$ for each agent is decomposed into a sum of a linear (time-varying) term $\sum_{j=1}^m a_j^i(k) x^j(k)$ and a nonlinear term $e^i(k)$. The linear term captures the effects of mixing the agent estimates, while the nonlinear term captures the nonlinear effects of the projection operation. This decomposition plays a crucial role in our analysis. As we will shortly see [cf. Lemma 3(d)], under the doubly stochasticity assumption on the weights, the nonlinear terms $e^i(k)$ are diminishing in time for each i , and therefore, the evolution of agent estimates is “almost linear”. Thus, the nonlinear term can be viewed as a non-persistent disturbance in the linear evolution of the estimates.

For notational convenience, let $w^i(k)$ denote

$$w^i(k) = \sum_{j=1}^m a_j^i(k) x^j(k). \quad (7)$$

In this notation, the iterate $x^i(k+1)$ and the projection error $e^i(k)$ are given by

$$x^i(k+1) = P_{X_i}[w^i(k)], \quad (8)$$

$$e^i(k) = x^i(k+1) - w^i(k). \quad (9)$$

In the following lemma, we show some relations for the sums $\sum_{i=1}^m \|x^i(k) - x\|^2$ and $\sum_{i=1}^m \|w^i(k) - x\|^2$, and $\sum_{i=1}^m \|x^i(k) - x\|$ and $\sum_{i=1}^m \|w^i(k) - x\|$ for an arbitrary vector x in the intersection of the agent constraint sets. Also, we prove that the errors $e^i(k)$ converge to zero as $k \rightarrow \infty$ for all i . The projection properties given in Lemma 1 and the doubly stochasticity of the weights play crucial roles in establishing these relations.

Lemma 3 Let the intersection set $X = \cap_{i=1}^m X_i$ be nonempty, and let Doubly Stochasticity assumption hold (cf. Assumption 3). Let $x^i(k)$, $w^i(k)$, and $e^i(k)$ be defined by Eqs. (7)–(9). Then, we have the following.

(a) For all $x \in X$ and all k , we have

- (i) $\|x^i(k+1) - x\|^2 \leq \|w^i(k) - x\|^2 - \|e^i(k)\|^2$ for all i ,
- (ii) $\sum_{i=1}^m \|w^i(k) - x\|^2 \leq \sum_{i=1}^m \|x^i(k) - x\|^2$,
- (iii) $\sum_{i=1}^m \|w^i(k) - x\| \leq \sum_{i=1}^m \|x^i(k) - x\|$.

(b) For all $x \in X$, the sequences $\left\{ \sum_{i=1}^m \|w^i(k) - x\|^2 \right\}$ and $\left\{ \sum_{i=1}^m \|x^i(k) - x\|^2 \right\}$ are monotonically nonincreasing with k .

(c) For all $x \in X$, the sequences $\left\{ \sum_{i=1}^m \|w^i(k) - x\| \right\}$ and $\left\{ \sum_{i=1}^m \|x^i(k) - x\| \right\}$ are monotonically nonincreasing with k .

(d) The errors $e^i(k)$ converge to zero as $k \rightarrow \infty$, i.e.,

$$\lim_{k \rightarrow \infty} e^i(k) = 0 \quad \text{for all } i.$$

Proof. (a) For any $x \in X$ and i , we consider the term $\|x^i(k+1) - x\|^2$. Since $X \subseteq X_i$ for all i , it follows that $x \in X_i$ for all i . Since we also have $x^i(k+1) = P_{X_i}[w^i(k)]$, we have from Lemma 1(b) that

$$\|x^i(k+1) - x\|^2 \leq \|w^i(k) - x\|^2 - \|x^i(k+1) - w^i(k)\|^2 \quad \text{for all } x \in X \text{ and } k \geq 0,$$

which yields the relation in part (a)(i) in view of relation (9).

By the definition of $w^i(k)$ in Eq. (7) and the stochasticity of the weight vector $a^i(k)$ [cf. Assumption 3(a)], we have for every agent i and any $x \in X$,

$$w^i(k) - x = \sum_{j=1}^m a_j^i(k) (x^j(k) - x) \quad \text{for all } k \geq 0. \quad (10)$$

Thus, for any $x \in X$, and all i and k ,

$$\|w^i(k) - x\|^2 = \left\| \sum_{j=1}^m a_j^i(k) (x^j(k) - x) \right\|^2 \leq \sum_{j=1}^m a_j^i(k) \|x^j(k) - x\|^2,$$

where the inequality holds since the vector $\sum_{j=1}^m a_j^i(k)(x^j(k) - x)$ is a convex combination of the vectors $x^j(k) - x$ and the squared norm $\|\cdot\|^2$ is a convex function. By summing the preceding relations over $i = 1, \dots, m$, we obtain

$$\sum_{i=1}^m \|w^i(k) - x\|^2 \leq \sum_{i=1}^m \sum_{j=1}^m a_j^i(k) \|x^j(k) - x\|^2 = \sum_{j=1}^m \left(\sum_{i=1}^m a_j^i(k) \right) \|x^j(k) - x\|^2.$$

Using the doubly stochasticity of the weight vectors $a^i(k)$, i.e., $\sum_{i=1}^m a_j^i(k) = 1$ for all j and k [cf. Assumption 3(b)], we obtain the relation in part (a)(ii),

$$\sum_{i=1}^m \|w^i(k) - x\|^2 \leq \sum_{i=1}^m \|x^i(k) - x\|^2 \quad \text{for all } x \in X \text{ and } k \geq 0.$$

Similarly, from relation (10) and the doubly stochasticity of the weights, we obtain for all $x \in X$ and all k ,

$$\sum_{i=1}^m \|w^i(k) - x\| \leq \sum_{i=1}^m \sum_{j=1}^m a_j^i(k) \|x^j(k) - x\| = \sum_{j=1}^m \|x^j(k) - x\|,$$

thus showing the relation in part (a)(iii).

(b) For any $x \in X$, the nonincreasing properties of the sequences $\left\{ \sum_{i=1}^m \|w^i(k) - x\|^2 \right\}$ and $\left\{ \sum_{i=1}^m \|x^i(k) - x\|^2 \right\}$ follow by combining the relations in parts (a)(i)–(ii).

(c) Since $x^i(k+1) = P_{X_i}(w^i(k))$ for all i and $k \geq 0$, using the nonexpansiveness property of the projection operation [cf. Eq. (2)], we have

$$\|x^i(k+1) - x\| \leq \|w^i(k) - x\| \quad \text{for all } x \in X_i, \text{ all } i \text{ and } k.$$

Summing the preceding relations over all $i \in \{1, \dots, m\}$ yields for all k ,

$$\sum_{i=1}^m \|x^i(k+1) - x\| \leq \sum_{i=1}^m \|w^i(k) - x\| \quad \text{for all } x \in X. \quad (11)$$

The nonincreasing property of the sequences $\{\sum_{i=1}^m \|w^i(k) - x\|\}$ and $\{\sum_{i=1}^m \|x^i(k) - x\|\}$ follows from the preceding relation and the relation in part (a)(iii).

(d) By summing the relations in part (a)(i) over $i = 1, \dots, m$, we obtain for any $x \in X$,

$$\sum_{i=1}^m \|x^i(k+1) - x\|^2 \leq \sum_{i=1}^m \|w^i(k) - x\|^2 - \sum_{i=1}^m \|e^i(k)\|^2 \quad \text{for all } k \geq 0.$$

Combined with the inequality $\sum_{j=1}^m \|w^j(k) - x\|^2 \leq \sum_{j=1}^m \|x^j(k) - x\|^2$ of part (a)(ii), we further obtain

$$\sum_{i=1}^m \|e^i(k)\|^2 \leq \sum_{i=1}^m \|x^i(k) - x\|^2 - \sum_{i=1}^m \|x^i(k+1) - x\|^2 \quad \text{for all } k \geq 0.$$

Summing these relations over $k = 0, \dots, s$ for any $s > 0$ yields

$$\sum_{k=0}^s \sum_{i=1}^m \|e^i(k)\|^2 \leq \sum_{i=1}^m \|x^i(0) - x\|^2 - \sum_{i=1}^m \|x^i(s+1) - x\|^2 \leq \sum_{i=1}^m \|x^i(0) - x\|^2.$$

By letting $s \rightarrow \infty$, we obtain

$$\sum_{k=0}^{\infty} \sum_{i=1}^m \|e^i(k)\|^2 \leq \sum_{i=1}^m \|x^i(0) - x\|^2,$$

implying $\lim_{k \rightarrow \infty} \|e^i(k)\| = 0$ for all i . ■

We next consider the evolution of the estimates $x^i(k+1)$ generated by method (3) over a period of time. In particular, we relate the estimates $x^i(k+1)$ to the estimates $x^i(s)$ generated earlier in time s with $s < k+1$ by exploiting the decomposition of the estimate evolution in Eqs. (5)–(6). In this, we use the transition matrices $\Phi(k, s)$ from time s to time k (see Section 3.4). As we will shortly see, the linear part of the dynamics is given in terms of the transition matrices, while the nonlinear part involves combinations of the transition matrices and the error terms from time s to time k .

