Good report! J.M. (as there are lots of small typos!)

Application of Energy-Momentum Method to Rotating Strings

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Abstract

The stability of axial motions of nonlinear elastic strings has been studied using the energy-momentum method, which is basically the same as the constructions of Liapunov functions in [1]. The dynamics of the whirling strings has also been formulated in a more geometric setting in this report . Though this problem has not been soved for some boundary conditions.

1 Introduction

The dynamics of rotating strings is so complex that it is not fully understood today even though it has been received extensive studies. The linearized solutions were obtained by D. Bernoulli and L. Euler in the 16th century. According to the solutions of the linearized system, a string can only rotate at certain angular velocities which is not consistent with experimental results. In 1955 Kolodner [5] studied the problem of free swirling of a heavy chain and showed that it can rotate at any angular velocity $\omega > \omega_1$ and for each $\omega_n < \omega \leq \omega_{n+1}$, there are exactly n distinct modes of rotation. Here $\omega_n, n = 1, 2, ...$ are the eigenvalues of the linearized equations. The dynamics of rotating strings depends on the boundary conditions. Caughey [2, 3] derived the planar modes and their stabilities for certain ω for two different boundary conditions for certain ω. In [4], Caughey also derived the planar modes and their stabilities for the case when the string is elastic.

In this report some geometric properties of the dynamics of rotating strings were investigated. We formulated the configuration spaces, the Hamiltonians and the Lagrangians. We studied the symmetries of the Hamiltonians including constructing the momentum maps.

Another interesting problem is the axial motion of a closed elastic string. Healy [1] derived the stability of a class of axial motions using Liapunov functions constructed from the conservation of energy and circulation. His method of studying stability is actually the energy-momntum method, though he did not mention that. In this report we also formulate this procedure explicitly using the energy-momentum method by constructing the Hamiltonian and the momentum maps.

2 The Energy-Momentum Method

Let P be a Poisson manifold, G be a Lie group acting on P, $\mathfrak{g}$ be the Lie algebra. Sppose z_e is a relative equilibrium and is regular, i.e., $g_{z_e} = \{0\}$. Let $\mathbf{J} : P \to \mathfrak{g}^*$ be the momentum map. Suppose $\mu = \mathbf{J}(z_e)$ is a generic point, i.e., its orbit is of maximal dimension.

Let $S \subset T_{z_e}P$ satisfy

i) $S \subset Ker D\mathbf{J}(z_e)$,
ii) S is transverse to the G_μ-orbit within $ker D\mathbf{J}(z_e)$,

then the energy momentum method is

i) find $\xi \in \mathfrak{g}$ such that $\delta H_\xi(z_e) = 0$,
ii) test $\delta^2 H_\xi(z_e)$ for definiteness on S.

The following Energy-Momentum Theorem is due to Simo, Posbergh and Marsden.

Theorem 2.1 *If $\delta^2 H_\xi(z_e)$ is definite, then z_e is G_μ orbitally stable in $\mathbf{J}^{-1}(\mu)$ and G-orbitally stable in P.*

One should be careful about applying the Energy-Momentum Theorem to the infinite dimensional Poisson manifold. There is an example in three-dimensional elasticity theory given by Ball and Marsden that $\delta^2 H(z_e)$ is positive definite, but z_e is not a local minimum of H. Two possible versions of the Energy-Momentum Method are

i) Arnold's convexity hypotheses,
ii) employing Sobolev spaces such that the energy norm defined by $\delta^2 H_\xi(z_e)$ to be equivalent to a Sobolev norm.

3 Application of the energy-momentum method to the stability of axial motions of elastic strings

The stability of axial motions for a closed loop elastic string was studied by Healey [1]. His method of constructing the Liapunov function using the conserved quantities such as the conservation of energy and the conservation of circulation is basically the energy momentum method. In this section we approach the problem in a more geometric way by constructing the configuration space, the Lagrangian, the Hamiltonian and the momentum maps. Then we formulate Healey's procedure of proving the stability by the energy-momentum method.

Now we state the physical problem. Consider a closed loop of elastic string. Let l and ρ be the natural length and the mass per unit length. Without loss of generality, suppose $l = 1$, $\rho = 1$. Let s denote the arclength of a material pointin the string($s \in \mathbb{R}(mod 1)$).

