Constrained Lagrangian Circuits: Final Report for CDS 205

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1 Introduction

It is often true in mechanical systems that the conversion between Lagrangian and Hamiltonian descriptions via the Legendre transformation is invertible. However, although there have been multiple different approaches for applying Lagrangian mechanics to electrical circuits (see e.g. [CM74], [YM06]), the transformation is almost always degenerate. One such approach is to abandon the standard method of using Hamilton's Principle to generate a symplectic structure on the cotangent bundle of the configuration manifold, and instead use the Hamilton-d'Alembert-Pontryagin Principle to generate a Dirac structure on the direct sum of the tangent and cotangent bundles. This report will follow the early part of that approach, which was taken by Sina Ober-Blöbaum as part her work on variational integration methods presented at the Fifth Annual Structured Integrators Workshop held at Caltech on 7-8 May 2009 [OB09], and summarize one method for constructing a Lagrangian on electrical circuits. In addition, it will answer a hanging question about a physical meaning behind a particular submanifold of the configuration space.

2 A Brief Review of Circuits

Charge q, current i, and voltage v are taken to be familiar quantities. Electric flux ϕ is defined as the time derivative of the voltage across a circuit element.

The following definitions are taken from [NR05]. A *node* is a point in the circuit where two or more elements meet. A *path* is a trace of adjacent elements with no elements included more than once. A *branch* is a path that connects two nodes. A *loop* is a path that begins and ends at the same node. A *mesh* is a loop that does not enclose any other loops. A *planar circuit* is a circuit that can be drawn on a plane without crossing branches.

Unless otherwise noted, the term "branch" will be used interchangeably with the term "element" or "circuit element".

Kirchoff's Voltage Law (KVL) states that the sum of the voltage drops around any loop is identically zero. Kirchoff's Current Law (KCL) states that the sum of currents entering a node is exactly equal to the sum total of currents exiting the node.

For this analysis, we will look at a restriction to ideal linear devices—inductors satisfy Equation 1, capacitors satisfy Equation 2, and resistors satisfy Equation 3.

$$\phi(t) = Li(t) \tag{1}$$
$$i(t) = C\phi(t) \tag{2}$$
$$v(t) = Ri(t) \tag{3}$$

The instantaneous power dissipated/absorbed/generated by a circuit element is given by Equation 4, the product of the voltage across the element and the current through it.

$$P = VI \tag{4}$$

3 Developing a Lagrangian Formulation for Circuits

In taking a geometrical approach to analyzing circuits, we define the configuration manifold to be the charge space $Q \subset \mathbb{R}^n$ of the circuit branches, with points on the manifold denoted $q \in Q$. For simplicity of later analysis, we enumerate the elements by letting the inductors be the first n_L dimensions of Q, the capacitors be the next n_C dimensions, the resistors the next n_R dimensions, and the voltage sources the final n_V dimension, where n_L, n_C, n_R, and n_V are the number of inductors, capacitors, resistors, and voltage sources, respectively. In addition, we give each of those devices an assumed current flow direction. For a particular charge configuration q, the tangent space $T_qQ \subset \mathbb{R}^n$ is the space of all possible currents i passing through the the branches. The corresponding cotangent space $T_q^*Q \subset \mathbb{R}^n$ is the space of electric fluxes ϕ.

Having already provided directed labels for the n branches, we now arbitrarily enumerate the m nodes in the circuit (assuming one has been defined as ground). Then the incidence matrix $K \in \mathbb{R}^{n,m}$ is an matrix describing how the branches and nodes are connected.

$$
\begin{aligned}
K_{i,j} &= \begin{cases} -1 & \text{branch current i flows into node j} \\ +1 & \text{branch current i flows out of node j} \\ 0 & \text{else} \end{cases} \qquad (5) \\
&= \left[K_L^T, K_C^T, K_R^T, K_V^T\right]^T
\end{aligned}
$$

where K_α is the incidence matrix for the inductive, capacitive, resistive, and source branches, respectively. It is clear to see that this matrix can be used to write the KCL equations as:

$$K^T i\left(t\right) = 0 \qquad (6)$$

Note that by KCL, we can define an integrable distribution on the tangent bundle TQ via Equation 7.