Recall that the transition matrices are defined as follows:

$$\Phi(k, s) = A(s)A(s+1) \cdots A(k-1)A(k) \quad \text{for all } s \text{ and } k \text{ with } k \geq s,$$

where

$$\Phi(k, k) = A(k) \quad \text{for all } k,$$

and each $A(s)$ is a matrix whose i -th column is the vector $a^i(s)$. Using these transition matrices and the decomposition of the estimate evolution of Eqs. (5)–(6), the relation between $x^i(k+1)$ and the estimates $x^1(s), \dots, x^m(s)$ at time $s \leq k$ is given by

$$x^i(k+1) = \sum_{j=1}^m [\Phi(k, s)]_j^i x^j(s) + \sum_{r=s+1}^k \left(\sum_{j=1}^m [\Phi(k, r)]_j^i e^j(r-1) \right) + e^i(k). \quad (12)$$

Here we can view $e^j(k)$ as an external perturbation input to the system.

We use this relation to study the “steady-state” behavior of a related process. In particular, we define an auxiliary sequence $\{y(k)\}$, where $y(k)$ is given by

$$y(k) = \frac{1}{m} \sum_{i=1}^m w^i(k) \quad \text{for all } k. \quad (13)$$

Since $w^i(k) = \sum_{j=1}^m a_j^i(k) x^j(k)$, under the doubly stochasticity of the weights, it follows that

$$y(k) = \frac{1}{m} \sum_{j=1}^m x^j(k) \quad \text{for all } k. \quad (14)$$

Furthermore, from the relations in (12) using the doubly stochasticity of the weights, we have

$$y(k) = \frac{1}{m} \sum_{j=1}^m x^j(s) + \frac{1}{m} \sum_{r=s+1}^k \left(\sum_{j=1}^m e^j(r-1) \right). \quad (15)$$

We now show that the limiting behavior of the agent estimates $x^i(k)$ is the same as the limiting behavior of $y(k)$ as $k \rightarrow \infty$. We establish this result using the assumptions on the multi-agent model of Section 3.3.

Lemma 4 Let the intersection set $X = \cap_{i=1}^m X_i$ be nonempty. Also, let Assumptions 2, 3, 4, and 5 hold. We then have

$$\lim_{k \rightarrow \infty} \|x^i(k) - y(k)\| = 0, \quad \lim_{k \rightarrow \infty} \|w^i(k) - y(k)\| = 0 \quad \text{for all } i.$$

Proof. By Lemma 3(d), we have $e^i(k) \rightarrow 0$ as $k \rightarrow \infty$ for all i . Therefore, for any $\epsilon > 0$, we can choose some integer s such that $\|e^i(k)\| \leq \epsilon$ for all $k \geq s$ and for all i . Using the relations in Eqs. (12) and (15), we obtain for all i and $k \geq s+1$,

$$\begin{aligned} \|x^i(k) - y(k)\| &= \left\| \sum_{j=1}^m \left([\Phi(k-1, s)]_j^i - \frac{1}{m} \right) x^j(s) \right. \\ &\quad \left. + \sum_{r=s+1}^{k-1} \sum_{j=1}^m \left([\Phi(k-1, r)]_j^i - \frac{1}{m} \right) e^j(r-1) + \left(e^i(k-1) - \frac{1}{m} \sum_{j=1}^m e^j(k-1) \right) \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^m \left| [\Phi(k-1, s)]_j^i - \frac{1}{m} \right| \|x^j(s)\| \\
&+ \sum_{r=s+1}^{k-1} \sum_{j=1}^m \left| [\Phi(k-1, r)]_j^i - \frac{1}{m} \right| \|e^j(r-1)\| + \|e^i(k-1)\| + \frac{1}{m} \sum_{j=1}^m \|e^j(k-1)\|.
\end{aligned}$$

Using the estimates for $\left| [\Phi(k-1, s)]_j^i - \frac{1}{m} \right|$ of Proposition 1(a), we have

$$\begin{aligned}
\|x^i(k) - y(k)\| &\leq 2 \frac{1 + \eta^{-B_0}}{1 - \eta^{B_0}} (1 - \eta^{B_0})^{\frac{k-1-s}{B_0}} \sum_{j=1}^m \|x^j(s)\| \\
&+ \sum_{r=s+1}^{k-1} 2 \frac{1 + \eta^{-B_0}}{1 - \eta^{B_0}} (1 - \eta^{B_0})^{\frac{k-1-r}{B_0}} \sum_{j=1}^m \|e^j(r-1)\| \\
&+ \|e^i(k-1)\| + \frac{1}{m} \sum_{j=1}^m \|e^j(k-1)\|. \tag{16}
\end{aligned}$$

Since $\|e^i(k)\| \leq \epsilon$ for all $k \geq s$ and for all i , from the preceding inequality we obtain

$$\begin{aligned}
\|x^i(k) - y(k)\| &\leq 2 \frac{1 + \eta^{-B_0}}{1 - \eta^{B_0}} (1 - \eta^{B_0})^{\frac{k-1-s}{B_0}} \sum_{j=1}^m \|x^j(s)\| \\
&+ 2m\epsilon \frac{1 + \eta^{-B_0}}{1 - \eta^{B_0}} \frac{1}{1 - (1 - \eta^{B_0})^{\frac{1}{B_0}}} + 2\epsilon.
\end{aligned}$$

Thus, by taking the limit superior as $k \rightarrow \infty$, we see that

$$\limsup_{k \rightarrow \infty} \|x^i(k) - y(k)\| \leq 2m\epsilon \frac{1 + \eta^{-B_0}}{1 - \eta^{B_0}} \frac{1}{1 - (1 - \eta^{B_0})^{\frac{1}{B_0}}} + 2\epsilon,$$

which by the arbitrary choice of ϵ , implies $\lim_{k \rightarrow \infty} \|x^i(k) - y(k)\| = 0$ for all i .

Consider now $\sum_{i=1}^m \|w^i(k) - y(k)\|$. By using $w^i(k) = \sum_{j=1}^m a_j^i(k) x^j(k)$ [cf. Eq. (7)] and the stochasticity of the vector $a^i(k)$, we have

$$\sum_{i=1}^m \|w^i(k) - y(k)\| \leq \sum_{i=1}^m \sum_{j=1}^m a_j^i(k) \|x^j(k) - y(k)\|.$$

By exchanging the order of the summations over i and j , and using the doubly stochasticity of $a^i(k)$, we further obtain

$$\sum_{i=1}^m \|w^i(k) - y(k)\| \leq \sum_{j=1}^m \left(\sum_{i=1}^m a_j^i(k) \right) \|x^j(k) - y(k)\| = \sum_{j=1}^m \|x^j(k) - y(k)\|. \tag{17}$$

Since $\lim_{k \rightarrow \infty} \|x^j(k) - y(k)\| = 0$ for all j , we have

$$\lim_{k \rightarrow \infty} \sum_{i=1}^m \|w^i(k) - y(k)\| = 0,$$

implying $\lim_{k \rightarrow \infty} \|w^i(k) - y(k)\| = 0$ for all i . ■

We next show that the agents reach a consensus asymptotically, i.e., the agent estimates $x^i(k)$ converge to the same point as k goes to infinity.

Proposition 2 (*Consensus*) Let the set $X = \cap_{i=1}^m X_i$ be nonempty. Also, let Assumptions 2, 3, 4, and 5 hold. For all i , let the sequence $\{x^i(k)\}$ be generated by the projected consensus algorithm (3). We then have for some $\tilde{x} \in X$,

$$\lim_{k \rightarrow \infty} \|x^i(k) - \tilde{x}\| = 0 \quad \lim_{k \rightarrow \infty} \|w^i(k) - \tilde{x}\| = 0, \quad \text{for all } i.$$

Proof. The proof idea is to consider the sequence $\{y(k)\}$, defined in Eq. (15), and show that it has a limit point in the set X . By using this and Lemma 4, we establish the convergence of each $w^i(k)$ and $x^i(k)$ to \tilde{x} .

