1999 (?)

CDS 280
Course Project
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Formal Instability of 3D Invisid flows

References

1. Abarbanel, H.D.I and D.D. Holm (1987)
   Nonlinear Analysis of inviscid flows in three dimensions: incompressible and baratropic fluids
   phys. Fluids 30. 3369-3382

2. Marsden, J.E. and T.S. Ratiu
   Introduction to Mechanics and Symmetry

3. Marsden, J.E.
   Lectures on Mechanics

Outline

I. Background
   1. The energy-momentum method on $(P, \{,\})$
   2. The Lagrangian specification (l, V)
   3. Conservation laws
   4. The canonical Hamilton's equations (l, v)

II. Stability analysis using the energy-momentum method

I-1. The energy-momentum method

momentum map P Poisson manifold, $\mathfrak{g}$ Lie algebra acts on P canonically. $\mathbb{J}: P \to \mathfrak{g}^*$

$$\langle \mathbb{J}(z), \xi \rangle = J(\xi)z, \quad \xi \in \mathfrak{g} \quad z \in P$$

where $J: \mathfrak{g} \to \mathcal{F}(P)$ $\quad X_{J(\xi)} = \xi_P$

The energy-momentum method

Hamiltonian system $\dot{z} = X_H(z)$

1. $X_H(z_e)$ is in the group direction
2. Let $A = H - \langle \mathbb{J}, \xi \rangle$
3. $\delta A(z_e) = 0$
4. $\delta^2 A(z_e)$ definite

Then z_e is formally stable

- By Simo, Posbergh and Marsden (1990), Simo, Lewis and Marsden (1991)
- Step 4 can be modified by the Arnold convexity analysis (useful for ∞-D systems)

I-2. The Lagrangian specification of 3D inviscid flows

Assumptions : Incompressible, homogeneous ($\rho = 1$) inviscid

Configuration space $\quad Q = G = \mathrm{Diff}_{vol}(\mathcal{D}) \quad \mathcal{D} \subset \mathbb{R}^3$

l - position of fluid particle at $t = 0$

x - position .. .. .. $t = t$

$$x = \eta(l, t), \quad l = l(x, t) = \eta^{-1}(x, t)$$

$$V(l, t) = \frac{\partial \eta}{\partial t} \qquad v = v(x, t) = V(l(x,t), t)$$

$$(\eta, \dot{\eta}) \in TG, \quad v \in \mathfrak{g}^*, \quad l \in G$$

$$(l, v) \in G \times \mathfrak{g}^* \cong T^*G$$

Canonical Ham. eqns

(l, v) Lagrangian specification $\qquad T^*G$

$\uparrow$ Lie-Poisson

Reconstruction $\quad v \quad$ Eulerian specification $\qquad \downarrow$ reduction $\quad \mathfrak{g}^*$

Euler-Poicaré eqns

Equations of motion on $\mathfrak{g}^*$

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + v \times \mathrm{curl}\, R(x)$$

$$\nabla \cdot v = 0$$

steady flow (Eulerian equilibrium states)

$$\begin{cases} v \cdot \nabla v_e + \nabla p_e = 0 \\ \nabla \cdot v_e = 0 \end{cases}$$

NOT Lagrangian equilibrium states

I-3. Conservation laws

Kelvin's thm $$\frac{d}{dt}\oint_{\gamma(t)} v\cdot ds = 0$$

$$\frac{d}{dt}\iint_{A(t)} \omega\cdot n\, dA = 0$$

Ertel's thm Let $S(x,t)$ be conserved along fluid particles. i.e. $$\frac{\partial S}{\partial t} + v\cdot\nabla S = 0$$

Then $\omega\cdot\nabla S$ is conserved along fluid particles

i.e. $$\frac{\partial}{\partial t}(\omega\cdot\nabla S) + v\cdot(\omega\cdot\nabla S) = 0$$