We view the string at time t as a manifold embedded in $\mathbb{R}^3$ denoted by M_t. Then M_{t_1} is diffeomorphic to M_{t_2} for arbitrary $t_1, t_2 \in \mathbb{R}$. M_t is also diffeomorphic to $\mathbb{R}(mod 1)$. Hence the configuration space Q is the embedding of M_t in $\mathbb{R}^3$, i.e.,

$$Q = Emb_{\mathbb{R}^3}(M_t)$$

$$s \mapsto \mathbf{r}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}, \tag{1}$$

where $\mathbf{r}(s,t)$ denotes the spatial position of the material point s.
Specifically, we require that $Q = \mathcal{H}^2(\mathbb{R}(mod 1))$, where

$$\mathcal{H}^k(\mathbb{R}(mod 1) = \{\mathbf{u} = \sum_{n=-\infty}^{\infty} \mathbf{a}_n e^{i2\pi ns}, \mathbf{a}_n \in \mathbb{C}^3, \bar{\mathbf{a}}_n = \mathbf{a}_n, ||\mathbf{u}|| = \sum_{n=-\infty}^{\infty} n^{2k}|\mathbf{a}_n|^2 < \infty\},$$

$k = 1, 2, \ldots$.

The inner product of $\mathbf{u} = \sum_{n=-\infty}^{\infty} \mathrm{a}_n e^{i2\pi ns}$ and $\mathbf{v} = \sum_{n=-\infty}^{\infty} \mathrm{b}_n e^{i2\pi ns}$ in $\mathcal{H}^k$ is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle_k = \sum_{n=-\infty}^{\infty} n^{2k} \mathrm{a}_{-n} \mathrm{b}_n.$$

The spaces $\mathcal{H}^k$ are norm equivalent to the ususal Sobolev spaces $W^{2,k}((0,1), \mathbb{R}^3)$.

Let $\mathbf{v} = \mathbf{r}_t$, then we have

$$TQ = \mathcal{H}^2(\mathbb{R}(mod1)) \times \mathcal{H}^1(\mathbb{R}(mod1)).$$

Since TQ is a Hilbert space, we can identify the tangent bundle of Q as

$$TQ = \mathcal{H}^2(\mathbb{R}(mod1)) \times \mathcal{H}^1(\mathbb{R}(mod1)).$$

We assume that there is a potential of elastic forces for the string $W : \mathbb{R}^+ \to \mathbb{R}$. Suppose $W(1) = 0$ and $W'' > 0$ on $(0, \infty)$.

$$T = W'(|\mathbf{r}_s|).$$

The Lagrangian of the system is $L : TQ \to \mathbb{R}$,

$$L(\mathbf{r}, \mathbf{v}) = \int_0^1 (\frac{1}{2}\mathbf{v} \cdot \mathbf{v} - W(|\mathbf{r}_s|))ds.$$

The Hamiltonian of the system is $H : T^*Q \to \mathbb{R}$,

$$H(\mathbf{r}, \mathbf{v}) = \int_0^1 (\frac{1}{2}\mathbf{v} \cdot \mathbf{v} + W(|\mathbf{r}_s|))ds.$$

Now we derive the Hamilton's equations for the system.