To show that $\{y(k)\}$ has a limit point in the set X , we first consider the sequence

$$\sum_{j=1}^m \text{dist}(y(k), X_j).$$

Since $x^j(k) \in X_j$ for all j and $k \geq 0$, we have

$$\sum_{j=1}^m \text{dist}(y(k), X_j) \leq \sum_{j=1}^m \|y(k) - x^j(k)\|.$$

Taking the limit as $k \rightarrow \infty$ in the preceding relation and using Lemma 4, we conclude

$$\lim_{k \rightarrow \infty} \sum_{j=1}^m \text{dist}(y(k), X_j) = 0. \quad (18)$$

For a given $x \in X$, using Lemma 3(c), we have

$$\sum_{i=1}^m \|x^i(k) - x\| \leq \sum_{i=1}^m \|x^i(0) - x\| \quad \text{for all } k \geq 0.$$

This implies that the sequence $\{\sum_{i=1}^m \|x^i(k) - x\|\}$, and therefore each of the sequences $\{x^i(k)\}$ are bounded. Since for all i

$$\|y(k)\| \leq \|x^i(k) - y(k)\| + \|x^i(k)\| \quad \text{for all } k \geq 0,$$

using Lemma 4, it follows that the sequence $\{y(k)\}$ is bounded. In view of Eq. (18), this implies that the sequence $\{y(k)\}$ has a limit point \tilde{x} that belongs to the set $X = \cap_{j=1}^m X_j$. Furthermore, because $\lim_{k \rightarrow \infty} \|w^i(k) - y(k)\| = 0$ for all i , we conclude that \tilde{x} is also a limit point of the sequence $\{w^i(k)\}$ for all i . Since the sum sequence $\left\{ \sum_{i=1}^m \|w^i(k) - \tilde{x}\| \right\}$

is nonincreasing by Lemma 3(c) and since each $\{w^i(k)\}$ is converging to \tilde{x} along a subsequence, it follows that

$$\lim_{k \rightarrow \infty} \sum_{i=1}^m \|w^i(k) - \tilde{x}\| = 0,$$

implying $\lim_{k \rightarrow \infty} \|w^i(k) - \tilde{x}\| = 0$ for all i . Using this, together with the relations $\lim_{k \rightarrow \infty} \|w^i(k) - y(k)\| = 0$ and $\lim_{k \rightarrow \infty} \|x^i(k) - y(k)\| = 0$ for all i (cf. Lemma 4), we conclude

$$\lim_{k \rightarrow \infty} \|x^i(k) - \tilde{x}\| = 0 \quad \text{for all } i.$$

■

3.6 Convergence Rate

In this section, we establish a convergence rate result for the iterates $x^i(k)$ generated by the projected consensus algorithm (3) for the case when the weights are time-invariant and equal, i.e., $a^i(k) = (1/m, \dots, 1/m)'$ for all i and k . In our multi-agent model, this case corresponds to a fixed and complete connectivity graph, where each agent is connected to every other agent. We provide our rate estimate under an interior point assumption on the sets X_i , stated in Assumption 1.

We first establish a bound on the distance from the vectors of a convergent sequence to the limit point of the sequence. This relation holds for constant uniform weights, and it is motivated by a similar estimate used in the analysis of alternating projections methods in Gubin *et al.* [11] (see the proof of Lemma 6 there).

Lemma 5 Let Y be a nonempty closed convex set in \mathbb{R}^n . Let $\{u(k)\} \subseteq \mathbb{R}^n$ be a sequence converging to some $\tilde{y} \in Y$, and such that $\|u(k+1) - y\| \leq \|u(k) - y\|$ for all $y \in Y$ and all k . We then have

$$\|u(k) - \tilde{y}\| \leq 2 \operatorname{dist}(u(k), Y) \quad \text{for all } k \geq 0.$$

Proof. Let $B(x, \alpha)$ denote the closed ball centered at a vector x with radius α , i.e., $B(x, \alpha) = \{z \mid \|z - x\| \leq \alpha\}$. For each l , consider the sets

$$S_l = \bigcap_{k=0}^l B\left(P_Y[u(k)], \operatorname{dist}(u(k), Y)\right).$$

The sets S_l are convex, compact, and nested, i.e., $S_{l+1} \subseteq S_l$ for all l . The nonincreasing property of the sequence $\{u(k)\}$ implies that $\|u(k+s) - P_Y[u(k)]\| \leq \|u(k) - P_Y[u(k)]\|$ for all $k, s \geq 0$; hence, the sets S_l are also nonempty. Consequently, their intersection $\bigcap_{l=0}^{\infty} S_l$ is nonempty and every point $y^* \in \bigcap_{l=0}^{\infty} S_l$ is a limit point of the sequence $\{u(k)\}$. By assumption, the sequence $\{u(k)\}$ converges to $\tilde{y} \in Y$, and therefore, $\bigcap_{l=0}^{\infty} S_l = \{\tilde{y}\}$. Then, in view of the definition of the sets S_l , we obtain for all k ,

$$\|u(k) - \tilde{y}\| \leq \|u(k) - P_Y[u(k)]\| + \|P_Y[u(k)] - \tilde{y}\| \leq 2 \operatorname{dist}(u(k), Y).$$

■

We now establish a convergence rate result for constant uniform weights. In particular, we show that the projected consensus algorithm converges with a geometric rate under the Interior Point assumption.

Proposition 3 Let Assumptions 1, 2, 3, 4, and 5 hold. Let the weight vectors $a^i(k)$ in algorithm (3) be given by $a^i(k) = (1/m, \dots, 1/m)'$ for all i and k . For all i , let the sequence $\{x^i(k)\}$ be generated by the algorithm (3). We then have

$$\sum_{i=1}^m \|x^i(k) - \tilde{x}\|^2 \leq \left(1 - \frac{1}{4R^2}\right)^k \sum_{i=1}^m \|x^i(0) - \tilde{x}\|^2 \quad \text{for all } k \geq 0,$$

where $\tilde{x} \in X$ is the limit of the sequence $\{x^i(k)\}$, and $R = \frac{1}{\delta} \sum_{i=1}^m \|x^i(0) - \bar{x}\|$ with \bar{x} and δ given in the Interior Point assumption.

Proof. Since the weight vectors $a^i(k)$ are given by $a^i(k) = (1/m, \dots, 1/m)'$, it follows that

$$w^i(k) = w(k) = \frac{1}{m} \sum_{j=1}^m x^j(k) \quad \text{for all } i,$$

[see the definition of $w^i(k)$ in Eq. (7)]. For all $k \geq 0$, using Lemma 2(b) with the identification $x^i = x^i(k)$ for each $i = 1, \dots, m$, and $\hat{x} = w(k)$, we obtain

$$\text{dist}(w(k), X) \leq \frac{1}{\delta m} \left(\sum_{j=1}^m \|x^j(k) - \bar{x}\| \right) \left(\sum_{j=1}^m \text{dist}(w(k), X_j) \right),$$

where the vector \bar{x} and the scalar δ are given in Assumption 1. Since $\bar{x} \in X$, the sequence $\{\sum_{i=1}^m \|x^i(k) - \bar{x}\|\}$ is nonincreasing by Lemma 3(c). Therefore, we have $\sum_{i=1}^m \|x^i(k+1) - \bar{x}\| \leq \sum_{i=1}^m \|x^i(0) - \bar{x}\|$ for all k . Defining the constant $R = \frac{1}{\delta} \sum_{i=1}^m \|x^i(0) - \bar{x}\|$ and substituting in the preceding relation, we obtain

$$\begin{aligned} \text{dist}(w(k), X) &\leq \frac{R}{m} \left(\sum_{j=1}^m \text{dist}(w(k), X_j) \right) \\ &= \frac{R}{m} \sum_{j=1}^m \|w(k) - x^j(k+1)\|, \end{aligned} \tag{19}$$

where the second relation follows in view of the definition of $x^j(k+1)$ [cf. Eq. (8)].