Corollary $$\frac{d}{dt}\left[(\omega\cdot\nabla)^n S\right] = 0$$

Corollary $\Omega = |D|^{-1}\,\omega\cdot\nabla l$ is conserved

l is conserved $D = \nabla l$ $|D| \equiv \det(D)$

Corollary $F : \mathbb{R}^3 \to \mathbb{R}$ is smooth & $C = \int_{\mathcal{D}} F(\Omega)\, d$

then $\frac{dC}{dt} = 0$

I-4. The canonical Hamilton's equations

Let $D = D_i^b = \frac{\partial l^b}{\partial x^i} = \nabla l \qquad |D| \equiv \det(D)$

Let $H = \int_{\mathcal{D}} \left[\frac{1}{2} |D| v^2 + p(|D| - 1) \right] d^3x \; = H(l, v)$

$\{F(l, v), G(l, v)\}$ – canonical Poisson bracket

$$= \int_{\mathcal{D}} \left[\frac{\delta F}{\delta l} \cdot D \cdot \frac{\delta G}{\delta v} - \frac{\delta G}{\delta l} \cdot D \cdot \frac{\delta F}{\delta v} + \omega \cdot \left(\frac{\delta F}{\delta v} \times \frac{\delta G}{\delta v} \right) \right]$$

$$\cdot |D|^{-1} d^3x$$

The Hamilton's equations (canonical) $\qquad \omega = \nabla \times v$

$$\frac{\partial v}{\partial t}(x, t) = -\nabla\left(\frac{v^2}{2} + p\right) + v \times \omega$$

$$\frac{\partial l}{\partial t}(x, t) = -v \cdot \nabla l$$

$$\frac{\partial |D|}{\partial t} = -\operatorname{div}(v|D|) \quad \rightarrow \quad \nabla \cdot v = 0$$

$$\frac{\partial \Omega}{\partial t}(x, t) = -v \cdot \nabla \Omega \qquad \Omega := |D|^{-1} \omega \cdot \nabla l \; = \omega \cdot \nabla l$$

<u>Remark</u> 1. homogeneous. incompressible flow $|D| = 1$

2. Eulerian equilibrium states $v = v(x) \quad p = p(x)$

But $l = l(x, t)$!

II. Stability analysis using the energy-momentum method

II-1 Eulerian equilibrium states

$$0 = -\nabla\left(\frac{\bar{v}^2}{2} + \bar{p}\right) + \bar{v} \times \bar{\omega}$$

$$0 = \left(\frac{\partial}{\partial t} + \bar{v}\cdot\nabla\right)\bar{l}\,(x,t)$$

$$0 = \nabla \cdot \bar{v}$$

$$0 = \left(\frac{\partial}{\partial t} + \bar{v}\cdot\nabla\right)\Omega(x,t) \qquad \Omega = \omega \cdot \nabla l$$

II-2 $\Omega = |D|^{-1}\omega\cdot\nabla l$ is a momentum map

$(l, v) \in P = G \times \mathfrak{g}^* \cong T^*G$

identify Ω w/ an element in $\mathfrak{g}^*$

$\Omega : P \to \mathfrak{g}^*$ Let $\xi \in \mathfrak{g}$

then $\langle \Omega(l,v), \xi \rangle = \int_{\mathcal{D}} (\Omega, \xi)\, d^3x$

To use the energy-momentum method, instead of letting $A = H - \langle \Omega, \xi \rangle$, we define

$$A = H + C \qquad C = \int_{\mathcal{D}} F(\Omega)\,|D|\,d^3x$$

II-3. $$A = H + C = \int_{\mathcal{D}} \left(\tfrac{1}{2} D V^2 + p(D-1)\right) d^3x + \int_{\mathcal{D}} F(\Omega)|D|\, d^3x$$

