Forced Variational Collision Integrators

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A Variational Principle for Elastic Collisions

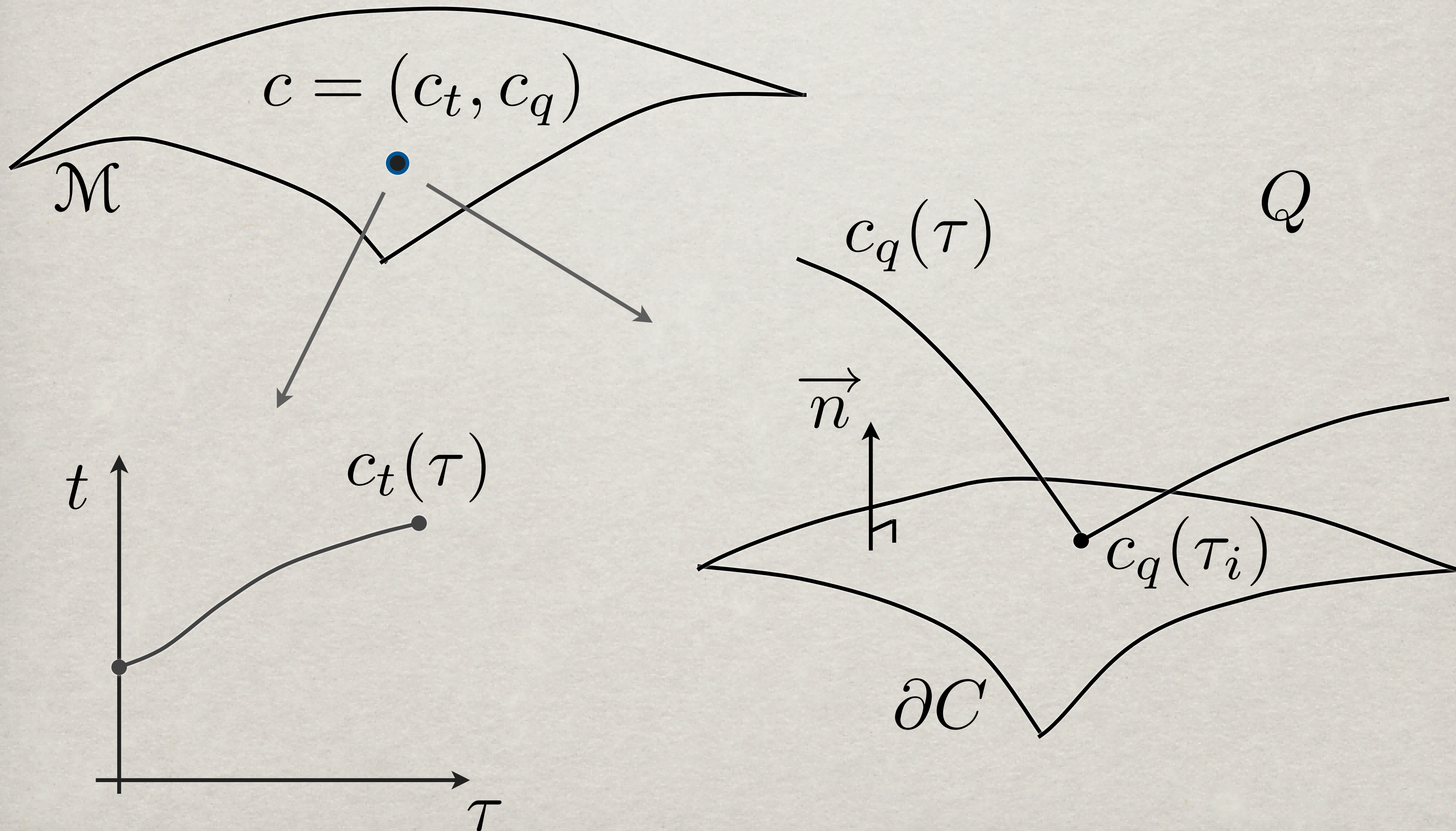
- Begin with a configuration manifold Q with a boundary $C \subset Q$ s.t. ∂C is the contact set, and a regular Lagrangian $L : TQ \to \mathbb{R}$