$$\begin{aligned}
\int_0^1 \frac{\partial \mathbf{r}}{\partial t} \cdot \delta \mathbf{v} ds &= \int_0^1 \frac{\delta H}{\delta \mathbf{v}} \cdot \delta \mathbf{v} ds \\
&= \lim_{\epsilon \to 0} [H(\mathbf{r}, \mathbf{v} + \epsilon \delta \mathbf{v}) - H(\mathbf{r}, \mathbf{v})] \frac{1}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 [\frac{1}{2}(\mathbf{v} + \epsilon \delta \mathbf{v}) \cdot (\mathbf{v} + \epsilon \delta \mathbf{v}) - \frac{1}{2} \mathbf{v} \cdot \mathbf{v}] ds \\
&= \int_0^1 \mathbf{v} \cdot \delta \mathbf{v} ds,
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{\partial \mathbf{v}}{\partial t} \cdot \delta \mathbf{r} ds &= -\int_0^1 \frac{\delta H}{\delta \mathbf{r}} \cdot \delta \mathbf{r} ds \\
&= -\lim_{\epsilon \to 0} [H(\mathbf{r} + \epsilon \delta \mathbf{r}, \mathbf{v}) - H(\mathbf{r}, \mathbf{v})] \frac{1}{\epsilon} \\
&= -\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 [W(|\mathbf{r}_s + \epsilon (\delta \mathbf{r})_s|) - W(|\mathbf{r}_s|)] ds
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^1 \frac{dW}{d(|\mathbf{r}_s|)} \cdot \frac{\partial |\mathbf{r}_s|}{\partial \mathbf{r}} \cdot (\delta \mathbf{r})_s ds \\
&= -\int_0^1 W'(|\mathbf{r}_s|) \frac{\mathbf{r}_s}{|\mathbf{r}_s|} \cdot d(\delta \mathbf{r}) \\
&= -W'(|\mathbf{r}_s|) \frac{\mathbf{r}_s}{|\mathbf{r}_s|} (\delta \mathbf{r})|_0^1 + \int_0^1 [W'(|\mathbf{r}_s|) \frac{\mathbf{r}_s}{|\mathbf{r}_s|} \cdot \delta \mathbf{r} ds \\
&= \int_0^1 [W'(|\mathbf{r}_s|) \frac{\mathbf{r}_s}{|\mathbf{r}_s|}]_s \cdot \delta \mathbf{r} ds.
\end{aligned}
$$

Hence the Hamilton's equations for the system is

$$
\frac{\partial \mathbf{r}}{\partial t} = \mathbf{v}, \tag{2}
$$

$$
\frac{\partial \mathbf{v}}{\partial t} = (W'(|\mathbf{r}_s|) \frac{\mathbf{r}_s}{|\mathbf{r}_s|})_s. \tag{3}
$$

The Poisson structure on T^*Q is canonical, i.e.

$$
\{F, G\} = \int_0^1 (\frac{\delta F}{\delta \mathbf{r}} \cdot \frac{\delta G}{\delta \mathbf{v}} - \frac{\delta F}{\delta \mathbf{v}} \cdot \frac{\delta G}{\delta \mathbf{r}}) ds,
$$

where $F, G : T^*Q \to \mathbb{R}$.

The Hamilton's equations is

$$
\dot{F} = \{F, H\}.
$$

Let $P = T^*Q$, then $(P, \{,\})$ is a Poisson manifold. Now we check some symmetries for the system. Let $\mathfrak{g}$ acts on Q by $\mathfrak{g} : Q \to Q$, $s \mapsto s' = s + \xi$. Let $\mathfrak{g}$ acts on T^*Q by the cotagent lift. Then

$$
\begin{aligned}
H \circ \xi &= H(\mathbf{r}(s + \xi, t), \mathbf{v}(s + \xi)) \\
&= \int_0^1 (\frac{1}{2} \mathbf{v}(s + \xi, t) \cdot \mathbf{v}(s + \xi, t) + W(|\mathbf{r}(s + \xi, t)_s|)) ds \\
&= \int_\xi^{1+\xi} (\frac{1}{2} \mathbf{v}(s', t) \cdot \mathbf{v}(s', t) + W(|\mathbf{r}(s', t)_{s'}|)) ds' \\
&= \int_0^1 (\frac{1}{2} \mathbf{v}(s, t) \cdot \mathbf{v}(s, t) + W(|\mathbf{r}(s, t)_s|)) ds \\
&= H
\end{aligned}
$$

Hence,

$$
H \circ \xi = H,
$$

i.e., H is $\mathfrak{g}$-invariant.

