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Report on: Reduction and Hamiltonian Structures on Duals of Lie Algebras specialized to Heavy top

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The following quotation from E.T. Whittaker give us the general phylolophicy underlying the paper we are reporting on " We have seen that the integration of a dynamical system which is solvble by quadratures can generally be effected by transforming it into another dynamical system with fewer degrees of freedom". (Chapter XI Fourth Ed.)
(Page 288)

From another point of view, particularly field theory, it has been observed that symetry properties of the Lagrangian or Hamiltonian for such systems imply the existence of conserved quantities. the culmination of such ideas is given by Noether's Theorem: Let there be a continous transformation of both cordinates and simultaneusly of field functions, which depend on s parameters ω_k ($k=1,\dots,s$) $x_\nu \to x'_\nu = f_\nu(x;\omega)$ $u_\alpha(x) \to u'_\alpha(x') = U_\alpha(u_\beta(x);\omega)$. and which makes the variation of the action $A = \int \mathcal{L}(x)dx$ equal zero $\delta A = 0$. Then there exist s dynamical invariants C_k (ie function which are independent of time, of the field functions and their derivatives) which may be represented in the form of spatial integrals $C_k = \int dx\, \Theta^0_{(k)}(x)$ of the zero components of certain "four vectors"

$$\Theta^\nu_{(k)} = \frac{\partial \mathcal{L}}{\partial u_{\alpha;\nu}} \left(u_{\alpha;\mu} X^\mu_k - \Psi_{\alpha;k} \right) - X^\nu_k \mathcal{L}(x)$$

and $X^\mu_k = \left.\dfrac{\partial f^\mu(x;\omega)}{\partial \omega_k}\right|_{\omega=0}$ $\qquad \Psi_{\alpha;k} = \left.\dfrac{\partial u_\alpha}{\partial \omega_k}\right|_{\omega=0}$.

(See N.N Bogoliubov, D.V. Shirkov; Quantum Fields)
(Page 19)

One of the great achioment of Symplectic Manifold theory has been to unify both ideas into a single theory which provide means for obtention of conserved dynamical quantities in both discrete and continous dynamical systems.

This is achived by a generalization of Noether's theorem to symplectic manifold theory and the development of a new notion that aplies to both cases: The momentum mapping. the definition of the momentum mapping is the following:

Def Let (P,ω) be a connected symplectic manifold and $\Phi: G\times P \to P$ a symplectic action of the Lie group G on P; that is, for each $g \in G$ the map $\Phi_g : P \Rightarrow P;\ x \to \Phi(g,x)$ is symplectic. We say that a mapping $J: P \Rightarrow g^*$ (the dual of the Lie algebra of G) is a momentum mapping for the action provided that for every $\xi \in g$

$d\hat{J}(\xi) = i_{\xi_P}\omega$ where $\hat{J}(\xi): P \Rightarrow R$ is defined by $\hat{J}(\xi)(x) = J(x)\cdot\xi$ and ξ_P is the infinitesimal generator of the action corresponding to ξ. In other words, J is a momentum mapping provided that $X_{\hat{J}(\xi)} = \xi_P$ for all $\xi \in g$. Some times (P,ω,Φ,J) is called a Hamiltonian G space. (Abraham, Marsden Page 276)

The following theorem is then how the momentum mapping is used as a form for computation of conserved dynamical quantities:

Theorem: Let Φ be a symplectic action of G on (P,ω) with a momentum mapping J. Suppose that $H: P \to R$ is invariant under the action, that is, $H(x) = H(\Phi_g(x))$ for all $x \in P$, $g \in G$. Then J is an <u>integral</u> for X_H; that is, if F_t is the flow of X_H, $J(F_t(x)) = J(x)$. (Abraham, Marsden Page 277).

The next step would be to find specific examples into which the theory would apply and develop more general theory. One of the achivement that the paper we are reporting on has accomplish is exactly this one. Most of the discussion now would be based on the paper and the development it has obtained. for the specific example of the heavy top.

The relevant parts of the paper for the heavy top are sections 2, 4, 5.1, 5.2.

Section 2.1 deals with three different posibilities of working with the equation of motion of the heavy top. These are the space coordinates, body coordinates and material coordinates. The material coordinates are actual coordinates relative to a fixed set of basis of points in the body while the space coordinates are the coordinates (material) of the body as it move in space. The body coordinates are the coordinates related to a basis fixed in the body as it moves in space. The most important conclusion from this section is "the configuration space of the heavy top may be identified with $SO(3)$. Consequently the phase space of the top is the cotangent bundle $T^*(SO(3))$".

