Swimming in Magnetic Fields: A Case Study

CDS 205 - Geometric Mechanics, Final Project

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Abstract – *A planar mechanical system consisting of two charged masses at the ends of two rigid bodies connected at a pivot is examined. The system is subjected to a uniform magnetic field and the pivot is driven by an internal torque such that the system exhibits a "swimming" motion. The Kaluza-Klein approach to charged particles is discussed and extended to the case of mechanical systems with point charges. Analytic equations of motion are compared to Working Model experiments to validate the approach.*

1 A Charged Particle in a Magnetic Field

We open by discussing the Hamiltonian and Lagrangian formulations for a charged particle in a magnetic field, following the methods presented in [2].

1.1 Hamiltonian Formulations

A charged particle in $\mathbb{R}^3$ moving in the presence of a magnetic field $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$ has equations of motion given by the Lorentz force law,

$$m\frac{d\mathbf{v}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B}, \tag{1}$$

where $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ [2]. As a magnetic field must be divergence free, we can view $\mathbf{B}$ as being the vector field associated with a closed two-form on $\mathbf{R}^3$, or,

$$\mathbf{i}_\mathbf{B}(dx \wedge dy \wedge dz) = B,$$

and so

$$B = B_x dy \wedge dz - B_y dx \wedge dz + B_z dx \wedge dy.$$

On the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ with elements $(\mathbf{x}, \mathbf{v})$, the canonical symplectic form is

$$\Omega = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}).$$

Applying proposition 6.6.2 of [2], we have a noncanonical symplectic form on this space,

$$\Omega_B = \Omega - \pi_Q^*\frac{e}{c}B = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}) - \frac{e}{c}B. \tag{2}$$

We take the Hamiltonian of the system to be the kinetic energy

$$H = \frac{m}{2}\|\mathbf{v}\|^2 \tag{3}$$

and write the associated Hamiltonian vector field as

$$X_H(u, v, w) = (u, v, w, \dot{u}, \dot{v}, \dot{w}).$$

Then Hamilton's equation relative to the symplectic form Ω_B, $\mathbf{d}H = \mathbf{i}_{X_H}\Omega_B$, becomes

$$\begin{aligned} m(\dot{x}d\dot{x} + \dot{y}d\dot{y} + \dot{z}d\dot{z}) &= m(u\,d\dot{x} - \dot{u}\,dx + v\,d\dot{y} - \dot{v}\,dy + w\,d\dot{z} - \dot{w}\,dz) \\ &\quad -\frac{e}{c}\left[B_x(v\,dz - w\,dy) - B_y(u\,dz - w\,dx) + B_z(u\,dy - v\,dx)\right]. \end{aligned} \tag{4}$$

This is equivalent to $\dot{x} = u$, $\dot{y} = v$, $\dot{z} = w$, and

$$\begin{aligned} m\dot{u} &= \frac{e}{c}(B_z v - B_y w) \\ m\dot{v} &= \frac{e}{c}(B_x w - B_z u) \\ m\dot{w} &= \frac{e}{c}(B_y u - B_x v). \end{aligned}$$

These equations are the same as Eq. 1, and so we can write this system as a Hamiltonian system with a noncanonical symplectic form.

In the case where B is not only closed but exact and we can write $B = \mathbf{d}A$ we can also map this system to one with a new Hamiltonian and a canonical symplectic form. We consider the map $t_A : \mathbb{R}^3 \to \mathbb{R}^3$, $(\mathbf{x}, \mathbf{v}) \mapsto (\mathbf{x}, \mathbf{p})$, where $\mathbf{p} = m\mathbf{v} + \frac{e}{c}\mathbf{A}$ and $\mathbf{A}^\flat = A$. Then by the momentum shifting lemma we have

$$t_A^*\Omega = \Omega - \pi_Q^*\mathbf{d}A,$$

or, noting how we arrived at Ω_B above,

$$t_A^*\Omega = \Omega - \pi_Q^* B = \Omega_B.$$

Hence t_A pulls the canonical form back to Ω_B. Pushing the Hamiltonian forward to the symplectic space $(\mathbf{R}^3 \times \mathbf{R}^3, \Omega)$ then gives us the new Hamiltonian

$$H_A = (t_A)_* H = \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|^2. \tag{5}$$