It is clear that $\mathfrak{g} \cong \mathbb{R}$, so we identify $\mathfrak{g}$ with $\mathbb{R}$. Now we construct the momentum map as

$$\mathbf{J} : P = T^*Q \to \mathbb{R} = \mathfrak{g}^*$$

$$\mathbf{J}(\mathbf{r}, \mathbf{v}) := \oint \mathbf{v} \cdot d\mathbf{r} = \int_0^1 \mathbf{v} \cdot \mathbf{r}_s ds$$

It is also clear that $\mathbf{J}$ is $\mathfrak{g}$-invariant, i.e.,

$$\mathbf{J} \circ \xi = \mathbf{J}$$

Now we prove that $\langle \mathbf{J}, \xi \rangle = \xi \cdot \mathbf{J} : P \to \mathbb{R}$ is a Casimir function.

$$\begin{aligned}
\{\langle \mathbf{J}, \xi \rangle, H\} &= \int_0^1 \left(\frac{\delta(\xi \mathbf{J})}{\delta \mathbf{r}} \cdot \frac{\delta H}{\delta \mathbf{v}} - \frac{\delta(\xi \mathbf{J})}{\delta \mathbf{v}} \cdot \frac{\delta H}{\delta \mathbf{r}}\right) ds \\
&= \xi \int_0^1 [(-\mathbf{v}_s) \cdot \mathbf{v} - (W'(|\mathbf{r}_s|)\frac{\mathbf{r}_s}{|\mathbf{r}_s|})_s \cdot \mathbf{r}_s] ds \\
&= -\xi W'(|\mathbf{r}_s|)|\mathbf{r}_s| - \xi \int_0^1 [\mathbf{v}_s \cdot \mathbf{v} - W'(|\mathbf{r}_s|)\frac{\mathbf{r}_s \cdot \mathbf{r}_{ss}}{|\mathbf{r}_s|}] ds \\
&= -\xi \int_0^1 [\frac{1}{2}\mathbf{v} \cdot \mathbf{v} - W(|\mathbf{r}_s|)]_s ds \\
&= 0.
\end{aligned}$$

Since $\langle \mathbf{J}, \xi \rangle = \xi \mathbf{J}$ is invariant along the trajectories, $\mathbf{J}$ is invariant along the trajectories. Let

$$J(\mathbf{r}, \mathbf{v}) = \mu.$$

Now we consider the following axial motions

$$\mathbf{r}(s,t) = \lambda \mathbf{p}(s + \frac{ct}{\lambda}), \mathbf{p}' \cdot \mathbf{p}' = 1. \tag{4}$$

We construct the new Hamiltonian

$$\tilde{H}(\mathbf{r}, \mathbf{v}) = H(\mathbf{r}, \mathbf{v}) - \langle \mathbf{J}(\mathbf{r}, \mathbf{v}) - \mu, \xi \rangle.$$

$$\delta \tilde{H}(\zeta, \eta) := \frac{d}{d\alpha} \tilde{H}(\mathbf{r} + \alpha\zeta, \mathbf{v} + \alpha\eta)|_{\alpha=0} = \int_0^1 [\zeta \cdot (W'(|\mathbf{r}_s|)\frac{\mathbf{r}_s}{|\mathbf{r}_s|} - \xi \mathbf{v}) + \eta \cdot (\xi \mathbf{r}_s - v)] ds$$

We require that $\delta \tilde{H} = 0$, so we have

$$\xi \mathbf{r}_s = \mathbf{v} \tag{5}$$

$$\xi \mathbf{v} = W'(|\mathbf{r}_s|)\frac{\mathbf{r}_s}{|\mathbf{r}_s|} \tag{6}$$

Substitute the axial motion (4) to (3) and the Hamilton equations (5) and (6), we get

$$\begin{aligned} \lambda &= \lambda(\mu^2) & (7) \\ c &= c(\mu) = \frac{\mu}{\lambda(\mu^2)} & (8) \\ \xi &= \xi(\mu) = \frac{c(\mu)}{\lambda(\mu^2)} & (9) \end{aligned}$$

where (7) is the solution of the algebraic equation

$$W'(\lambda) = \frac{\mu^2}{\lambda^3}.$$

Hence the axial motions can be denoted by

$$\mathcal{M}_\mu = \{(\mathrm{r},\mathrm{v}) | (\mathrm{r},\mathrm{v}) = (\lambda(\mu^2)\mathrm{p}, c(\mu)\mathrm{p}'), p \in \mathcal{H}^2, \mathrm{p}' \cdot \mathrm{p}' = 1\}.$$

It is clear that $\mathcal{M}_\mu$ is an infinite dimensional differentiable manifold in $P = T^*Q = \mathcal{H}^2 \times \mathcal{H}^1$.

