POISSON BRACKETS ON SUPERFLUIDS

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In this paper, a set of noncanonical Poisson brackets on superfluid ^{4}He are formed. These noncanonical brackets involve the physical variables P, for the momentum density ρ, for the mass density S, for the entropy density. Then, by the same method, a set of noncanonical brackets are found for rotating superfluid ^{4}He.

The Hamilton equations are found for two-fluid hydrodynamics in canonically conjugate variables, [1]. To find Hamilton's equations for

$$H = \int \left[\frac{\rho V_s^2}{2} + \mathrm{P} V_s + \mathcal{E}(\rho, S, \mathrm{P}) \right] dV$$

$$d\mathcal{E} = T\,dS + \mu\, d\rho + (V_n - V_s)\, d\mathrm{P}$$

$$j = \rho_s V_s + \rho_n V_n$$

the new variables α, β, γ, f are introduced such that

$$V_s = \nabla \alpha$$

$$\mathrm{P} = S \nabla \beta + f \nabla \alpha$$

V_s is defined as the superfluid component of the velocity, V_n is the normal component, and $\mathcal{E}$ is the thermodynamic energy. The new variables form the canonically conjugate pairs (ρ, α) (s, β), and (f, γ), [2]. Using these variables, the canonical Poisson bracket is formed,

$$[F, G] = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial G}{\partial z^j}$$

where z^i are the phase space coordinates, and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$ [3].

This is useful, but recent progress has been made by the use of noncanonical Poisson brackets formed on the physical variables [3] The resulting bracket takes the form

$$[F,G] = \int \frac{\delta F}{\delta \chi^i} \mathcal{O}^{ij} \frac{\delta G}{\delta \chi^j}$$

where $\delta F/\delta \chi^i$ is a functional derivative, and $\mathcal{O}^{ij}$ may be dependent on dynamical variables and their derivatives. The noncanonical Poisson bracket is the goal of what follows.

To arrive at the Poisson brackets for physical variables, the method described by Dyzaloshinskii and Volovick is used, [4]. This involves the use of quantum mechanics where the Poisson bracket is replaced by the commutators $[\hat{a},\hat{b}]$. The commutators are found for the physical variables and then are passed back to the classical limit according to

$$[\hat{a},\hat{b}] \Rightarrow \frac{\hbar}{i}\{a,b\}. \qquad (1)$$

This method makes finding the Poisson brackets easy because of the commutator relation

$$[\hat{a},[\hat{b},\hat{c}]]+[\hat{c},[\hat{a},\hat{b}]]+[\hat{b},[\hat{c},\hat{a}]] = 0 .$$

Thus, when the final bracket is obtained, it automatically satisfies Jacobi's identity, [4].

An example of this method is seen in the work by Landau [5] with temperature at absolute zero. He found the density and momentum operators to be

$$\hat{\rho}(r) = \sum_a m_a \delta(r_a - r)$$

$$\hat{p}(r) = \frac{\hbar}{2i} \sum_a \left(\nabla_a \delta(r_a - r) + \delta(r_a - r) \nabla_a \right).$$

These equalities lead to the commutator relations

$$[\hat{\rho}_1, \hat{\rho}_2] = 0$$

$$[\hat{p}_{k1}, \hat{\rho}_2] = -i\hbar\, \hat{\rho}_1 \nabla_{k1} \delta(r_1 - r_2)$$

$$[\hat{p}_{k1}, \hat{p}_{\ell 2}] = -i\hbar \left(\hat{p}_{\ell 1} \nabla_{k1} - \hat{p}_{k2} \nabla_{\ell 2} \right) \delta(r_1 - r_2).$$

Then passing over to the classical limit by (1), the brackets

$$\{\rho_1, \rho_2\} = 0$$

$$\{p_{k1}, \rho_2\} = \rho_1 \nabla_{k1} \delta(1 - 2)$$

$$\{p_{k1}, p_{\ell 2}\} = \left(p_{\ell 1} \nabla_{k1} - p_{k2} \nabla_{\ell 2} \right) \delta(1 - 2)$$

are formed. Calculations using these brackets will lead to the Euler equations, [4]. Similar operations lead to brackets for the spin operator.

The quantum mechanical commutators described above are associated with a Lie algebra of the corresponding groups of transformations. In this case, the momentum operator is the generator of the group of movements, and the spin operator is the generator of the group of rotations in spin space.