1.2 Lagrangian Formulations

We can also write this system in Lagrangian form. Let

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{q}}. \tag{6}$$

We see that this is the correct Lagrangian for this system by performing the Legendre transformation. First,

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A},$$

and

$$\begin{aligned} H(\mathbf{q},\mathbf{p}) &= \mathbf{p}\cdot\mathbf{q} - L(\mathbf{q},\dot{\mathbf{q}}) \\ &= \left(m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A}\right)\cdot\mathbf{q} - \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 - \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{q}} \\ &= \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 = \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|^2, \end{aligned}$$

the Hamiltonian given by Eq. 5 above. Thus the Euler-Lagrange equations for this Lagrangian are equivalent to Hamilton's equations (using the canonical symplectic form) for the Hamiltonian of Eq. 5.

This system can also be seen as the reduction of a larger system called the *Kaluza-Klein system*. The idea of this formulation is to augment the configuration space with a new cyclic variable whose conjugate momentum is charge. We consider the new configuration space $Q_K = \mathbb{R}^3 \times S^1$ which we parameterize by $(\mathbf{q},\theta)$. Define the connection one-form on Q_K,

$$\omega = A + \mathbf{d}\theta, \tag{7}$$

where A is defined as above. We then let the Kaluza-Klein Lagrangian be

$$L_K(\mathbf{q},\dot{\mathbf{q}},\theta,\dot{\theta}) = \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 + \frac{1}{2}(\mathbf{A}\cdot\dot{\mathbf{q}} + \dot{\theta})^2. \tag{8}$$

The momenta of this system are then

$$\mathbf{p} = \frac{dL_K}{d\dot{\mathbf{q}}} = m\dot{\mathbf{q}} + (\mathbf{A}\cdot\dot{\mathbf{q}} + \dot{\theta})\mathbf{A}$$

and

$$p = \mathbf{A}\cdot\dot{\mathbf{q}} + \dot{\theta}.$$

From the Euler-Lagrange equation for θ we see that p is a conserved quantity. In fact, p defines the charge by $p = \frac{e}{c}$ =constant. The Hamiltonian on T^*Q_K is then

$$\begin{aligned} H_K(\mathbf{q},\mathbf{p},\theta,p) &= \frac{1}{2m}\|\mathbf{p} - p\mathbf{A}\|^2 + \frac{1}{2}p^2 \\ &= \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|^2 + \frac{1}{2}\left(\frac{e}{c}\right)^2. \end{aligned} \tag{9}$$

This Hamiltonian differs from that of Eq. 5 by only the constant $p^2/2$ and so it is equivalent.

The mechanism of connections can help us to form the Lagrangian for more complex mechanical systems that interact with a magnetic field. This technique will be discussed in Section 3.

2 A Planar Mechanical System and Geometric Phase

The following example is taken from [1]. We consider two planar rigid bodies with moments of inertia I_1 and I_2 connected by a pin joint through their centers of mass. Let θ_1 and θ_2 be their orientations with respect to a fixed inertial reference direction. The shape space of this system can then be considered to be S^1 parameterized by the hinge angle $\psi = \theta_2 - \theta_1$. The configuration space is $S^1 \times S^1$ and, setting $\theta = \theta_1$, can be parameterized by (θ,ψ). The Lagrangian for this system (neglecting translational motion) is the kinetic energy,

$$L(\theta,\psi,\dot{\theta},\dot{\psi}) = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\psi})^2. \tag{10}$$