Let $(\mathbf{r}_a, \mathbf{v}_a) = (\lambda(\mu^2)\mathbf{p}_a, c(\mu)\mathbf{p}'_a) \in \mathcal{M}_{mu}$, then

$$\begin{aligned} \delta^2 \tilde{H}(\zeta,\eta) &= \frac{d^2}{d\alpha^2}\tilde{H}(\mathrm{r}_a + \alpha\zeta, \mathrm{v}_a + \alpha\eta)|_{\alpha=0} & (10) \\ &= \int_0^1 [|\eta - \xi(\mu)\zeta'|^2 + \{W''(\lambda(\mu^2)) - \frac{W'(\lambda(\mu^2))}{\lambda(\mu^2)}\}(\mathrm{p}'_a \cdot \zeta')^2] ds, & (11) \end{aligned}$$

By the assumptions on W, it can be shown that $\exists \Lambda_0 > 0$, such that

$$\frac{d}{d\lambda}(\frac{W(\lambda)}{\lambda}) > 0,$$

for $1 \leq \lambda < \Lambda_0$.

It can be derived from (11) that $\delta^2 \tilde{H}(\zeta,\eta) = 0$ if and only if $(\zeta,\eta) \in T_a\mathcal{M}_\mu$. Furthermore, it can also be shown that $T_a\mathcal{M}_\mu$ is also a closed subspace of the phase space $P = T^*Q = \mathcal{H}^2 \times \mathcal{H}^1$, which is a Hilbert space. So we have

$$P = T_a\mathcal{M}_{mu} \oplus T\_a\mathcal{M}_\mu{}^\perp.$$

Now we can use the Energy-Momentum theorem to prove the stability of the axial motions because this is the second version of the Energy-Momentum Theorem for infinite dimensional Poisson manifolds which was givee in Section 2.

4 Some Geometrical Properties for Rotating Strings

In this section we give geometrical formulations for the dynamics of rotating strings. We establish the configuration space, the Lagrangian, the Hamiltonian and the momentum maps.

We first suppose that the string is inextensible. Let $\Omega - [0, L]$, M_t be the manifolds that is diffeomorphic and isometric to Ω. Let Q be the embedding of M_t in $\mathbb{R}^3$, i.e.,

$$Q = \mathcal{F}^3_{len}(\Omega)$$

$$s \mapsto \mathbf{r}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}, \tag{12}$$

and $\mathbf{r}(s,t)$ satisfies $|\mathbf{r}_s| = 1$.

Now we consider several different boundary conditions. See (1), (3) and (1). It can be shown that the Lagrangian $L : TQ \to \mathbb{R}$ is

$$L(\mathbf{r}, \mathbf{r}_t, t) = \int_0^L \frac{1}{2}\rho|\mathbf{r}_t|^2 + \int_0^L \rho\mathbf{g}\cdot\mathbf{r}ds - \int_0^L T(s,t)ds,$$

where $\mathbf{r}_t$ is the velocity, ρ is the mass per unit length, $\mathbf{g}$ is the gravitational acceleration. $T(s,t)$ is the tension.

The Euler-Lagrange equations can be derive as

$$(\rho\mathbf{r}_t)_t = \rho\mathbf{g} + (T\mathbf{r}_s)_s,$$

with the constraints $|\mathbf{r}_s| = 1$.

It can be shown that the Hamiltonian $H : T^*Q \to \mathbb{R}$ is

$$H(\mathbf{r}, \mathbf{p}, t) = \int_0^L \frac{1}{2\rho}|\mathbf{p}|^2 - \int_0^L \rho\mathbf{g}\cdot\mathbf{r}ds + \int_0^L T(s,t)ds,$$

where $\mathbf{p} = \rho\mathbf{r}_t$.