$$
\begin{aligned}
\Delta_Q\left(q\right) &= \left\{i \in T_q Q | K^T i = 0\right\} \\
&= \mathcal{N}\left(K^T\right) \qquad (7) \\
& \qquad (8)
\end{aligned}
$$

Using the electrical power relationship $P = VI$ for a device with current I with a current drop V, along with the relationships of q to i and v to ϕ, and understanding that ideal inductors and capacitors are purely reactive (they dissipate no energy), it is easy to see that for an inductor with inductance L and current i, the stored magnetic energy is given by Equation 9 and the stored electric energy in a capacitor with capacitance C holding a charge q is given by Equation 10.

$$
\begin{aligned}
E_L &= \frac{1}{2} L i^2 \qquad (9) \\
E_C &= \frac{1}{2} \frac{1}{C} q^2 \qquad (10)
\end{aligned}
$$

Recall from mechanical systems that the kinetic energy is energy stored in the system that is dependent on the tangent-space variables and not the configuration-space variables, and potential energy depends only on position the configuration manifold. Then by direct analogy, in this chosen coordinate system we can consider the "kinetic" energy of an electrical circuit to be the magnetic energy stored in inductors and the "potential" energy to be the electric energy stored in capacitors. From this, we are led to define a Lagrangian in Equation 11 along with forcing terms in Equation 12, where $R\left(i\right)$ is the action of resistive elements on the incident current and V is the action of active voltage sources.

$$
\begin{aligned}
\mathcal{L} : TQ &\rightarrow \mathbb{R} \\
\mathcal{L}\left(q, i\right) &= E_L\left(i\right) - E_C\left(q\right) \qquad (11) \\
f\left(i\right) &= R\left(i\right) + V \qquad (12)
\end{aligned}
$$

To show that this is indeed a Lagrangian for circuits, we examine the Lagrange-d'Alembert-Pontryagin Principle:

$$\delta \int_0^T \left(\mathcal{L}(q,i) + \langle \phi, \dot{q} - v \rangle\right) dt + \int_0^T f \delta q dt = 0, \delta q \ \in \Delta_Q(q) \tag{13}$$

and note that by taking the variations we get the known circuit equations: conservation of charge ($\dot{q} = i$), KVL with node voltages v ($\dot{p} = \frac{\partial}{\partial q}\mathcal{L} + f + Kv$), Gauss's law ($\phi = \frac{\partial}{\partial i}\mathcal{L}$), and KCL ($K^T v = 0$).

Since we are dealing with linear circuit elements, we can introduce some notation to concisely write this expression. We define a diagonal inductance matrix $L \in \mathbb{R}^n$, where $L_{j,j} = L_j$, the inductance of the j^{th} circuit element, and a diagonal inverse capacitance (elastance) matrix $C \in \mathbb{R}^n$, where $C_{j,j} = \frac{1}{C_j}$, the inverse of the capacitance of the j^{th} element. In order to exactly match the definitions of electric and magnetic energies, in the case where a branch has no inductance or capacitance, we set the corresponding element of the inductance or capacitance matrix to be zero.

This allows us to concretely write the Lagrangian as

$$\mathcal{L}(q,i) = \frac{1}{2} i^T L i - \frac{1}{2} q^T C q \tag{14}$$

For the moment, if we attempt to convert directly to a Hamiltonian formulation of the circuit, we find a problem. Namely, computing the fiber derivative (Equation 15) shows that the Legendre transformation for this Lagrangian is non-degenerate exactly when the every device in the circuit is an inductor.

$$\begin{aligned} \mathbb{F}\mathcal{L}(q,i) &= \left(q, \frac{\partial}{\partial i}\mathcal{L}(q,i)\right) \\ &= (q, Li) \end{aligned} \tag{15}$$

In order to rectify this situation, we attempt reduction. In essence, this reduction step is required because of a large redundancy in the configuration-space variables. In actuality, solution curves of the system will evolve on some lower dimensional submanifold of Q. We will now seek out that submanifold and a constrained Lagrangian.