- Formulate the set of admissable trajectories as a smooth manifold, the **path space**:

$$
\begin{aligned}
\mathcal{M} &= \mathcal{T} \times \mathcal{Q}([0,1], \tau_i, \partial C, Q) \\
\mathcal{T} &= \{c_t \in C^\infty([0,1], \mathbb{R}) \mid c_t' > 0 \text{ in } [0,1]\} \\
\mathcal{Q}([0,1], \tau_i, \partial C, Q) &= \{c_q : [0,1] \to Q \mid c_q \text{ is } C^0, \text{ piecewise } C^2 \\
&\quad \exists \text{ one singularity in } c_q(\tau) \text{ at } \tau_i, \\
&\quad c_q(\tau_i) \in \partial C\}
\end{aligned}
$$

Fetecau R.C., Marsden J.E., Ortiz M., West M. "Nonsmooth Lagrangian mechanics and variational collision integrators". *SIAM Journal on Applied Dynamical Systems 2003; 2(3):381– 416.*

A Picture of Path Space



A Variational Principle for Elastic Collisions

- Define the **action map**, $\mathfrak{G} : \mathcal{M} \rightarrow \mathbb{R}$ s.t.

$$\mathfrak{G}(c_t, c_q) = \int_0^1 L\left(c_q(\tau), \frac{c_q'(\tau)}{c_t'(\tau)}\right) c_t'(\tau) d\tau$$

- Applying Hamilton's Principle of critical action yields

$$\int_0^{\tau_i} EL(c'') \cdot \delta c \, d\tau + \int_{\tau_i}^1 EL(c'') \cdot \delta c \, d\tau + \Theta_L(c') \Big|_{\tau_i^-}^{\tau_i^+} \cdot \hat{\delta} c(\tau_i) = 0$$

where $EL(c'') = \left[\frac{\partial L}{\partial q} c_t' - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] dc_q + \left[\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}} \frac{c_q'}{c_t'} - L\right)\right] dc_t$

$$\Theta_L(c') = \left[\frac{\partial L}{\partial \dot{q}}\right] dc_q - \left[\frac{\partial L}{\partial \dot{q}} \frac{c_q'}{c_t'} - L\right] dc_t$$

Implications of the Variational Principle

- By arbitrariness of variations δc of the path

$$\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) = 0 \quad \text{in} \quad [c_t(0), c_t(\tau_i)) \cup (c_t(\tau_i), c_t(1)]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = 0 \quad \text{in} \quad [c_t(0), c_t(\tau_i)) \cup (c_t(\tau_i), c_t(1)]$$

$$\left[\frac{\partial L}{\partial \dot{q}} q - E dt\right]_{\tau_i^-}^{\tau_i^+} = 0 \quad \text{on} \quad TQ_e|(\mathbb{R} \times \partial C)$$

Conservation of the Symplectic Form

- Assuming uniqueness of trajectories, associate the space of solution curves $q(t)$ with the space of initial conditions $(t_0, q_0, \dot{q}_0)$ on $\mathbb{R} \times TQ$

- Define the **Lagrangian Flow**

$$F_t(t_0, q_0, \dot{q}_0) = (t_0 + t, q(t_0 + t), \dot{q}(t_0 + t))$$

as well as the **restricted action**

$$\hat{\mathfrak{G}}_t(t_0, q_0, \dot{q}_0) = \int_{t_0}^{t_0+t} L\left(q(s), \dot{q}(s)\right) ds$$

Conservation of the Symplectic Form

- Using the notion of a curve $\lambda \mapsto (t_0^\lambda, q_0^\lambda, \dot{q}_0^\lambda)$ which passes through $(t_0, q_0, \dot{q}_0)$ at $\lambda = 0$, the equation

$$\begin{aligned} d\mathfrak{G}(c) \cdot \delta c \quad &= \int_0^{\tau_i} EL(c'') \cdot \delta c \, d\tau + \int_{\tau_i}^1 EL(c'') \cdot \delta c \, d\tau \\ &\quad + \bar{\Theta}_L(\tilde{c}) \cdot \delta \tilde{c} \Big|_0^{\tau_i^-} + \bar{\Theta}_L(\tilde{c}) \cdot \delta \tilde{c} \Big|_{\tau_i^+}^1 \end{aligned}$$