From the Euler-Lagrange equation for θ we obtain conservation of angular momentum,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0,$$

or $I_1\dot{\theta} + I_2(\dot{\theta} + \dot{\psi}) = \mu$ =constant. Equivalently,

$$d\theta + \frac{I_2}{I_1 + I_2}d\psi = \frac{\mu}{I_1 + I_2}dt.$$

The left side of this equation is the mechanical connection. We can integrate this equation to determine how θ changes as ψ changes,

$$\Delta\theta = -\frac{I_2}{I_1 + I_2}\Delta\psi + \frac{\mu}{I_1 + I_2}T,$$

where T is the amount of time taken for ψ to change by $\Delta\psi$ (note that there is a minor error in this formula in [1]). In particular, for a single complete revolution of body 2 relative to body 1 we have

$$\Delta\theta = \frac{I_2}{I_1 + I_2}\left(\frac{\mu}{I_2}T - 2\pi\right).$$

The term associated with zero angular momentum, $-\left(\frac{I_2}{I_1+I_2}\right)2\pi$, is the geometric phase associated with a complete revolution of body 2. The other term is the dynamic phase associated with the internal kinetic energy of the system.

3 A Planar Mechanical System with Charge

We consider here the system in Fig. 1. We have two identical rigid bodies each with mass M, moment of inertia I, and length l connected through their centers of mass (a special case of the system considered above). The orientation of body 1 is measured by the angle θ with respect to a reference direction, and the orientation of body 2 is measured by the angle φ from body 1. Connected to one end of body 1 and body 2 is a massless point with charge e_1 and e_2, respectively. We also have a uniform magnetic field of strength B directed out of the page. The position of the pivot is described by the vector $\mathbf{x} = (x, y) \in \mathbb{R}^2$. Hence the configuration space $Q = SE(2) \times S^1 \cong \mathbb{R}^2 \times S^1 \times S^1$, parameterized by $(\mathbf{x}, \theta, \varphi)$, and the shape space is S^1 parameterized by φ.

The Lagrangian for this system in the absence of the magnetic field is simply the kinetic energy,

$$L(\mathbf{x}, \theta, \varphi, \dot{\mathbf{x}}, \dot{\theta}, \dot{\varphi}) = M\|\dot{\mathbf{x}}\|^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}I(\dot{\theta} + \dot{\varphi})^2. \tag{11}$$

Regarding the magnetic field as a two-form on Q as above we have $B = Bdx \wedge dy$. We can also write $B = \mathbf{d}A$, where $A = -\frac{B}{2}ydx + \frac{B}{2}xdy$ and $\mathbf{A} = (-\frac{B}{2}y, \frac{B}{2}x)$.

We now consider the maps $t_1, t_2 : Q \to \mathbb{R}^2$ that give us the location of each charge in $\mathbb{R}^2$. We have

$$\begin{aligned} t_1(\mathbf{x}, \theta, \varphi) &= \mathbf{x} + \begin{bmatrix} \frac{l}{2}\cos\theta \\ \frac{l}{2}\sin\theta \end{bmatrix}, \\ t_2(\mathbf{x}, \theta, \varphi) &= \mathbf{x} + \begin{bmatrix} \frac{l}{2}\cos(\theta + \varphi) \\ \frac{l}{2}\sin(\theta + \varphi) \end{bmatrix}. \end{aligned}$$

Let $t_1(\mathbf{x}, \theta, \varphi) = \mathbf{q}_1$ and $t_2(\mathbf{x}, \theta, \varphi) = \mathbf{q}_2$.

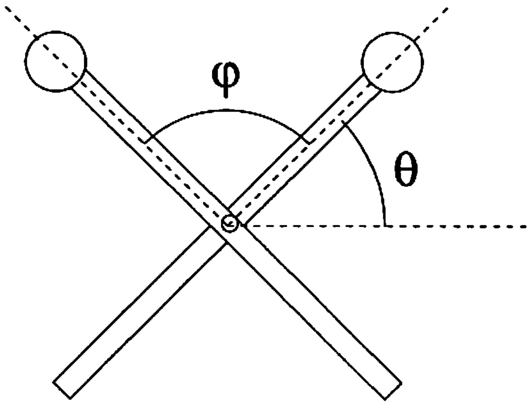

Figure 1: System under consideration and coordinates.