By any desired method, we can construct a new matrix $K_2 \in \mathbb{R}^{n,m-n}$ such that $\mathcal{R}(K) = \mathcal{R}(K_2)^{\perp}$ (that is, $[K|K_2] \sim I$). Next we define variables $(\tilde{q}, \tilde{\imath})$ on the submanifold tangent-bundle:

$$\begin{aligned} q &= K_2 \tilde{q} \\ i &= K_2 \tilde{\imath} \end{aligned}$$

for $\tilde{q} \in \tilde{Q} \subset \mathbb{R}^{n-m}$, $\tilde{\imath} \in T_{\tilde{q}}\tilde{Q} \subset \mathbb{R}^{n-m}$. In addition, by requiring preservation of the pairing between tangent and cotangent vectors, we generate constrained cotangent vectors by Equation 16.

$$
\begin{aligned}
\langle \tilde{p}, \tilde{\imath} \rangle &= \langle p, i \rangle \\
\tilde{p}^T \tilde{\imath} &= p^T K_2 i \\
\tilde{p} &= K_2^T p
\end{aligned}
\tag{16}
$$

We then define a constrained Lagrangian via the pullback:

$$
\begin{aligned}
\mathcal{L}^C : T\tilde{Q} &\to \mathbb{R} \\
\mathcal{L}^C \left(\tilde{q}, \tilde{\imath}\right) &:= \mathcal{L}\left(K_2 \tilde{q}, K_2 \tilde{\imath}\right)
\end{aligned}
\tag{17}
$$

If we now examine the Legendre transformation of the constrained Lagrangian

$$
\begin{aligned}
\mathbb{F}\mathcal{L}^C \left(\tilde{q}, \tilde{\imath}\right) &= \left(\tilde{q}, \frac{\partial}{\partial \tilde{\imath}} \mathcal{L}^C \left(\tilde{q}, \tilde{\imath}\right)\right) \\
\left(\tilde{q}, \tilde{p}\right) &= \left(\tilde{q}, K_2^T L K_2 \tilde{\imath}\right)
\end{aligned}
\tag{18}
$$

Evidently, the Legendre transformation on the constrained Lagrangian is degenerate iff $\mathcal{N}\left(K_2^T L K_2\right) \neq 0$. It is clear that $\mathcal{R}\left(K_2\right) \cap \mathcal{N}\left(L\right) = \{0\}$ is a necessary condition for non-degeneracy.

By design $\mathcal{R}\left(K\right) = \mathcal{R}\left(K_2\right)^{\perp}$, and applying linear algebra we get $\mathcal{R}\left(K\right) = \mathcal{N}\left(K^T\right)^{\perp}$, and so by transitivity

$$
\begin{aligned}
\mathcal{R}\left(K_2\right) &= \mathcal{N}\left(K^T\right) \\
&= \left\{ i \in T_q Q | K_L^T i_L + K_C^T i_C + K_R^T i_R + K_V^T i_V = 0 \right\}
\end{aligned}
\tag{19}
$$

In addition,

$$
\begin{aligned}
\mathcal{N}\left(L\right) &= \{ i \in T_q Q | i_L = 0 \} \\
K_L^T \mathcal{N}\left(L\right) &= \{0\}
\end{aligned}
$$

Therefore, a necessary condition for the Legendre transformation of the constrained Lagrangian $\mathcal{L}^C$ to be nondegenerate is given in Equation 20

$$
\begin{aligned}
\{0\} &= \{ i \in T_q Q | i_L = 0, K_C i_C + K_R i_R + K_V i_V = 0 \} \\
\{0\} &= \mathcal{N}\left(\left[K_C, K_R, K_V\right]\right)
\end{aligned}
\tag{20}
$$

4 Physical Interpretation of $T\tilde{Q}$

As described thus far, the only way to generate the K_2 matrix is by way of the incidence matrix. Also, this matrix is not unique, since it is constructed by simply choosing basis vectors of a subspace. This means that there is not necessarily any physically meaningful description of elements of the submanifold $\tilde{Q}$ for a particular K_2 matrix. It would be preferable to have a method for constructing the K_2 matrix directly from the circuit diagram, and which is designed so that $\tilde{Q}$ is meaningful. Recalling that K_2 is simply the mapping between $T_{\tilde{q}}\tilde{Q}$ and T_qQ, we reexamine at the conditions that govern the submanifold $\tilde{Q}$:

1. All branch currents $i \in Q$ must be representable by a linear combination of the reduced branch currents $\tilde{\imath} \in \tilde{Q}$.
2. There must be $n - m$ total reduced branch currents.