In section 2.2 a relationship between space and body coordinates is obtained. This relationship is a parametrization of $SO(3)$. Equation 2.2 present the matrix which parametrizes $SO(3)$ throu the Euler angles. Later on they will obtain a natural chart for the cotangent bundle $T^*(SO(3))$.

In section 2.3 properties of the Lie Algebra $so(3)$ and its dual are established, which would "simplify the computations and identify the geometrical structure of the Hamiltonian of the heavy top". Some of the important points made here are ① the identification of $so(3)$ with $\mathbb{R}^3$ with Lie Algebra product identified with the usual cross product ② Equational propierties that are related to the matrix and vector representation of the Lie Algebra. ③ Representation of the adjoint action which we are told is conjugation and of $Ad^*_{A^{-1}} m = Am$.

We are going to be given in section 2.4 the hamiltonian for the system. This is done in the following way: First there is a presentation of the different ways the velocity could be obtained for the given coordinate system. The second step is to compute the kinetic energy using the expressions of velocity already obtained. Then a computation of body and angular velocity is given. An inner product is given to facilitate the computation of the kinetic energy and then the computation of the inertia tensor is obtained. After this a presentation follows that establishes relations between the canonical variables for the two different system of coordinates in which the differential equations would be expressed. Then the potential energy is computed. And we see the expression of the Hamiltonian in equations 2.32, 2.33. More relations are given at the end of the section between the body coordinate parameters and spatial or material coordinate parameters. In section 2.5 and 2.6 we are given the Hamilton equations of motion in body coordinates and a presentation of the Poisson bracket. We can see right now that some of the advantages of this method are already ~~the~~ given. One of the main advantages perhaps is the simplification in the computations that is obtained. Vizualizing the Euler Angles as parameters of SO(3) permits use of abstract group properties which are advantageus in the obtention of the Hamiltonian.

In section 4 we are given a review of Poisson Manifold momentum maps and their relation to semidirect (of duals) product of Lie Algebras. Some of the definition which are presented are conventional definitions of the following left Lie Algebra, Poisson bracket, Poisson manifold, canonical map, Casimir functions, Lie group action Lie algebra mapping, momentum mapping, equivariant pairing, Hamiltonian actions. Two important points are presented ① "A casimir function is characterized by the property of being invariant under the coadjoint action", ② " An important property of equivariant momentum maps is that they are canonical."

On section 4.4 the two fundamental theorems of the paper are given and proved. These are:

"Theorem 1: The maps $\tilde{J}_L, \tilde{J}_R : T^*G \times V^* \to S_\pm^*$, $J_L(\alpha_g, a) = (T_e^* R_g(\alpha_g), \Phi(g^{-1})^* a)$, $J_R(\alpha_g, a) = (T_e^* L_g(\alpha_g), \Phi(g)^* a)$ are canonical; in fact, these maps are reductions of the momentum maps by the action of V and are themself momentum maps for the action (left or right) of $G \times V$ on the Poisson manifold $T^*G \times V^*$."

"Theorem 2: Let $H: T^*G \times V^* \to \mathbb{R}$ be left invariant under the action of T^*G of the stabilizer G_a for every $a \in V$. Then H induces a Hamiltonian $H_L \in F(S_-^*)$ defined by $H(T_e^* L_g(\alpha_g), \Phi(g)^* a) = H(\alpha_g, a)$ thus yielding Lie-Poisson equation on S_-^*. The curve $\alpha_a(t) \in T^*G$ is a solution of Hamilton's equation defined by H_a on T^*G if and only if $J_R(\alpha_a(t), a)$ is a solution of the Hamiltonian system defined by H_L on S_-^*. In particular, the evolution of $a \in V^*$ is given by $\Phi(g_a(t))^* a$ where $g_a(t)$ is the projection of $\alpha_a(t)$ on G."

We can observe that these two theorems give us the fundamental way of computing integral invariants using the representation of duals of Lie Algebras and some solutions to Hamilton equation. Properties are given which relate the actual solution of the canonical equations to the Lie Algebra solution.

Sections 5.1, 5.2 are aplications of the above two theorems. We observe inmediatly the advantages obtained by using the above two theorems for the computations of dynamical conserved quantities and how Casimir function are infact together with the momentum map the desired quantities.