The Hamilton's equations are

$$\mathbf{r}_t = \frac{\mathbf{p}}{\rho} \tag{13}$$

$$\mathbf{p}_t = \rho\mathbf{g} + (T(s,t)\mathbf{r}_s)_s \tag{14}$$

$$|\mathbf{r}_s| = 1 \tag{15}$$

Let $\mathfrak{g}$ be the rotation around the z-axis, then $\mathfrak{g} \cong \mathbb{R}$. We define the momentum map as

$$\mathbf{J}_z(\mathbf{r}, \mathbf{p}) = \int_0^L (\mathbf{p}\times\mathbf{r})\cdot\mathbf{z}_0 ds,$$

where $\mathbf{z}_0$ is the unit vector in the z-direction. It can be shown that J is conserved.

If the string is linearly elastic, then we have the stress-strain law

$$T - T_0 = EA(|\mathbf{r}_s| - 1),$$

The Lagrangian is

$$L(\mathbf{r}, \mathbf{r}_t, t) = \int_0^L \frac{1}{2}\rho|\mathbf{r}_t|^2 + \int_0^L \rho\mathbf{g}\cdot\mathbf{r}ds - \int_0^L W(|\mathbf{r}_s|)ds,$$

where $W(|\mathbf{r}_s| = (T0 - EA)|\mathbf{r}_s| + \frac{1}{2}EA|\mathbf{r}_s|^2$.

The Hamiltonian is

$$H(\mathbf{r}, \mathbf{p}, t) = \int_0^L \frac{1}{2\rho}|\mathbf{p}|^2 - \int_0^L \rho\mathbf{g}\cdot\mathbf{r}ds + \int_0^L W(|\mathbf{r}_s|)ds,$$

where $\mathbf{p} = \rho\mathbf{r}_t$.

The Hamilton's equations are

$$\mathbf{r}_t = \frac{\mathbf{p}}{\rho} \tag{16}$$

$$\mathbf{p}_t = \rho\mathbf{g} + (W'(|\mathbf{r}_s|)\frac{\mathbf{r}_s}{|\mathbf{r}_s|})_s \tag{17}$$

$$|\mathbf{r}_s| = 1 \tag{18}$$

The momentum map for this case is the same as for the inextensible case.

5 Sammary

We gave the geometrical formulations for the axial motions of nolinear elastic closed loops and for the rotating strings. The future works is to investigate stability of the planar modes using the energy-momentum method.

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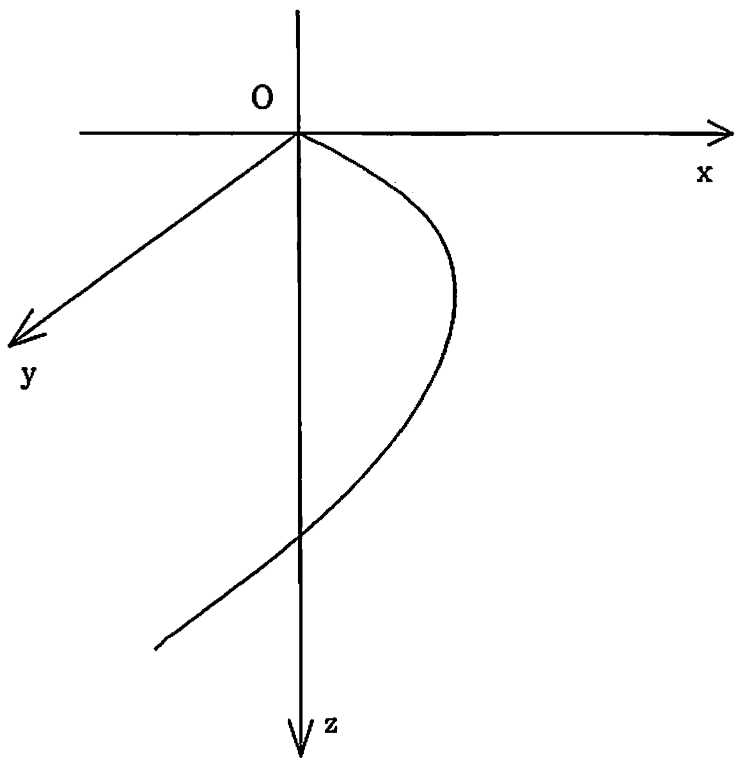