  becomes

$$\begin{aligned} d\mathfrak{G}_t((t_0, q_0, \dot{q}_0)) \cdot (\delta t_0, \delta q_0, \delta \dot{q}_0) \; &= \bar{\Theta}_L(F_t(t_0, q_0, \dot{q}_0)) \cdot \frac{d}{d\lambda} F_t(t_0^\lambda, q_0^\lambda, \dot{q}_0^\lambda) \Big|_{\lambda=0} \\ &\quad - \bar{\Theta}_L((t_0, q_0, \dot{q}_0)) \cdot \frac{d}{d\lambda} (t_0^\lambda, q_0^\lambda, \dot{q}_0^\lambda) \Big|_{\lambda=0} \end{aligned}$$

Conservation of the Symplectic Form

- Taking an exterior derivative yields

$$0 = dd\hat{\mathfrak{G}}_t = F_t^*(d\bar{\Theta}_L) - d\bar{\Theta}_L$$

- This indicates conservation of the **Lagrangian symplectic form** in the extended sense

$$F_t^*(\Omega_L) = \Omega_L$$

$$\text{where} \quad \Omega_L = -d\bar{\Theta}_L = \omega_L + dE \wedge dt$$

Extension to Discrete Mechanics

- The Lagrangian, path space, trajectory, and action map all get redefined in the discrete setting

$$
\begin{aligned}
L_d(q0, q1, h) &\approx \int_0^h L(q, \dot{q})dt \\
\mathcal{T}_d &= \{t_d \in C^\infty([0,1],[0,1]) \mid t_d \text{ onto}, t_d' > 0 \text{ in } [0,1]\} \\
\mathcal{Q}(\tilde{\alpha}, \partial C, Q) &= \{q_d : \{t_0, \ldots, t_{i-1}, \tilde{\tau}, t_i, \ldots, t_N\} \to Q, q_d(\tilde{\tau}) \in \partial C\} \\
(\alpha, q_d) &= (\alpha, \{q_0, \ldots, q_{i-1}, \tilde{q}, q_i, \ldots, q_N\}) \\
\mathfrak{G}_d(\alpha, q_d) &= \sum_0^{i-2} L_d(q_k, q_{k+1}, h) + \sum_{k=i}^{N-1} L_d(q_k, q_{k+1}, h) \\
&\quad + L_d(q_{i-1}, \tilde{q}, \alpha h) + L_d(\tilde{q}, q_i, (1-\alpha)h)
\end{aligned}
$$

Taking (a lot of) Discrete Variations

$$d\mathfrak{G}_d(\alpha, q_d) \cdot (\delta\alpha, \delta q_d) =$$

$$\sum_{k=1}^{i-2} EL_d(q_{k-1}, q_k, q_{k+1}) \cdot \delta q_k + \sum_{k=i+1}^{N-1} EL_d(q_{k-1}, q_k, q_{k+1}) \cdot \delta q_k$$