As in the case of the Kaluza-Klein formulation for a charged particle in a magnetic field, we now augment the configuration space with two new cyclic variables whose conjugate momenta will be the two charges. Let

$$Q_K = Q \times S^1 \times S^1,$$

and parameterize this larger space by $(\mathbf{x}, \theta, \varphi, \psi_1, \psi_2)$. Define two connections

$$\begin{aligned} \omega_1 &= A + \mathrm{d}\psi_1, && (12) \\ \omega_2 &= A + \mathrm{d}\psi_2. && (13) \end{aligned}$$

We now define the Kaluza-Klein Lagrangian for this system,

$$\begin{aligned} L_K(\mathbf{x}, \theta, \varphi, \psi_1, \psi_2, \dot{\mathbf{x}}, \dot{\theta}, \dot{\varphi}, \dot{\psi}_1, \dot{\psi}_2) &= L + \frac{1}{2}\left\| \left\langle \omega_1, \left(\mathbf{q}_1, \dot{\mathbf{q}}_1, \psi_1, \dot{\psi}_1 \right) \right\rangle \right\|^2 \\ &\quad + \frac{1}{2}\left\| \left\langle \omega_2, \left(\mathbf{q}_2, \dot{\mathbf{q}}_2, \psi_2, \dot{\psi}_2 \right) \right\rangle \right\|^2 \\ &= M\|\dot{\mathbf{x}}\|^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} I(\dot{\theta} + \dot{\varphi})^2 \\ &\quad + \frac{1}{2}\left((t_1^* \mathbf{A}) \cdot \dot{\mathbf{q}}_1 + \dot{\psi}_1 \right)^2 \\ &\quad + \frac{1}{2}\left((t_2^* \mathbf{A}) \cdot \dot{\mathbf{q}}_2 + \dot{\psi}_2 \right)^2. \end{aligned}$$

The conserved momenta are

$$\begin{aligned} p_1 = \frac{\partial L_K}{\partial \dot{\psi}_1} &= (t_1^* \mathbf{A}) \cdot \dot{\mathbf{q}}_1 + \dot{\psi}_1, \\ p_2 = \frac{\partial L_K}{\partial \dot{\psi}_2} &= (t_2^* \mathbf{A}) \cdot \dot{\mathbf{q}}_2 + \dot{\psi}_2. \end{aligned}$$

These are related to the charges by $p_1 = \frac{e_1}{c}$, $p_2 = \frac{e_2}{c}$.

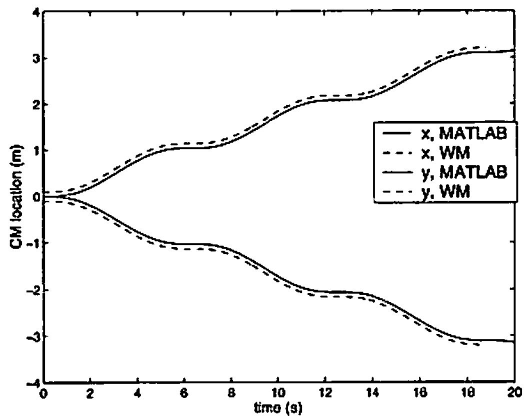

Figure 2: Comparison of Working Model and MATLAB simulations for swimming motion. Working Model initial conditions offset to make plots distinguishable.

The Euler-Lagrange equations for x, y, and θ then yield

$$\ddot{x} = \frac{B}{2Mc}\left[\left(\dot{y}+\frac{l}{2}\dot{\theta}\cos\theta\right)e_1+\left(\dot{y}+\frac{l}{2}(\dot{\theta}+\dot{\varphi})\cos(\theta+\varphi)\right)e_2\right], \tag{14}$$

$$\ddot{y} = \frac{-B}{2Mc}\left[\left(\dot{x}-\frac{l}{2}\dot{\theta}\sin\theta\right)e_1+\left(\dot{x}-\frac{l}{2}(\dot{\theta}+\dot{\varphi})\sin(\theta+\varphi)\right)e_2\right], \tag{15}$$

$$\ddot{\theta} = -\frac{1}{2}\ddot{\varphi}-\frac{Bl}{2c}\left[(\dot{x}\cos\theta+\dot{y}\sin\theta)e_1+(\dot{x}\cos(\theta+\varphi)+\dot{y}\sin(\theta+\varphi))e_2\right]. \tag{16}$$

We omit the equation for φ here because we view φ as being actively controlled and wish to examine the effect of this on the other states.