By Proposition 2, we have $w(k) \rightarrow \tilde{x}$ for some $\tilde{x} \in X$ as $k \rightarrow \infty$. Furthermore, by Lemma 3(c) and the relation $w^i(k) = w(k)$ for all i and k , we have that the sequence $\{\|w(k) - x\|\}$ is nonincreasing for any $x \in X$. Therefore, the sequence $\{w(k)\}$ satisfies the conditions of Lemma 5, and by using this lemma we obtain

$$\|w(k) - \tilde{x}\| \leq 2 \text{dist}(w(k), X) \quad \text{for all } k.$$

Combining this relation with Eq. (19), we further obtain

$$\|w(k) - \tilde{x}\| \leq \frac{2R}{m} \sum_{i=1}^m \|w(k) - x^i(k+1)\|.$$

Taking the square of both sides and using the convexity of the square function $(\cdot)^2$, we have

$$\|w(k) - \tilde{x}\|^2 \leq \frac{4R^2}{m} \sum_{i=1}^m \|w(k) - x^i(k+1)\|^2. \quad (20)$$

Since $x^i(k+1) = P_{X_i}[w(k)]$ for all i and k , using Lemma 3(a) with the substitutions $x = \tilde{x} \in X$ and $e^i(k) = x^i(k+1) - w(k)$ for all i , we see that

$$\sum_{i=1}^m \|w(k) - x^i(k+1)\|^2 \leq m \|w(k) - \tilde{x}\|^2 - \sum_{i=1}^m \|x^i(k+1) - \tilde{x}\|^2 \quad \text{for all } k.$$

Using this relation in Eq. (20), we obtain

$$\|w(k) - \tilde{x}\|^2 \leq \frac{4R^2}{m} \left(m \|w(k) - \tilde{x}\|^2 - \sum_{i=1}^m \|x^i(k+1) - \tilde{x}\|^2 \right).$$

Rearranging the terms and using the relation $m \|w(k) - \tilde{x}\|^2 \leq \sum_{i=1}^m \|x^i(k) - \tilde{x}\|^2$ [cf. Lemma 3(a) with $w(k) = w^i(k)$ and $x = \tilde{x}$], we obtain

$$\sum_{i=1}^m \|x^i(k+1) - \tilde{x}\|^2 \leq \left(1 - \frac{1}{4R^2}\right) \sum_{i=1}^m \|x^i(k) - \tilde{x}\|^2,$$

which yields the desired result. ■

4 Constrained Optimization

We next consider the problem of optimizing the sum of convex objective functions corresponding to m agents connected over a time-varying topology. The goal of the agents is to cooperatively solve the constrained optimization problem

$$\text{minimize} \quad \sum_{i=1}^m f_i(x) \quad (21)$$

$$\text{subject to} \quad x \in \bigcap_{i=1}^m X_i, \quad (22)$$

where each $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function, representing the local objective function of agent i , and each $X_i \subseteq \mathbb{R}^n$ is a closed convex set, representing the local constraint set of agent i . We assume that the local objective function f_i and the local constraint set X_i are known to agent i only.

To keep our discussion general, we do not assume differentiability of any of the functions f_i . Since each f_i is convex over the entire \mathbb{R}^n , the function is differentiable almost everywhere (see [2] or [25]). At the points where the function fails to be differentiable, a subgradient exists and can be used in the role of a gradient. In particular, for a given convex function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and a point \bar{x} , a *subgradient of the function F at \bar{x}* is a vector $s_F(\bar{x}) \in \mathbb{R}^n$ such that

$$F(\bar{x}) + s_F(\bar{x})'(x - \bar{x}) \leq F(x) \quad \text{for all } x. \quad (23)$$

The set of all subgradients of F at a given point \bar{x} is denoted by $\partial F(\bar{x})$, and it is referred to as the *subdifferential set* of F at \bar{x} .

4.1 Distributed Projected Subgradient Algorithm

We introduce a distributed subgradient method for solving problem (21) using the assumptions imposed on the information exchange among the agents in Section 3.3. The main idea of the algorithm is the use of consensus as a mechanism for distributing the computations among the agents. In particular, each agent i starts with an initial estimate $x^i(0) \in X_i$ and updates its estimate. An agent i updates its estimate by combining the estimates received from its neighbors, by taking a subgradient step to minimize its objective function f_i , and by projecting on its constraint set X_i . Formally, each agent i updates according to the following rule:

$$v^i(k) = \sum_{j=1}^m a_j^i(k) x^j(k) \quad (24)$$

$$x^i(k+1) = P_{X_i} [v^i(k) - \alpha_k d_i(k)], \quad (25)$$

where the scalars $a_j^i(k)$ are nonnegative weights and the scalar $\alpha_k > 0$ is a stepsize. The vector $d_i(k)$ is a subgradient of the agent i local objective function $f_i(x)$ at $x = v^i(k)$.

We refer to the method (24)-(25) as the *projected subgradient algorithm*. To analyze this algorithm, we find it convenient to re-write the relation for $x^i(k+1)$ in an equivalent form. This form helps us identify the linear effects due to agents mixing the estimates [which will be driven by the transition matrices $\Phi(k, s)$], and the nonlinear effects due to taking subgradient steps and projecting. In particular, we re-write the relations (24)–(25) as follows:

$$v^i(k) = \sum_{j=1}^m a_j^i(k) x^j(k)$$

$$x^i(k+1) = v^i(k) - \alpha_k d_i(k) + \phi^i(k) \quad (26)$$

$$\phi^i(k) = P_{X_i} [v^i(k) - \alpha_k d_i(k)] - (v^i(k) - \alpha_k d_i(k)). \quad (27)$$

The evolution of the iterates is complicated due to the nonlinear effects of the projection operation, and even more complicated due to the projections on different sets. In our subsequent analysis, we study two special cases: 1) when the constraint sets are the same [i.e., $X_i = X$ for all i], but the agent connectivity is time-varying; and 2) when the

constraint sets X_i are different, but the agent communication graph is fully connected. In the analysis of both cases, we use a basic relation for the iterates $x^i(k)$ generated by the method in (27). The relation is established in the following lemma.

Lemma 6 Let Assumptions 2 and 3 and hold. Let $\{x^i(k)\}$ be the iterates generated by the algorithm (24)-(25). We have for any $z \in X = \cap_{i=1}^m X_i$ and all $k \geq 0$,

$$\begin{aligned} \sum_{i=1}^m \|x^i(k+1) - z\|^2 &\leq \sum_{i=1}^m \|x^i(k) - z\|^2 + \alpha_k^2 \sum_{i=1}^m \|d_i(k)\|^2 - 2\alpha_k \sum_{i=1}^m (f_i(v^i(k)) - f_i(z)) \\ &\quad - \sum_{i=1}^m \|\phi^i(k)\|^2. \end{aligned}$$

Proof. Since $x^i(k+1) = P_{X_i}[v^i(k) - \alpha_k d_i(k)]$, it follows from Lemma 1(b) and from the definition of the projection error $\phi^i(k)$ in (27) that for any $z \in X$,

$$\|x^i(k+1) - z\|^2 \leq \|v^i(k) - \alpha_k d_i(k) - z\|^2 - \|\phi^i(k)\|^2.$$

By expanding the term $\|v^i(k) - \alpha_k d_i(k) - z\|^2$, we obtain

$$\|v^i(k) - \alpha_k d_i(k) - z\|^2 = \|v^i(k) - z\|^2 + \alpha_k^2 \|d_i(k)\|^2 - 2\alpha_k d_i(k)'(v^i(k) - z).$$

Since $d_i(k)$ is a subgradient of $f_i(x)$ at $x = v^i(k)$, we have

$$d_i(k)'(v^i(k) - z) \geq f_i(v^i(k)) - f_i(z).$$

By combining the preceding relations, we obtain

$$\|x^i(k+1) - z\|^2 \leq \|v^i(k) - z\|^2 + \alpha_k^2 \|d_i(k)\|^2 - 2\alpha_k (f_i(v^i(k)) - f_i(z)) - \|\phi^i(k)\|^2.$$

Since $v^i(k) = \sum_{j=1}^m a_j^i(k) x^j(k)$, using the convexity of the norm square function and the stochasticity of the weights $a_j^i(k)$, $j = 1, \dots, m$, it follows that

$$\|v^i(k) - z\|^2 \leq \sum_{j=1}^m a_j^i(k) \|x^j(k) - z\|^2.$$

Combining the preceding two relations, we obtain

$$\begin{aligned} \|x^i(k+1) - z\|^2 &\leq \sum_{j=1}^m a_j^i(k) \|x^j(k) - z\|^2 + \alpha_k^2 \|d_i(k)\|^2 - 2\alpha_k (f_i(v^i(k)) - f_i(z)) \\ &\quad - \|\phi^i(k)\|^2. \end{aligned}$$

By summing the preceding relation over $i = 1, \dots, m$, and using the doubly stochasticity of the weights, i.e.,

$$\sum_{i=1}^m \sum_{j=1}^m a_j^i(k) \|x^j(k) - z\|^2 = \sum_{j=1}^m \left(\sum_{i=1}^m a_j^i(k) \right) \|x^j(k) - z\|^2 = \sum_{j=1}^m \|x^j(k) - z\|^2,$$

we obtain the desired relation. ■

4.1.1 Convergence when $X_i = X$ for all i

We first study the case when all constraint sets are the same, i.e., $X_i = X$ for all i . The next assumption formally states the conditions we adopt in the convergence analysis.

