CDS 205 PROJECT PROPOSAL

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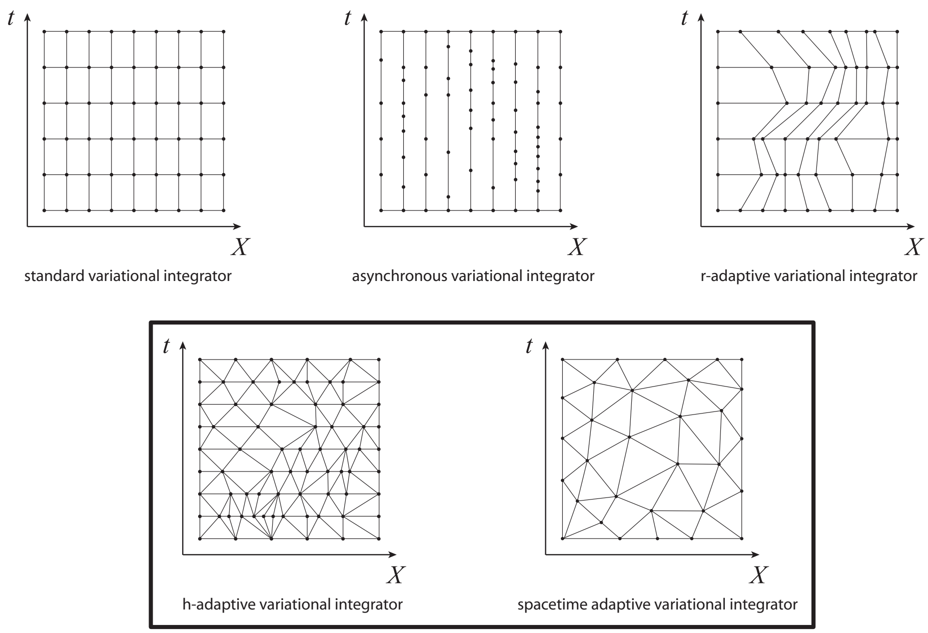
Goal: develop and understand spatially- and temporally- adaptive variational integrators for mechanical PDEs that do not require a constant number of degrees of freedom in the material domain. This problem is different from (but related to) r-adaptive or asynchronous variational integrators, which adapt DOFs in either the material or time domain but always maintain a constant number of material DOFs. In general, allowing a variable number of material DOFs allows the material domain to be remeshed in an *arbitrary* way over the course of the simulation. For the course project I will focus on a highly simplified model of 1D elastic body dynamics. My overall approach is a direct discretization of Hamilton's principle on a spacetime mesh. Rather than trying to adapt this mesh to meet some performance criterion, I will simply assume that a "good" mesh is known a priori (for now). The figure below is a cartoon of spacetime meshes for the 1D elasticity problem using different variational schemes; the types of problems I am interested are highlighted in the bottom row.



1 Lagrangian

Consider a 1D elastic body immersed in $\mathbb{R}^3$ with reference configuration $\mathcal{B} = [0, 1]$ and uniform mass density $\rho = 2$. The immersion $q : \mathcal{B} \to \mathbb{R}^3$ gives the configuration of the body at time t. The total kinetic energy at time t can therefore be written as

$$K_t = \int_{\mathcal{B}} \frac{\rho}{2} \left\| \frac{\partial q_t(X)}{\partial t} \right\|^2 dV.$$

We use a simple elastic potential

$$U_t = \frac{1}{2} \int_{\mathcal{B}} \rho \left(\left\| \frac{\partial q_t(X)}{\partial X} \right\|^2 - 1 \right) dV$$

where the leading factor $1/2$ is merely for convenience. The resulting Lagrangian is then simply

$$L_t = \mathcal{V}_{\mathcal{B}} + \int_{\mathcal{B}} \|\nabla q_t\|^2 \, dV$$

where in the final expression $\mathcal{V}_{\mathcal{B}}$ is the volume of the body, ∇ represents partial derivatives with respect to both space *and* time, and $\|\cdot\|$ is the usual Lorentzian metric on $\mathbb{R}^{3,1}$. The trajectory of q in time is given by the extrema of the corresponding action integral, i.e., solutions of

$$\delta\mathcal{S} = \delta \int_{t_0}^{t_f} L_t dt = 0$$

where there is zero variation in q at t_0 and t_f.

2 Discrete Lagrangian

Consider a spacetime domain with coordinates X and t along the material and temporal axes, respectively. Let V, E, and F be the sets of vertices, edges, and faces in a simplicial mesh of this domain. For the moment, assume that we are only interested in configurations in $\mathbb{R}$ (instead of $\mathbb{R}^3$). We can then describe the discrete configuration of the body via the map $q_D : \mathsf{V} \to \mathbb{R}$. Suppose that we interpolate $\hat{q}$ piecewise linearly over each simplex to get a map $\hat{q} : \mathcal{B} \to \mathbb{R}^3$. We can then express a *discrete* action $\mathcal{S}_D$ as the sum over each simplex of the action of the interpolated configuration:

$$\mathcal{S}_D = \sum_{f \in \mathsf{F}} \int_f L.$$

Note that the integral over f is actually an integral over space *and* time. In fact, this integral is very straightforward to compute. Let A_{qX}, A_{qt} and A_{Xt} be the *projected* areas of f along the t, X, and q axes, respectively. It is easy to show that on f

$$\frac{\partial \hat{q}}{\partial t} = \frac{A_{qX}}{A_{Xt}}$$

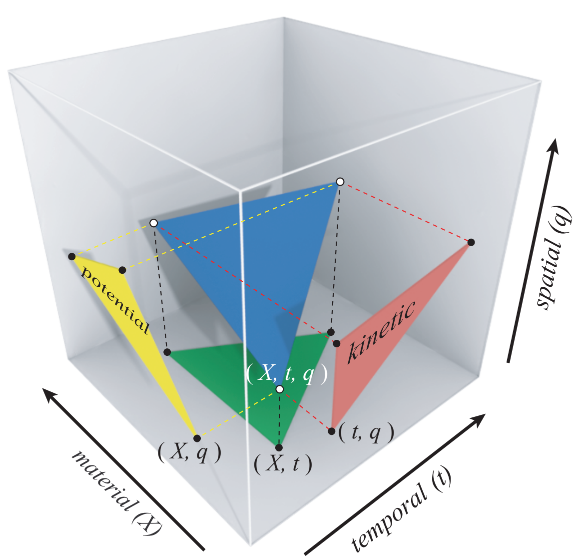
and

$$\frac{\partial \hat{q}}{\partial X} = \frac{A_{qt}}{A_{Xt}},$$

hence integrating these quantities over f (which has area A_{Xt} in the material/time domain) gives us

$$\mathcal{S}_D(f) = A_{qt}^2 - A_{qX}^2 + A_{Xt},$$

where the final term accounts for $\mathcal{V}_{\mathcal{B}}$ in the continuous Lagrangian.



3 Numerical Implementation

Our discrete action is quadratic in q, hence the system expressing extremization of this action with respect to q is linear. From here we can proceed in a number of ways – for instance, we can specify the values of q at t_0 and t_f and solve for the remaining values of q such that the action is extremized. The remainder of this project entails implementing the system described above and analyzing the behavior of the resulting solutions.