4 Experimental Results

The system was simulated both directly using Working Model 2D (which we consider to be analogous to "experiment" as it does not require knowledge of the equations of motion) and using the equations of motion derived here to verify the validity of this approach. In all simulations we drove φ according to

$$\varphi(t) = \frac{5\pi}{8} - \frac{\pi}{8}\cos t,$$

i.e. the bodies started out $\frac{\pi}{2}$ radians apart and then were cyclically driven through $\frac{\pi}{8}$ radians relative to one another.

4.1 Case 1: Swimming

Setting $e_1 = -e_2 = e$ produces a "swimming" motion in the direction given by $\theta(0)+\frac{1}{2}\varphi(0)$. Fig. 2 is a comparison of MATLAB and Working Model results for this motion with $\theta(0) = \frac{\pi}{2}$. Agreement is excellent; we needed to offset the initial conditions of the Working Model experiment to make the two plots distinguishable.

A movie of this simulation is available at `http://www.cds.caltech.edu/~waydo/205_project/swimming.avi`.

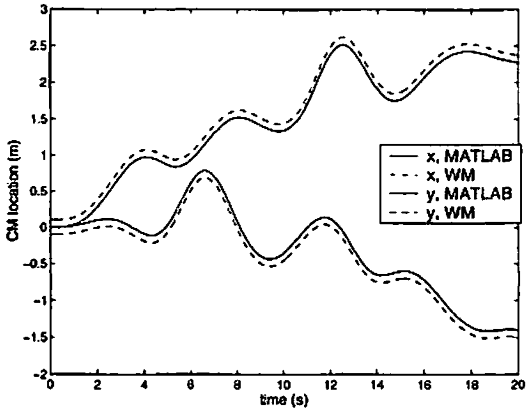

Figure 3: Comparison of Working Model and MATLAB simulations for wandering motion. Working Model initial conditions offset to make plots distinguishable.

4.2 Case 2: Wandering

Setting $e_1 = e_2 = e$ produces a "wandering" motion, so called here because it causes the system to wander about without a strong sense of direction as in the case of swimming. Fig. 3 is a comparison of MATLAB and Working Model results for this motion with $\theta(0) = \frac{pi}{2}$. Agreement is again excellent and we again offset the initial conditions to make the two plots distinguishable.

A movie of this simulation is available at `http://www.cds.caltech.edu/~waydo/205_project/wandering.avi`.

5 Summary and Conclusions

An overview of several analysis techniques for a charged particle in a magnetic field were presented and discussed. One such technique was extended to the case of a simple mechanical system that has point charges located on it. This technique would easily extend to more complex mechanical system with more charged components - one would simply need to include a connection one-form corresponding to each charged component in the formulation of the Kaluza-Klein Lagrangian as presented here.

The original intent of this work was to examine the role that geometric phases might have in the "swimming" motion of this mechanical system. However, simply examining how to formulate the Lagrangian for this system turned out to be a more difficult task than expected. It still appears that geometric phase plays a role here as cyclic inputs in the shape variable φ are capable of producing a net drift of the system. The phase relationship does not appear to be as simple as that of the connected rigid bodies examined here or the particle on a rotating hoop discussed in [2], particularly because the time parameterization of the curve in shape space as well as the curve itself affects the drift in the group variables. An interesting future problem would be to rigorously examine how the group variables drift as a function of the curve traversed in shape space.

Another future direction would be to transform the Lagrangian derived here back to the Hamiltonian formulation and examine this system from the two Hamiltonian points of view discussed above - a kinetic energy Hamiltonian with a momentum-shifted symplectic form and a more complex Hamiltonian with the canonical symplectic form.

6 Acknowledgements

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References

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[2] J. Marsden and T. Ratui. *Introduction to Mechanics and Symmetry*. Springer-Verlag New York, Inc., 2^{th} edition, 1999.