Assumption 6

- (a) The constraint sets X_i are the same, i.e., $X_i = X$ for a closed convex set X .
- (b) The subgradient sets of each f_i are bounded over the set X , i.e., there is a scalar $L > 0$ such that for all i ,

$$\|d\| \leq L \quad \text{for all } d \in \partial f_i(x) \quad \text{and all } x \in X.$$

The subgradient boundedness assumption in part (b) holds for example when the set X is compact (see [2]).

In proving our convergence results, we use a property of the infinite sum of products of the components of two sequences. In particular, for a scalar $\beta \in (0, 1)$ and a scalar sequence $\{\gamma_k\}$, we consider the ‘‘convolution’’ sequence $\sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell = \beta^k \gamma_0 + \beta^{k-1} \gamma_1 + \dots + \beta \gamma_{k-1} + \gamma_k$. We have the following result.

Lemma 7 Let $0 < \beta < 1$ and let $\{\gamma_k\}$ be a positive scalar sequence. Assume that $\lim_{k \rightarrow \infty} \gamma_k = 0$. Then

$$\lim_{k \rightarrow \infty} \sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell = 0.$$

In addition, if $\sum_k \gamma_k < \infty$, then

$$\sum_k \sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell < \infty.$$

Proof. Let $\epsilon > 0$ be arbitrary. Since $\alpha_k \rightarrow 0$, there is an index K such that $\alpha_k \leq \epsilon$ for all $k \geq K$. For all $k \geq K + 1$, we have

$$\sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell = \sum_{\ell=0}^K \beta^{k-\ell} \gamma_\ell + \sum_{\ell=K+1}^k \beta^{k-\ell} \gamma_\ell \leq \max_{0 \leq t \leq K} \gamma_t \sum_{\ell=0}^K \beta^{k-\ell} + \epsilon \sum_{\ell=K+1}^k \beta^{k-\ell}.$$

Since $\sum_{\ell=K+1}^k \beta^{k-\ell} \leq \frac{1}{1-\beta}$ and

$$\sum_{\ell=0}^K \beta^{k-\ell} = \beta^k + \dots + \beta^{k-K} = \beta^{k-K} (1 + \dots + \beta^K) \leq \frac{\beta^{k-K}}{1-\beta},$$

it follows that for all $k \geq K + 1$,

$$\sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell \leq \max_{0 \leq t \leq K} \gamma_t \frac{\beta^{k-K}}{1-\beta} + \frac{\epsilon}{1-\beta}.$$

Therefore,

$$\limsup_{k \rightarrow \infty} \sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell \leq \frac{\epsilon}{1-\beta}.$$

Since ϵ is arbitrary, we conclude that $\limsup_{k \rightarrow \infty} \sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell = 0$, implying

$$\lim_{k \rightarrow \infty} \sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell = 0.$$

Suppose now $\sum_k \gamma_k < \infty$. Then, for any integer $M \geq 1$, we have

$$\sum_{k=0}^M \left(\sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell \right) = \sum_{\ell=0}^M \gamma_\ell \sum_{t=0}^{M-\ell} \beta^t \leq \sum_{\ell=0}^M \gamma_\ell \frac{1}{1-\beta},$$

implying that

$$\sum_{k=0}^{\infty} \left(\sum_{\ell=0}^k \beta^{k-\ell} \gamma_\ell \right) \leq \frac{1}{1-\beta} \sum_{\ell=0}^{\infty} \gamma_\ell < \infty.$$

■

Our goal is to show that the agent disagreements $\|x^i(k) - x^j(k)\|$ converge to zero. To measure the agent disagreements $\|x^i(k) - x^j(k)\|$ in time, we consider their average $\frac{1}{m} \sum_{j=1}^m x^j(k)$, and consider the agent disagreement with respect to this average. In particular, we define

$$y(k) = \frac{1}{m} \sum_{j=1}^m x^j(k) \quad \text{for all } k.$$

In view of Eq. (26), we have

$$y(k+1) = \frac{1}{m} \sum_{i=1}^m v^i(k) - \frac{\alpha_k}{m} \sum_{i=1}^m d_i(k) + \frac{1}{m} \sum_{i=1}^m \phi^i(k).$$

When the weights are doubly stochastic, since $v^i(k) = \sum_{j=1}^m a_j^i(k) x^j(k)$, it follows that

$$y(k+1) = y(k) - \frac{\alpha_k}{m} \sum_{i=1}^m d_i(k) + \frac{1}{m} \sum_{i=1}^m \phi^i(k). \quad (28)$$

Under Assumption 6, the assumptions on the agent weights and connectivity stated in Section 3.3, and some conditions on the stepsize α_k , the next lemma studies the convergence properties of the sequences $\{\|x^i(k) - y(k)\|\}$ for all i .

Lemma 8 Let Assumptions 2, 3, 4, 5, and 6 hold. Let $\{x^i(k)\}$ be the iterates generated by the algorithm (24)-(25) and consider the auxiliary sequence $\{y(k)\}$ defined in (28).

(a) If the stepsize satisfies $\lim_{k \rightarrow \infty} \alpha_k = 0$, then

$$\lim_{k \rightarrow \infty} \|x^i(k) - y(k)\| = 0 \quad \text{for all } i.$$

(b) If the stepsize satisfies $\sum_{k \rightarrow \infty} \alpha_k^2 < \infty$, then

$$\sum_{k=1}^{\infty} \alpha_k \|x^i(k) - y(k)\| < \infty \quad \text{for all } i.$$

Proof. (a) Using the relations in (27) and the transition matrices $\Phi(k, s)$, we can write for all i , and for all k and s with $k > s$,

$$\begin{aligned} x^i(k+1) &= \sum_{j=1}^m [\Phi(k, s)]_j^i x^j(s) - \sum_{r=s}^{k-1} \sum_{j=1}^m [\Phi(k, r+1)]_j^i \alpha_r d_j(r) - \alpha_k d_i(k) \\ &\quad + \sum_{r=s}^{k-1} \sum_{j=1}^m [\Phi(k, r+1)]_j^i \phi^j(r) + \phi^i(k). \end{aligned}$$

Similarly, using the transition matrices and relation (28), we can write for $y(k+1)$ and for all k and s with $k > s$,

$$y(k+1) = y(s) - \frac{1}{m} \sum_{r=s}^{k-1} \sum_{j=1}^m \alpha_r d_j(r) - \frac{\alpha_k}{m} \sum_{i=1}^m d_i(k) + \frac{1}{m} \sum_{r=s}^{k-1} \sum_{j=1}^m \phi^j(r) + \frac{1}{m} \sum_{j=1}^m \phi^j(k).$$

Therefore, since $y(s) = \frac{1}{m} \sum_{j=1}^m x^j(s)$, we have for $s = 0$,

$$\begin{aligned} \|x^i(k) - y(k)\| &\leq \sum_{j=1}^m \left| [\Phi(k-1, 0)]_j^i - \frac{1}{m} \right| \|x^j(0)\| \\ &\quad + \sum_{r=0}^{k-2} \sum_{j=1}^m \left| [\Phi(k-1, r+1)]_j^i - \frac{1}{m} \right| \alpha_r \|d_j(r)\| \\ &\quad + \alpha_{k-1} \|d_i(k-1)\| + \frac{\alpha_{k-1}}{m} \sum_{j=1}^m \|d_j(k-1)\| \\ &\quad + \sum_{r=0}^{k-2} \sum_{j=1}^m \left| [\Phi(k-1, r+1)]_j^i - \frac{1}{m} \right| \|\phi^j(r)\| \\ &\quad + \|\phi^i(k-1)\| + \frac{1}{m} \sum_{j=1}^m \|\phi^j(k-1)\|. \end{aligned}$$

Using the estimate for $\left| [\Phi(k, s)]_j^i - \frac{1}{m} \right|$ of Proposition 1, we have for all $k \geq s$,

$$\left| [\Phi(k, s)]_j^i - \frac{1}{m} \right| \leq C \beta^{k-s} \quad \text{for all } i, j,$$

with $C = 2 \frac{1+\eta^{-B_0}}{1-\eta^{B_0}}$ and $\beta = (1 - \eta^{B_0})^{\frac{1}{B_0}}$. Hence, using this relation and the subgradient boundedness, we obtain for all i and $k \geq 2$,

$$\|x^i(k) - y(k)\| \leq mC\beta^{k-1} \sum_{j=1}^m \|x^j(0)\| + mCL \sum_{r=0}^{k-2} \beta^{k-r} \alpha_r + 2\alpha_{k-1}L$$

$$+C \sum_{r=0}^{k-2} \beta^{k-r} \sum_{j=1}^m \|\phi^j(r)\| + \|\phi^i(k-1)\| + \frac{1}{m} \sum_{j=1}^m \|\phi^j(k-1)\|. \quad (29)$$