One approach that allows you to construct such a set of branch currents is a well-established method from electrical engineering known as "mesh analysis," which is applicable in planar circuits. This procedure is essentially a direct application of KVL that allows an analyst to describe a large, complicated circuit with a smaller number of currents than there are devices. Even though there are fewer independent variables than in the full circuit, once the mesh currents are known, you can easily compute any of the branch currents or voltages by applying superposition and the constituency equations.

To perform mesh analysis:

1. Locate and enumerate all of the meshes in the circuit.
2. Define a unique directed "mesh current" for each mesh.
3. By applying superposition, write down the the KVL equation for each mesh current.
4. Simultaneously solve the resulting set of equations.

For example, consider the circuit shown in Figure 1. The mesh equations are:

$$\begin{aligned}
0 &= R_5 i_1 + R_6 i_1 + R_4 (i_1 - i_2) + R_3 (i_1 - i_3) \\
0 &= R_7 i_2 + R_9 i_2 + R_8 (i_2 - i_4) + R_4 * (i_2 - i_1) \\
0 &= R_2 i_3 + R_3 (i_3 - i_1) + R_1 (i_3 - i_4) \\
-I_1 &= i_4
\end{aligned}$$

For the sake of numeric calculations assume $I_1 = 1\text{A}$, $R_n = n\Omega$. Then solving the mesh equations yields:

$$\begin{aligned}
i_1 &= -0.1031\text{A} \\
i_2 &= -0.3004\text{A} \\
i_3 &= -0.2182\text{A} \\
i_4 &= -1\text{A}
\end{aligned}$$

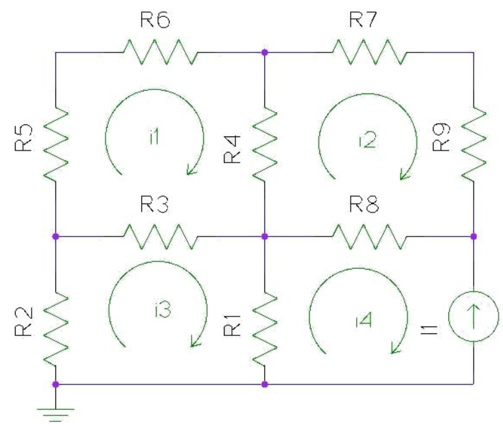

Figure 1: Mesh Analysis Example Circuit

where the negative currents indicate that the assumed sense of the current was wrong. Then, as a sanity check, the voltage across R_1 is $V_{R_1} = (i_3 - i_4)R_1 = 0.7818V$. If you instead solve 16 KVL and KCL equations on the elements (colloquially termed the "big gun" method [Ecc06]), you will also find $V_{R_1} = 0.7818V$.

It is clear that we can generate all of the branch currents from the mesh currents. However, we still need to have the correct number of mesh currents for them to serve as a basis on the tangent space of the submanifold.

Theorem 4.1. *For connected planar circuits with n branches and $m+1$ nodes and no singleton cutsets, the number of mesh currents is exactly $n - m$.*

Proof. The idea behind the proof is to inductively build the circuit an element at a time, at each stage keeping careful track of the number of nodes, branches, and meshes.

Since the circuit has no singleton cutsets, we can take a loop that encloses the entire circuit. Let the number of elements in the loop be k. Then there are $(k-1)+1$ nodes and 1 mesh. So for this circuit the equality holds.

Now consider how we may add new elements to a circuit in which the the equality holds.

1. We can add an element that does not connect to the circuit.

2. We can add an element that connects to an existing node on one terminal, with the other terminal floating.

3. We can add an element that connects both terminals to two existing nodes.

4. We can break the circuit at a node and insert the element.

Since we know the circuit is connected, we can choose to only add elements in an order that they connect to the existing subcircuit. Thus we can safely ignore moves of this type.

If we add an element that connects at one terminal, then we are creating exactly one new branch, one new node, and zero new meshes. The equality holds.

If we add an element that connects both terminals to existing nodes, then we are creating exactly one new branch, zero new nodes, and either creating one new mesh or dividing an existing mesh into two meshes. The equality holds.

If we break the circuit at a node and insert an element, we are creating one new branch, one new node, and zero new meshes. The equality holds.