Figure 1: Figure 1, $T(L,t)=0,\mathbf{r}(0,t)=0$

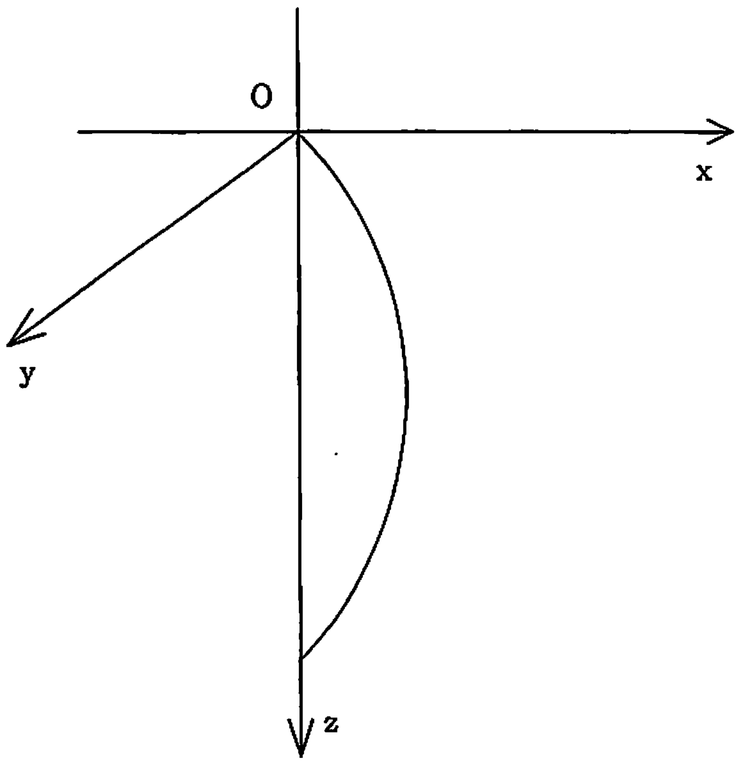

Figure 2: Figure 2, $\mathbf{r}(0,t)=0, \mathbf{r}(L,t)=0$

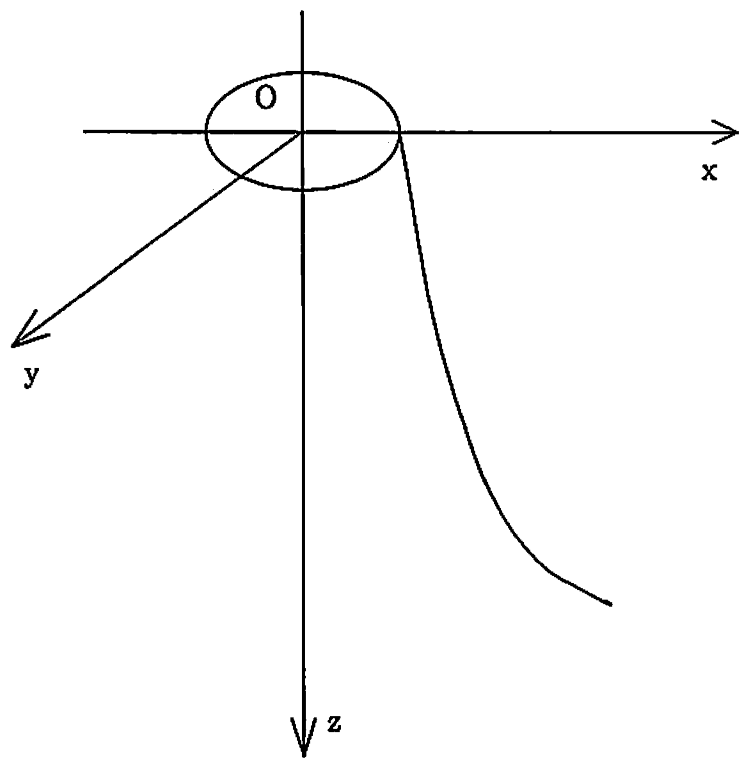

Figure 3: Figure 3, $T(L,T)=0, \mathbf{r}(0,t)=(\epsilon,0,0)$