We next show that the errors $\phi^i(k)$ satisfy $\|\phi^i(k)\| \leq \alpha_k L$ for all i and k . In view of the relations in (27), since $x^j(k) \in X_j = X$ for all k and j , and the vector $a^i(k)$ is stochastic for all i and k , it follows that $v^i(k) \in X$ for all i and k . Furthermore, by the projection property in Lemma 1(b), we have for all i and k ,

$$\begin{aligned} \|x^i(k+1) - v^i(k)\|^2 &\leq \|v^i(k) - \alpha_k d_i(k) - v^i(k)\|^2 - \|x^i(k+1) - (v^i(k) - \alpha_k d_i(k))\|^2 \\ &\leq \alpha_k^2 L^2 - \|\phi^i(k)\|^2, \end{aligned}$$

where in the last inequality we use $\|d_i(k)\| \leq L$ (see Assumption 6). It follows that $\|\phi^i(k)\| \leq \alpha_k L$ for all i and k . By using this in relation (29), we obtain

$$\|x^i(k) - y(k)\| \leq mC\beta^{k-1} \sum_{j=1}^m \|x^j(0)\| + 2mCL \sum_{r=0}^{k-2} \beta^{k-r} \alpha_r + 4\alpha_{k-1}L. \quad (30)$$

By taking the limit superior in relation (30) and using the facts $\beta^k \rightarrow 0$ (recall $0 < \beta < 1$) and $\alpha_k \rightarrow 0$, we obtain for all i ,

$$\limsup_{k \rightarrow \infty} \|x^i(k) - y(k)\| \leq 2mCL \limsup_{k \rightarrow \infty} \sum_{r=0}^{k-2} \beta^{k-r} \alpha_r$$

Finally, since $0 < \beta < 1$ and $\lim_{k \rightarrow \infty} \alpha_k = 0$, by Lemma 7 we have

$$\lim_{k \rightarrow \infty} \sum_{r=0}^{k-2} \beta^{k-r} \alpha_r = 0.$$

In view of the preceding two relations, it follows that $\lim_{k \rightarrow \infty} \|x^i(k) - y(k)\| = 0$ for all i .

(b) By multiplying the relation in (30) with α_k , we obtain

$$\alpha_k \|x^i(k) - y(k)\| \leq mC\alpha_k \beta^{k-1} \sum_{j=1}^m \|x^j(0)\| + 2mCL \sum_{r=0}^{k-2} \beta^{k-r} \alpha_k \alpha_r + 4\alpha_k \alpha_{k-1}L.$$

By using $\alpha_k \beta^{k-1} \leq \alpha_k^2 + \beta^{2(k-1)}$ and $2\alpha_k \alpha_r \leq \alpha_k^2 + \alpha_r^2$ for any k and r , we have

$$\alpha_k \|x^i(k) - y(k)\| \leq mC\beta^{2(k-1)} \sum_{j=1}^m \|x^j(0)\| + mCA\alpha_k^2 + mCL \sum_{r=0}^{k-2} \beta^{k-r} \alpha_r^2 + 2L(\alpha_k^2 + \alpha_{k-1}^2),$$

where $A = \sum_{j=1}^m \|x^j(0)\| + \frac{L}{(1-\beta)}$. Therefore, by summing and grouping some of the terms, we obtain

$$\sum_{k=1}^{\infty} \alpha_k \|x^i(k) - y(k)\| \leq mC \left(\sum_{k=1}^{\infty} \beta^{2(k-1)} \right) \sum_{j=1}^m \|x^j(0)\|$$

$$+ \sum_{k=1}^{\infty} (mCA\alpha_k^2 + 2L(\alpha_k^2 + \alpha_{k-1}^2)) + mCL \sum_{k=1}^{\infty} \sum_{r=0}^{k-2} \beta^{k-r} \alpha_r^2.$$

In the preceding relation, the first term is summable since $0 < \beta < 1$. The second term is summable since $\sum_k \alpha_k^2 < \infty$. The third term is also summable by Lemma 7. Hence, $\sum_{k=1}^{\infty} \alpha_k \|x^i(k) - y(k)\| < \infty$. ■

Using Lemmas 6 and 8, we next show that the iterates $x^i(k)$ converge to an optimal solution when we use a stepsize converging to zero fast enough.

Proposition 4 Let Assumptions 2, 3, 4, 5, and 6 hold. Let $\{x^i(k)\}$ be the iterates generated by the algorithm (24)-(25) with the stepsize satisfying $\sum_k \alpha_k = \infty$ and $\sum_k \alpha_k^2 < \infty$. In addition, assume that the optimal solution set X^* is nonempty. Then, there exists an optimal point $x^* \in X^*$ such that

$$\lim_{k \rightarrow \infty} \|x^i(k) - x^*\| = 0 \quad \text{for all } i.$$

Proof. From Lemma 6, we have for $z \in X$ and all k ,

$$\begin{aligned} \sum_{i=1}^m \|x^i(k+1) - z\|^2 &\leq \sum_{j=1}^m \|x^j(k) - z\|^2 + \alpha_k^2 \sum_{i=1}^m \|d_i(k)\|^2 \\ &\quad - 2\alpha_k \sum_{i=1}^m (f_i(v^i(k)) - f_i(z)) - \sum_{i=1}^m \|\phi^i(k)\|^2. \end{aligned}$$

By dropping the nonpositive term on the right hand side, and by using the subgradient boundedness, we obtain

$$\begin{aligned} \sum_{i=1}^m \|x^i(k+1) - z\|^2 &\leq \sum_{j=1}^m \|x^j(k) - z\|^2 + \alpha_k^2 mL^2 - 2\alpha_k \sum_{i=1}^m (f_i(v^i(k)) - f_i(y(k))) \\ &\quad - 2\alpha_k (f(y(k)) - f(z)). \end{aligned} \quad (31)$$

In view of the subgradient boundedness and the stochasticity of the weights, it follows

$$|f_i(v^i(k)) - f_i(y(k))| \leq L \|v^i(k) - y(k)\| \leq L \sum_{j=1}^m a_j^i(k) \|x^j(k) - y(k)\|,$$

implying, by the doubly stochasticity of the weights, that

$$\sum_{i=1}^m |f_i(v^i(k)) - f_i(y(k))| \leq L \sum_{j=1}^m \left(\sum_{i=1}^m a_j^i(k) \right) \|x^j(k) - y(k)\| = L \sum_{j=1}^m \|x^j(k) - y(k)\|.$$

By using this in relation (31), we see that for any $z \in X$, and all i and k ,

$$\sum_{i=1}^m \|x^i(k+1) - z\|^2 \leq \sum_{j=1}^m \|x^j(k) - z\|^2 + \alpha_k^2 mL^2 + 2\alpha_k L \sum_{j=1}^m \|x^j(k) - y(k)\|$$

$$-2\alpha_k (f(y(k)) - f(z)).$$

By letting $z = z^* \in X^*$, and by re-arranging the terms and summing these relations over some arbitrary window from K to N with $K < N$, we obtain for any $z^* \in X^*$,

$$\begin{aligned} \sum_{i=1}^m \|x^i(N+1) - z^*\|^2 + 2 \sum_{k=K}^N \alpha_k (f(y(k)) - f(z^*)) &\leq \sum_{i=1}^m \|x^i(K) - z^*\|^2 \\ &+ mL^2 \sum_{k=K}^N \alpha_k^2 + 2L \sum_{k=K}^N \alpha_k \sum_{j=1}^m \|x^j(k) - y(k)\|. \end{aligned} \quad (32)$$

By letting $K = 1$ and $N \rightarrow \infty$ in relation (32), and using $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$ and $\sum_{k=1}^{\infty} \alpha_k \sum_{j=1}^m \|x^j(k) - y(k)\| < \infty$ [which follows by Lemma 8], we obtain

$$\sum_{k=1}^{\infty} \alpha_k (f(y(k)) - f(z^*)) < \infty.$$

Since $x^j(k) \in X$ for all j , we have $y(k) \in X$ for all k . Since $z^* \in X^*$, it follows that $f(y(k)) - f^* \geq 0$ for all k . This relation, the assumption that $\sum_{k=1}^{\infty} \alpha_k = \infty$, and $\sum_{k=1}^{\infty} \alpha_k (f(y(k)) - f(z^*)) < \infty$ imply

$$\liminf_{k \rightarrow \infty} (f(y(k)) - f^*) = 0. \quad (33)$$

We next show that each sequence $\{x^i(k)\}$ converges to the same optimal point. By dropping the nonnegative term involving $f(y(k)) - f(z^*)$ in (32), we have

$$\sum_{i=1}^m \|x^i(N+1) - z^*\|^2 \leq \sum_{i=1}^m \|x^i(K) - z^*\|^2 + mL^2 \sum_{k=K}^N \alpha_k^2 + 2L \sum_{k=K}^N \alpha_k \sum_{j=1}^m \|x^j(k) - y(k)\|.$$

