HAMILTONIAN STRUCTURE FOR THE 2D SURFACE QUASI-GEOSTROPHIC EQUATION (CDS 205 FINAL PROJECT REPORT, 2005)

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1. Introduction

We try to study the following 2D surface quasi-geostrophic system (SQG) from a Hamiltonian point of view.

$$\begin{aligned} D_t\theta &= 0 \\ u &= \nabla^{\perp}\psi \\ -\left(-\triangle\right)^{1/2}\psi &= \theta \end{aligned} \tag{1.1}$$

where $D_t = \partial_t + u\cdot\nabla$, $\nabla^{\perp} = (-\partial_2, \partial_1)^T$, and $(-\triangle)^{1/2}$ is the pseudodifferential operator corresponding to the multiplier $|k|$. This system is the boundary dynamics of the 0-th order approximation of the motion of the atmosphere or the ocean on the earth surface when the rotation of earth is dominating. θ is called the potential temperature, whose distribution indicates the temperature at different locations on the earth surface.

The surface quasigeostrophic (SQG) system proved to be quite successful when applied to atmospheric or oceanic flows and is important in geostrophy (see e.g. Pedlosky [Ped79]). Recently, it is found that the SQG equation bears much resemblance to the 3D Euler equations with respect to the mechanism of singularity formation (if there are singularities at all). It is worth mentioning that the singularity problem for the 3D Euler equation is one of the most outstanding open problems today. Therefore, it is important to study this SQG system, whose singularity problem is also open.

As is well known, the Hamiltonian formulation of the Euler equations yields many insights into the Euler dynamics. However, to my knowledge, such a formulation for the SQG system (1.1) is still missing. The purpose of this project is to investigate the possibility of a Hamiltonian formulation for the SQG system (1.1), and furthermore, whether such a formulation would help study the SQG singularity problem.

In Section 2 I will derive the 2D SQG system. Then in Section 3 I will derive the Hamiltonian structure for the 2D SQG system and give a brief discussion about it. Finally, in Section 4 I will give a brief summary.

2. The Quasigeostrophic Equations and Its Hamiltonian Structure

We consider some stratified fluid on the earch surface, e.g. the ocean or the atmosphere. When the rotation effect is strong, one can derive various approximate equations that govern the overall behavior of this fluid. In this section we will

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derive one of such approximate systems which has been successfully equiped with a Hamiltonian structure.

2.1. **Rotating fluid.** Denote by Ω the angular velocity of the earth's rotation. For any particle, denote by u_I its velocity in an inertial frame, and by u_R its velocity in the rotating frame moving with the earth. Then we have

$$u_I = u_R + \Omega \times r.$$

It follows that

$$\begin{aligned} (D_t u_I)_I &= (D_t u_R)_I + \Omega \times u_I \\ &= (D_t u_R)_R + \Omega \times u_R + \Omega \times (u_R + \Omega \times r) \\ &= (D_t u_R)_R + 2\Omega \times u_R + \Omega \times \Omega \times r \end{aligned}$$

where $2\Omega \times u_R$ is the Coriolis' force, and $\Omega \times \Omega \times r$ is the centripedal force.

Thus the 3D Euler equations in the rotating frame becomes

$$D_t u + 2\Omega \times u = \nabla \Phi$$

where $\nabla\Phi$ contains $-\nabla p$, the gravity term, and the centripedal term.

Now let the fluid under consideration be at latitude $\theta \in [0, \pi/2]$, and define the Coriolis' parameter $f = 2|\Omega| \sin\theta$. It is easy to see that if we take the frame (e_1, e_2, e_3) such that e_3 is perpendicular to the earth surface, we have

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_t + (u_1\partial_1 + u_2\partial_2 + u_3\partial_3) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + f \begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_1\Phi \\ \partial_2\Phi \\ \partial_3\Phi \end{pmatrix}. \tag{2.1}$$

2.2. **The QG system.** The QG system is the limiting equation of the primitive system when the Rossby number ($\sim 1/f$) approaches 0.

The primitive model adds a new scalar function, called the potential temperature, into the rotating system. It reads

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \theta \end{pmatrix}_t + (u_1\partial_1 + u_2\partial_2 + u_3\partial_3) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \theta \end{pmatrix} + f \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \theta \end{pmatrix} = f \begin{pmatrix} \partial_1\psi \\ \partial_2\psi \\ \partial_3\psi \\ 0 \end{pmatrix}.$$

Where $\psi = \Phi/f$. Here it is assumed that the size of Φ is comparable to f, since otherwise the dynamics would not be interesting when taking $f \to \infty$.

Now taking $f \to \infty$, it can be shown that (Beale-Bourgeois [BB94], Iftimie [Ift9x]) the limiting system is the following:

$$\begin{aligned} \partial_t(\triangle\psi) + (-\partial_2\psi, \partial_1\psi)^T \cdot \tilde{\nabla}\triangle\psi &= 0 \\ \theta &= \partial_3\psi \end{aligned} \tag{2.2}$$

where $\triangle$ is the Laplacian in 3D, while $\tilde{\nabla}$ is a 2D operator. This system is called the QG equations.

Remark 2.1. As mentioned in Iftimie [Ift9x], since the QG equation conserves $\triangle\psi$, the whole system behaves like the 2D Euler equations. Therefore it is easy to prove the global well-posedness of (2.2).