For the general case, let G be the group where it is desired to relate the hydrodynamic variables a,b,c..., to A,B,C..., which are elements of an algebra of the group G. The method for finding the bracket is the same as that in L.1.6, [6]. The bracket is given by

$$[\xi, \eta] = T_e(Ad\ \eta)\cdot\xi \quad , \qquad [6].$$

According to the notation in [4],

$$A \rightarrow g A g^{-1} \tag{2a}$$

$$\delta A = [\delta g A] \tag{2b}$$

$$\delta g = \lambda \int dV \alpha^k(x) L^k(x) \tag{2c}$$

where $\alpha^k(x)$ are local infinitely small "angles," L^k are generators of the local transformation, and λ is a constant. These yields the bracket

$$\lambda [L^k(x), A(y)] = \frac{\delta A(y)}{\delta \alpha^k(x)} .$$

For the superfluid case, the dynamics must be changed to two-fluid dynamics, where one fluid corresponds to the usual case, and the other corresponds to the condensate phase of the liquid. This means that the number of independent variables includes the phase $\varphi(x)$. Actually, the energy of the system does not depend upon the phase itself but upon its gradient. The velocity of the superfluid is defined by

$$V_s = \nabla \varphi$$

$$\mathrm{curl}\, V_s = 0 .$$

Now the bracket including the phase $\varphi(x)$ is found by the method described above. Because the condensate phase does not move like a normal liquid but rather as a body in one quantum state, it is necessary to add a group of gauge transformations to the group of movements. The group corresponding to the Poisson brackets of the phase is the group of gauge transformations which corresponds to phase shifts $\varphi \Rightarrow \varphi + \chi$. The action of the group is upon the wavefunction

$$\psi(1,2,\ldots) \Rightarrow \psi(1,2,\ldots)\, \exp\left(i \frac{m}{\hbar}(\chi_1 + \chi_2 + \ldots)\right) .$$

By methods described above,

$$\delta\psi = \int dV \frac{1}{m} \rho(r) \left(i \frac{m}{\hbar} \chi(r)\right) \psi(1,2,\ldots) \tag{3a}$$

$$\delta g = \frac{-i}{\hbar} \int dV \rho(r)\, \delta \varphi(r) \tag{3b}$$

$\rho(r)$ is the mass density, and by comparing (2) and (3), it is evident that $\rho(r)$ is the generator of the gauge group. So,

$$\{\rho_1 , \varphi_2\} = - \frac{\delta \varphi_2}{\delta \varphi_1} .$$

At the movement $x^k \Rightarrow x^k + u^k(x)$ the phase varies as

$$\varphi \Rightarrow \varphi - u^k \nabla_k \varphi .$$

It follows that the Poisson brackets for current p and phase φ are

$$\{ p_{k1}, \varphi_2 \} = \frac{\delta \varphi_2}{\delta u_1^k}$$

It is checked in [4] that these brackets lead to equations of motion equivalent to those obtained by Landau. The brackets are written out completely in the form

$$\begin{aligned}\{F,G\} = &\big[(\delta G/\delta\rho)\partial_\ell \rho + (\delta G/\delta\varphi)\,\varphi_{,\ell} \\ &+ (\delta G/\delta p_k)(p_\ell \partial_k + \partial_\ell p_k) + (\delta G/\delta s)\partial_\ell s\big]\,\delta F/\delta p_\ell \\ &+ (\delta G/\delta p_k)(\rho\partial_k \delta F/\delta\rho - \varphi_{,k}\,\delta F/\delta\varphi + s\partial_k \delta F/\delta s)\} \\ &+ [(\delta F/\delta\rho)\,\delta G/\delta\varphi - (\delta G/\delta\rho)\,\delta F/\delta\varphi] ,\end{aligned}$$

where $p = \rho\nabla\varphi + s\nabla\beta + f\nabla\gamma$ is the momentum density, ρ is the mass density, s is the entropy density, φ the phase parameter and F and G are functionals of these variables, with added notation $\partial_\ell = \partial/\partial x_\ell$, $\varphi_{,\ell} = \partial\varphi/\partial x_\ell$, [2].

Now the same procedure is followed for the case of rotating superfluid ^{4}He. The rotating case is the same as normal ^{4}He except that

$$\mathrm{Curl}\ V_s \neq 0 .$$

Thus,

$$V_s = \nabla \alpha + a$$

for some vector a. a is the vortical part of the superfluid velocity. Therefore, another variable a is added to the list of independent variables, and the bracket for a must be found. The end result is given by

$$\{F,G\} = \{[(\delta G/\delta\rho)\partial_\ell \rho + (\delta G/\delta\varphi)\varphi_{,\ell} + (\delta G/\delta a_k)(a_{k,\ell} + a_\ell \partial_k)$$

$$+ (\delta G/\delta N_k)(N_\ell \partial_k + \partial_\ell N_k)]\,\delta F/\delta N_\ell$$

$$+ (\delta G/\delta s)\partial_\ell s\,\delta F/\delta P_l + (\delta G/\delta P_k)[(P_\ell \partial_k + \partial_\ell P_k)\frac{\delta F}{\delta P_l}$$

$$+ s\,\partial_k\,\delta F/\delta s],$$

where $N_k = -\rho\varphi_{,k} - d_j a_{j,k} + (d_j a_k)_{,j}$

with d being the conjugate pair for a, [2].

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