Since $\sum_k \alpha_k^2 < \infty$ and $\sum_{k=1}^{\infty} \alpha_k \sum_{j=1}^m \|x^j(k) - y(k)\| < \infty$, it follows that the sequence $\{x^i(k)\}$ is bounded for each i , and

$$\limsup_{N \rightarrow \infty} \sum_{i=1}^m \|x^i(N+1) - z^*\|^2 \leq \liminf_{K \rightarrow \infty} \sum_{i=1}^m \|x^i(K) - z^*\|^2 \quad \text{for all } i.$$

Thus, the scalar sequence $\{\sum_{i=1}^m \|x^i(k) - z^*\|\}$ is convergent for every $z^* \in X^*$. By Lemma 8, we have $\lim_{k \rightarrow \infty} \|x^i(k) - y(k)\| = 0$. Therefore, it also follows that $\{y(k)\}$ is bounded and the scalar sequence $\{\|y(k) - z^*\|\}$ is convergent for every $z^* \in X^*$. Since $y(k)$ is bounded, it must have a limit point, and in view of $\liminf_{k \rightarrow \infty} f(y(k)) = f^*$ [cf. Eq. (33)] and the continuity of f (due to convexity of f over \mathbb{R}^n), one of the limit points of $\{y(k)\}$ must belong to X^* ; denote this limit point by x^* . Since the sequence $\{\|y(k) - x^*\|\}$ is convergent, it follows that $y(k)$ can have a unique limit point, i.e., $\lim_{k \rightarrow \infty} y(k) = x^*$. This and $\lim_{k \rightarrow \infty} \|x^i(k) - y(k)\| = 0$ imply that each of the sequences $\{x^i(k)\}$ converges to the same $x^* \in X^*$. ■

4.1.2 Convergence for uniform weights

We next consider a version of the projected subgradient algorithm (24)–(25) for the case when the agents use uniform weights, i.e., $a_j^i(k) = \frac{1}{m}$ for all i, j , and $k \geq 0$. We show that the estimates generated by the method converge to an optimal solution of problem (21) under some conditions. In particular, we adopt the following assumption in our analysis.

Assumption 7 For each i , the local constraint set X_i is a compact set, i.e., there exists a scalar $B > 0$ such that

$$\|x\| \leq B \quad \text{for all } x \in X_i \text{ and all } i.$$

An important implication of the preceding assumption is that, for each i , the subgradients of the function f_i at all points $x \in X_i$ are uniformly bounded, i.e., there exists a scalar $L > 0$ such that

$$\|g\| \leq L \quad \text{for all } g \in \partial f_i(x), \text{ all } x \in X_i \text{ and all } i. \quad (34)$$

Under this and the interior point assumption on the intersection set $X = \cap_{i=1}^m X_i$ (cf. Assumption 1), we have the following result.

Proposition 5 Let Assumptions 1 and 7 and hold. Let $\{x^i(k)\}$ be the iterates generated by the algorithm (24)–(25) with the weight vectors $a^i(k) = (1/m, \dots, 1/m)'$ for all i and k , and the stepsize satisfying $\sum_k \alpha_k = \infty$ and $\sum_k \alpha_k^2 < \infty$. Then, the sequences $\{x^i(k)\}$, $i = 1, \dots, m$, converge to the same optimal point, i.e.,

$$\lim_{k \rightarrow \infty} x^i(k) = x^* \quad \text{for some } x^* \in X^* \text{ and all } i.$$

Proof. By Assumption 7, each set X_i is compact, which implies that the intersection set $X = \cap_{i=1}^m X_i$ is compact. Since each function f_i is continuous (due to being convex over \mathbb{R}^n), it follows from Weierstrass' Theorem that problem (21) has an optimal solution, denoted by $z^* \in X$. By using Lemma 6 with $z = z^*$, we have for all i and $k \geq 0$,

$$\begin{aligned} \sum_{i=1}^m \|x^i(k+1) - z^*\|^2 &\leq \sum_{i=1}^m \|x^i(k) - z^*\|^2 + \alpha_k^2 \sum_{i=1}^m \|d_i(k)\|^2 \\ &\quad - 2\alpha_k \sum_{i=1}^m (f_i(v^i(k)) - f_i(z^*)) - \sum_{i=1}^m \|\phi^i(k)\|^2. \end{aligned} \quad (35)$$

For any $k \geq 0$, define the vector $s(k)$ by

$$s(k) = \frac{\epsilon}{\epsilon + \delta} \bar{x} + \frac{\delta}{\epsilon + \delta} \hat{x}(k),$$

where $\hat{x}(k) = \frac{1}{m} \sum_{i=1}^m x^i(k)$, $\epsilon = \sum_{j=1}^m \text{dist}(\hat{x}(k), X_j)$, and δ is the scalar given in Assumption 1 (cf. Lemma 2). By using the subgradient boundedness [see (34)] and adding and subtracting the term $2\alpha_k \sum_{i=1}^m f_i(s(k))$ in Eq. (35), we obtain

$$\begin{aligned} \sum_{i=1}^m \|x^i(k+1) - z^*\|^2 &\leq \sum_{i=1}^m \|x^i(k) - z^*\|^2 + \alpha_k^2 m L^2 - \sum_{i=1}^m \|\phi^i(k)\|^2 \\ &\quad - 2\alpha_k \sum_{i=1}^m (f_i(s(k)) - f_i(z^*)) - 2\alpha_k \sum_{i=1}^m (f_i(v^i(k)) - f_i(s(k))). \end{aligned}$$

Using the subgradient definition and the subgradient boundedness assumption, we further have

$$|f_i(v^i(k)) - f_i(s(k))| \leq L \|v^i(k) - s(k)\| \quad \text{for all } i \text{ and } k.$$

Combining these relations with the preceding and using the notation $f = \sum_{i=1}^m f_i$, we obtain

$$\begin{aligned} \sum_{i=1}^m \|x^i(k+1) - z^*\|^2 &\leq \sum_{i=1}^m \|x^i(k) - z^*\|^2 + \alpha_k^2 m L^2 - \sum_{i=1}^m \|\phi^i(k)\|^2 \\ &\quad - 2\alpha_k (f(s(k)) - f(z^*)) + 2\alpha_k L \sum_{i=1}^m \|v^i(k) - s(k)\|. \end{aligned} \quad (36)$$

Since the weights are all equal, from relation (24) we have $v^i(k) = \hat{x}(k)$ for all i and k . Using Lemma 2(b) with the substitution $s = s(k)$ and $\hat{x} = \hat{x}(k) = \frac{1}{m} \sum_{j=1}^m x^j(k)$, we obtain

$$\|v^i(k) - s(k)\| \leq \frac{1}{\delta m} \left(\sum_{j=1}^m \|x^j(k) - \bar{x}\| \right) \left(\sum_{j=1}^m \text{dist}(\hat{x}(k), X_j) \right) \quad \text{for all } i \text{ and } k.$$

Since $x^j(k) \in X_j$, we have $\text{dist}(\hat{x}(k), X_j) \leq \|\hat{x}(k) - x^j(k)\|$ for all j and k . Furthermore, since $\bar{x} \in X \subseteq X_j$ for all j , using Assumption 7, we obtain $\|x^j(k) - \bar{x}\| \leq 2B$. Therefore, for all i and k ,

$$\|v^i(k) - s(k)\| \leq \frac{2B}{\delta} \sum_{j=1}^m \text{dist}(\hat{x}(k), X_j) \leq \frac{2B}{\delta} \sum_{j=1}^m \|\hat{x}(k) - x^j(k)\|. \quad (37)$$

Moreover, we have $\hat{x}(k) = v^j(k)$ for all j and k , implying

$$\|x^j(k+1) - \hat{x}(k)\| = \|x^j(k+1) - (v^j(k) - \alpha_k d_j(k))\| + \alpha_k \|d_j(k)\|.$$