$\delta A = 0 \Rightarrow$ Eulerian equilibrium states

$$\delta^2 A = \int_{\mathcal{D}} \Big[|\bar{D}|(\delta v)^2 + 2\bar{v}\cdot\delta v\,\delta|D| + 2\,\delta p\,\delta|D| + \bar{F}_a(2\,\delta\Omega^a\delta D + |\bar{D}|\delta^2\Omega^a) + |\bar{D}|\bar{F}_{ab}\delta\Omega^a\delta\Omega^b + \delta^2 D\left(\tfrac{1}{2}\bar{v}^2 + \bar{p} + \bar{F}\right)\Big] d^3x$$

where $\bar{F} = F(\bar{\Omega})$, $\bar{F}_a = \dfrac{\partial \bar{F}}{\partial \bar{\Omega}^a}$, $\bar{F}_{ab} = \dfrac{\partial^2 \bar{F}}{\partial \bar{\Omega}^a \partial \bar{\Omega}^b}$

$D = 1 \Rightarrow \delta D = 0, \delta^2 D = 0$

$$\delta\Omega^a = (\delta\omega)^i D_{;i}^{a} + \omega^i (\delta D)_{;i}^{a}$$

$$(\delta\omega)^i = (\operatorname{curl}\delta v)^i, \quad (\delta D)_{;i}^{a} = \frac{\partial\, \delta l^a}{\partial x^i}$$

$$\int_{\mathcal{D}} |\delta v|^2 d^3x = \int_{\mathcal{D}} \delta\omega \cdot \left(-\frac{1}{\nabla^2}\right)\cdot \delta\omega\, d^3x$$

$$\delta^2 A = \int_{\mathcal{D}} \left[\delta\omega^i , (\delta D)^a_i\right] \begin{bmatrix} \overbrace{-\frac{\delta_{ij}}{\nabla^2} + D^c_i F_{cm} D^m_i}^{M_{ij}} & F_b \delta_{ij} + D^c_i F_{cb}\,\omega^j \\ F_a \delta_{ji} + D^c_j F_{ac}\,\omega^i & \omega^i F_{ab}\,\omega^i \end{bmatrix} \begin{bmatrix} \delta\omega^i \\ (\delta D)^b_j \end{bmatrix} d^3x$$

Let $M_{ij} = -\frac{\delta_{ij}}{\nabla^2} + D^a_i F_{ab} D^b_i$

① Suppose $M > 0$ Consider the following deformati

$$\omega^i \delta D^a_i = (\omega \cdot \nabla)\delta \ell^a = 0$$

then $\delta^2 A = \int_{\mathcal{D}} \left(\delta\omega^i M_{ij}\,\delta\omega^i + 2\,\delta\omega^i F_a \nabla_i \delta \ell^a\right) d^3x$

Let $\lambda > 0$ & $\psi^i_\lambda(x)$ be the eigenvalue and eigenfunctio of M_{ij}

$$\delta\omega^i(x) = \int c_\lambda \psi^i_\lambda(x)\, d\lambda$$

$$\delta^2 A = \int \Big(\lambda |c_\lambda|^2 \int_{\mathcal{D}} |\psi_\lambda|^2 d^3c + 2c^*_\lambda \int F_a \psi^*_\lambda(x) \cdot \nabla \delta \ell_a d$$
$$\Big) d\lambda < 0$$

If $\int_{\mathcal{D}} F_a \psi^*_\lambda(x) \cdot \nabla \delta \ell^a\, d^3x < -\frac{\lambda}{2} c_\lambda \int_{\mathcal{D}} |\psi_\lambda(x)|^2\, d^3x$

② M_{ij} is not definite. Then let $\delta l = 0$ $\delta\omega^i \neq 0$ giving indefinite $\delta^2 A$

Remark :

1. In ① we require the projection of $F_a \nabla \delta l^a$ onto $\psi_\lambda(x)$ is negative enough. i.e. $\delta\omega \cdot (F_a \nabla \delta l^a)$ is large enough. $(\delta\omega \cdot \nabla)\delta l$ is the vortex stretching of distortions in particle labels.

2. For 2D flow, the vortex stretching term is ze[r]
We have the Casimir function $C = \int_{\mathcal{D}} \Phi(\omega)\, d^{[2]}$
So that we can use the energy-Casimir method to analyze the stability without referri[ng] to the canonical variables (l, v), directly on $\mathfrak{g}^*$