By induction, we can generate the entire circuit in this manner, at all times preserving the equality. $\square$

5 Example

For the circuit shown in Figure 2, we have the constituency matrices

$$L = \begin{pmatrix} L_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{C_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{C_3} \end{pmatrix}$$

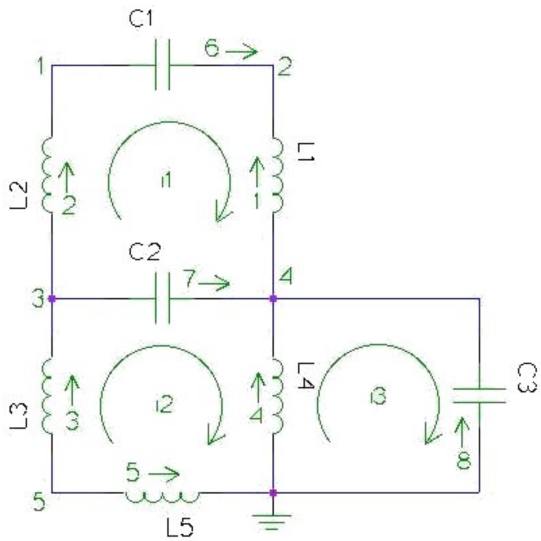

Figure 2: An Example Circuit for Analysis

in coordinates (where i_n is the branch current through node n, not the mesh current)

$$\begin{aligned}
q &= \left(q_L^T, q_C^T\right)^T \\
q_L &= \left(q_1, q_2, q_3, q_4, q_5\right)^T \\
q_C &= \left(q_6, q_7, q_8\right)^T \\
i &= \left(i_L^T, i_C^T\right)^T \\
i_L &= \left(i_1, i_2, i_3, i_4, i_5\right)^T \\
i_C &= \left(i_6, i_7, i_8\right)^T
\end{aligned}$$

and Lagrangian

$$\mathcal{L}\left(q, i\right) = \frac{1}{2} i^T L i - \frac{1}{2} q^T C q$$

The incidence matrix is

$$K = \begin{pmatrix} 0 & -1 & 0 & +1 & 0 \\ -1 & 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 & +1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The mesh equations are given by

$$\begin{aligned} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{pmatrix} &= K_2 \begin{pmatrix} \tilde{\imath}_1 \\ \tilde{\imath}_2 \\ \tilde{\imath}_3 \\ \tilde{\imath}_4 \end{pmatrix} \\ K_2 &= \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Then $\rho\left(K|K_2\right) = 8$, so $\mathcal{R}\left(K\right) = \mathcal{R}\left(K_2\right)^{\perp}$, as desired.

This yields a constrained Lagrangian of

$$\mathcal{L}^C\left(\tilde{q}, \tilde{\imath}\right) = \frac{1}{2}\tilde{\imath}^T \begin{pmatrix} L_1 + L_2 & 0 & 0 \\ 0 & L_3 + L_4 + L_5 & -L_4 \\ 0 & -L_4 & L_4 \end{pmatrix} \tilde{\imath} - \frac{1}{2}\tilde{q}^T \begin{pmatrix} \frac{1}{C_1} + \frac{1}{C_2} & -\frac{1}{C_2} & 0 \\ -\frac{1}{C_2} & \frac{1}{C_2} & 0 \\ 0 & 0 & \frac{1}{C_3} \end{pmatrix} \tilde{q}$$

In addition, for this circuit, since there are no resistors or voltage sources, and the K_C matrix

$$K_C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

is full rank, so we know that the Legendre transformation of the constrained Lagrangian is non-degenerate, so we can freely pass back-and-forth between a Lagrangian formulation and Hamiltonian formulation.

6 Conclusion

When attempting to formulate a Lagrangian function for circuit analysis, it is almost always the case that the Legendre transformation which converts the problem to a Hamiltonian formulation is degenerate. One approach to try to deal with this degeneracy involves applying known constraints and reformulating the problem on a lower-dimensional configuration manifold. In the special case of planar circuits, using mesh analysis allows you to generate the constrained Lagrangian and submanifold directly from the circuit.

Although this treatment has focused solely on linear circuits, it appears that by using the mesh analysis technique we can move immediately to analysis of nonlinear circuits, whereas without the technique it is not entirely clear how to generate the K_2 matrix.

In addition, it seems intuitively possible that using this method to analyze circuits from a geometric standpoint may allow the large circuit to be broken up into several smaller subcircuits with interaction terms. This has implications for circuit simulation, as well as analysis of electro-mechanical systems.

References

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