In view of the definition of the error term $\phi^i(k)$ in (27) and the subgradient boundedness, it follows

$$\|x^j(k+1) - \hat{x}(k)\| \leq \|\phi^j(k)\| + \alpha_k L,$$

which when substituted in relation (37) yields

$$\|v^i(k) - s(k)\| \leq \frac{2B}{\delta} \left(\alpha_k m L + \sum_{j=1}^m \|\phi^j(k)\| \right) \quad \text{for all } i \text{ and } k. \quad (38)$$

We now substitute the estimate (38) in Eq. (36) and obtain for all k ,

$$\begin{aligned} \sum_{i=1}^m \|x^i(k+1) - z^*\|^2 &\leq \sum_{i=1}^m \|x^i(k) - z^*\|^2 + \alpha_k^2 m L^2 - \sum_{i=1}^m \|\phi^i(k)\|^2 \\ &\quad - 2\alpha_k (f(s(k)) - f(z^*)) + \frac{4m^2 B L^2}{\delta} \alpha_k^2 \\ &\quad + \frac{4\alpha_k m B L}{\delta} \sum_{i=1}^m \|\phi^i(k)\|. \end{aligned} \quad (39)$$

Note that for each i , we can write

$$\begin{aligned} \frac{4\alpha_k m B L}{\delta} \|\phi^i(k)\| &= 2 \left(\frac{2\sqrt{2}\alpha_k m B L}{\delta} \right) \left(\frac{1}{\sqrt{2}} \|\phi^i(k)\| \right) \\ &\leq \left(\frac{2\sqrt{2}\alpha_k m B L}{\delta} \right)^2 + \frac{1}{2} \|\phi^i(k)\|^2. \end{aligned}$$

Therefore, by summing the preceding relations over i , we have for all k ,

$$\frac{4\alpha_k m B L}{\delta} \sum_{i=1}^m \|\phi^i(k)\| \leq \frac{8m^3 B^2 L^2}{\delta^2} \alpha_k^2 + \frac{1}{2} \sum_{i=1}^m \|\phi^i(k)\|^2,$$

which when substituted in Eq. (39) yields

$$\begin{aligned} \sum_{i=1}^m \|x^i(k+1) - z^*\|^2 &\leq \sum_{i=1}^m \|x^i(k) - z^*\|^2 + C \alpha_k^2 - \frac{1}{2} \sum_{i=1}^m \|\phi^i(k)\|^2 \\ &\quad - 2\alpha_k (f(s(k)) - f(z^*)), \end{aligned}$$

where $C = mL^2 + \frac{4m^2 B L^2}{\delta} + \frac{8m^3 B^2 L^2}{\delta^2}$. By re-arranging the terms and summing the preceding relations over k for $k = K, \dots, N$ for some arbitrary K and N with $K < N$, we obtain

$$\begin{aligned} \sum_{i=1}^m \|x^i(N+1) - z^*\|^2 &+ \frac{1}{2} \sum_{k=K}^N \sum_{i=1}^m \|\phi^i(k)\|^2 + 2 \sum_{k=K}^N \alpha_k (f(s(k)) - f(z^*)) \\ &\leq \sum_{i=1}^m \|x^i(K) - z^*\|^2 + C \sum_{k=K}^N \alpha_k^2. \end{aligned} \quad (40)$$

By setting $K = 0$ and letting $N \rightarrow \infty$, in view of $\sum_k \alpha_k^2 < \infty$, we see that

$$\frac{1}{2} \sum_{k=0}^{\infty} \sum_{i=1}^m \|\phi^i(k)\|^2 + 2 \sum_{k=0}^{\infty} \alpha_k (f(s(k)) - f(z^*)) < \infty.$$

Since by Lemma 2(a) we have $s(k) \in X$, the relation $\sum_{i=1}^m (f_i(s(k)) - f_i(z^*)) \geq 0$ holds for all k , thus implying that

$$\frac{1}{2} \sum_{k=0}^{\infty} \sum_{i=1}^m \|\phi^i(k)\|^2 < \infty,$$

$$\sum_{k=0}^{\infty} \alpha_k (f(s(k)) - f(z^*)) < \infty.$$

In view of the former of the preceding two relations, we have

$$\lim_{k \rightarrow \infty} \phi^i(k) = 0 \quad \text{for all } i,$$

while from the latter, since $\sum_k \alpha_k = \infty$ and $f(s(k)) - f^* \geq 0$ [because $s(k) \in X$ for all k], we obtain

$$\liminf_{k \rightarrow \infty} f(s(k)) = f^*. \quad (41)$$

Since $\phi^i(k) \rightarrow 0$ for all i and $\alpha_k \rightarrow 0$ [in view of $\sum_k \alpha_k^2 < \infty$], from Eq. (38) it follows that

$$\lim_{k \rightarrow \infty} \|v^i(k) - s(k)\| = 0 \quad \text{for all } i.$$

Finally, since $x^i(k+1) = v^i(k) - \alpha_k d_i(k) + \phi^i(k)$ [see (27)], in view of $\alpha_k \rightarrow 0$, $\|d_i(k)\| \leq L$, and $\phi^i(k) \rightarrow 0$, we see that $\lim_{k \rightarrow \infty} \|x^i(k+1) - v^i(k)\| = 0$ for all i . This and the preceding relation yield

$$\lim_{k \rightarrow \infty} \|x^i(k+1) - s(k)\| = 0 \quad \text{for all } i.$$

We now show that the sequences $\{x^i(k)\}, i = 1, \dots, m$, converge to the same limit point, which lies in the optimal solution set X^* . By taking limsup as $N \rightarrow \infty$ in relation (40) and then liminf as $K \rightarrow \infty$, (while dropping the nonnegative terms on the right hand side there), since $\sum_k \alpha_k^2 < \infty$, we obtain for any $z^* \in X^*$,

$$\limsup_{N \rightarrow \infty} \sum_{i=1}^m \|x^i(N+1) - z^*\|^2 \leq \liminf_{K \rightarrow \infty} \sum_{i=1}^m \|x^i(K) - z^*\|^2,$$

implying that the scalar sequence $\{\sum_{i=1}^m \|x^i(k) - z^*\|\}$ is convergent for every $z^* \in X^*$. Since $\|x^i(k+1) - s(k)\| \rightarrow 0$ for all i , it follows that the scalar sequence $\{\|s(k) - z^*\|\}$ is also convergent for every $z^* \in X^*$. In view of $\liminf_{k \rightarrow \infty} f(s_k) = f^*$ [cf. Eq. (41)], it follows that one of the limit points of $\{s_k\}$ must belong to X^* ; denote this limit by x^* . Since $\{\|s(k) - z^*\|\}$ is convergent for $z^* = x^*$, it follows that $\lim_{k \rightarrow \infty} s(k) = x^*$. This and $\|x^i(k+1) - s(k)\| \rightarrow 0$ for all i imply that each of the sequences $\{x^i(k)\}$ converges to a vector x^* , with $x^* \in X^*$. ■

5 Conclusions

We studied constrained consensus and optimization problems where agent i 's estimate is constrained to lie in a closed convex set X_i . For the constrained consensus problem, we presented a distributed projected consensus algorithm and studied its convergence properties. Under some assumptions on the agent weights and the connectivity of the network, we proved that each of the estimates converge to the same limit, which belongs to the intersection of the constraint sets X_i . We also showed that the convergence rate is geometric under an interior point assumption for the case when agent weights are

time-invariant and uniform. For the constrained optimization problem, we presented a distributed projected subgradient algorithm. We showed that with a stepsize converging to zero fast enough, the estimates generated by the subgradient algorithm converges to an optimal solution for the case when all agent constraint sets are the same and when agent weights are time-invariant and uniform.

The framework and algorithms studied in this paper motivate a number of interesting research directions. One interesting future direction is to extend the constrained optimization problem to include both local and global constraints, i.e., constraints known by all the agents. While global constraints can also be addressed using the “primal projection” algorithms of this paper, an interesting alternative would be to use “primal-dual” subgradient algorithms, in which dual variables (or prices) are used to ensure feasibility of agent estimates with respect to global constraints. Such algorithms have been studied in recent work [19] for general convex constrained optimization problems (without a multi-agent network structure).

Moreover, in this paper, we presented convergence results for the distributed subgradient algorithm for two cases: agents have time-varying weights but the same constraint set; and agents have time-invariant uniform weights and different constraint sets. When agents have different constraint sets, the convergence analysis relies on an error bound that relates the distances of the iterates (generated with constant uniform weights) to each X_i with the distance of the iterates to the intersection set under an interior point condition (cf. Lemma 2). This error bound is also used in establishing the geometric convergence rate of the projected consensus algorithm with constant uniform weights. These results can be extended using a similar analysis once an error bound is established for the general case with time-varying weights. We leave this for future work.

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