2.3. **Hamiltonian structure for (2.2).** Now we present the Hamiltonian structure obtain by Holm ([Hol86]). Let $q = \triangle\psi$. Then the main equation can be written as

$$\partial_t q + [\psi, q] = 0$$

where $[a,b] = \frac{\partial(a,b)}{\partial(x_1,x_2)}$ is the Jacobian bracket. Now introduce the Lie-Poisson bracket

$$\{F,G\} \equiv -\int_0^\infty \int_{\mathbb{R}^2} q \left[\frac{\delta F}{\delta q}, \frac{\delta G}{\delta q}\right]$$

and the Hamiltonian

$$H = \int_0^\infty dx_3 \int_{\mathbb{R}^2} |\nabla\psi|^2 \; dx_1 dx_2.$$

Now simple calculation yields

$$\frac{\delta H}{\delta q} = -\psi.$$

Therefore

$$\begin{aligned} \{H,q\} &= \int_0^\infty \int_{\mathbb{R}^2} q\,[\psi,\delta] \\ &= \int_0^\infty \int_{\mathbb{R}^2} q\psi_{x_1}\delta_{x_2} - q\psi_{x_2}\delta_{x_1} \\ &= -\left(q\psi_{x_1}\right)_{x_2} + \left(q\psi_{x_2}\right)_{x_1} \\ &= -[\psi,q]\,. \end{aligned}$$

We see that

$$\dot{q} = \{H,q\}$$

is just (2.2) and therefore the above setting gives the Hamiltonian structure of the QG system.

3. A Hamiltonian Structure for SQG

Although for the QG system (2.2), a Hamiltonian structure has been successfully equipped to it, for the SQG equation (1.1) it is still lacking.

First we derive the SQG system. In (2.2), since the quantity $q = \triangle\psi$ is just carried around by the flow, it will remain 0 if initially we take $\triangle\psi_0 \equiv 0$. In this case we are left with the Laplace equation

$$\triangle\psi = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}^+.$$

We assign fast decay boundary condition at infinity. On the bottom surface $x_3 = 0$, we use the evolution equation for θ

$$\theta_t + [\psi, \theta] = 0$$

as the boundary condition, where $\theta = \frac{\partial\psi}{\partial x_3}$. Since $\triangle\psi = 0$, it turns out that $\theta = -\left(-\tilde{\triangle}\right)^{1/2}\psi$, where $\tilde{\triangle}$ is the 2D Laplacian, on the bottom surface. This leads to the SQG system (1.1):

$$\begin{aligned} \theta_t + [\psi,\theta] &= 0 \\ \theta &= -\left(-\tilde{\triangle}\right)^{1/2}\psi. \end{aligned}$$

Remark 3.1. Although, as we mentioned in Section 2, the well-posedness of the QG system (2.2) is easy to obtain, it is argued in Held et. al. [HPGS95] that this fact does not mean that there cannot be singularities formation in the SQG system.

For the SQG system, the kinetic energy is

$$K = \int \left|\tilde{\nabla}\psi\right|^2 \ dx_1 dx_2$$

where $\tilde{\nabla}$ is the 2D gradient. It is easy to check that

$$K = \int \left|\theta\right|^2 \ dx_1 dx_2.$$

Now it is clear that the naive choices of the Hamiltonian $H = K$ or $H = K + \int \theta^2$ would not give the correct dynamics.

Instead we take $H = \int \psi\theta$. We take the same kind of Lie-Poisson bracket as in Section 2:

$$\{F, G\} = -\int \theta \left[\frac{\delta F}{\delta \theta}, \frac{\delta G}{\delta \theta}\right] \ dx_1 dx_2.$$

Since $\frac{\delta H}{\delta \theta} = \psi$ which can be easily checked, following the same line as the argument of Section 2 we see that

$$\dot{\theta} = \{H, \theta\}$$

gives the Hamiltonian structure of the SQG system.

Now we spend some time comparing SQG and 2D Euler. We know that for the 2D Euler system, the conserved quantity is the scalar vorticity ω. The 2D Euler equation can be written as

$$\partial_t \omega + [\psi, \omega] = 0$$

where ψ is the stream function, i.e.,

$$-\triangle \psi = \omega.$$

In this case, the Hamiltonian is simply the kinetic energy $H = \int |u|^2 \ dx_1 dx_2$. Noticing that it can also be written as

$$H = \int \psi\omega \ dx_1 dx_2,$$

we see an analogue between the Hamiltonian structure of the SQG equation and the 2D Euler equation.

Remark 3.2. The quantity $\int \psi\theta$ has been proved to be conserved by the QG dynamics in Constantin-Majda-Tabak [CMT94]. However, there it is described as "an additional positive definite conserved quantity without a direct analogue for the 3D Euler equations" ([CMT94], pp. 1501). Here we see that it really has an analogue for the 2D Euler equations. The kinetic energy $\int |u|^2$ in the 2D Euler dynamics corresponds to two instead of just one conserved quantity in the SQG dynamics, namely the kinetic energy $\int |u|^2 = \int \theta^2$, and the Hamiltonian $\int \psi\theta = \int \left((-\triangle)^{-1/2}\, \theta\right) \theta$.

4. Summary

In this short note we derived the Hamiltonian structure for the surface quasi-geostrophic equation (SQG). We found that this Hamiltonian structure bears direct resemblance with the Hamiltonian structure of the 2D Euler equations.

Remark 4.1. Since the derivation is surprisingly easy, I strongly believe that it should has been done before. I will continue searching through the literature for previous works.

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