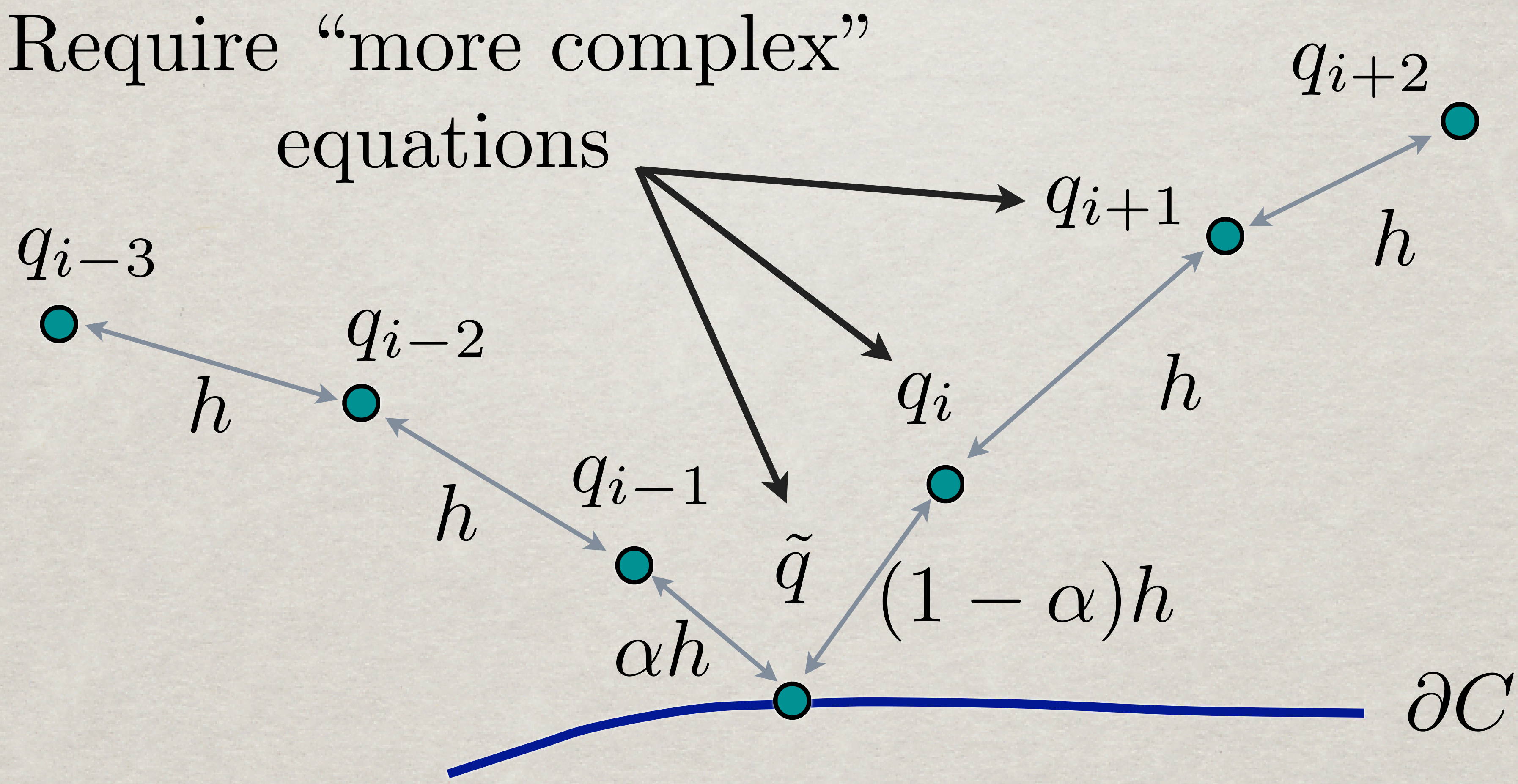
$$+ \left[D_2 L_d(q_{i-2}, q_{i-1}, h) + D_1 L_d(q_{i-1}, \tilde{q}, \alpha h)\right] \cdot \delta q_{i-1}$$

$$+h \left[D_3 L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3 L_d(\tilde{q}, q_i, (1-\alpha)h)\right] \cdot \delta\alpha$$

$$+i^*(D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1-\alpha)h)) \cdot \tilde{q}$$

$$+ \left[D_2 L_d(\tilde{q}, q_i, (1-\alpha)h) + D_1 L_d(q_i, q_{i+1}, h)\right] \cdot \delta q_i$$

Visualization of Variational Integration



← Use Discrete Euler-Lagrange Equations →

Implications of the Discrete Variational Principle

- The **Discrete Euler-Lagrange equations** hold "away" from the collision. That is,

$$D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) = 0$$
$$\forall k \in \{0, \ldots, i-2, i+1, \ldots, N\}$$

- The collision time and configuration, as defined by α and $\tilde{q}$, must satisfy

$$D_2 L_d(q_{i-2}, q_{i-1}, h) + D_1 L_d(q_{i-1}, \tilde{q}, \alpha h) = 0$$
$$\tilde{q} \in \partial C$$

Implications of the Discrete Variational Principle

- The post-collision configuration q_i must satisfy

$$D_3 L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3 L_d(\tilde{q}, q_i, (1-\alpha)h) = 0$$
$$i^*(D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1-\alpha)h)) = 0$$

- The final step in returning to the uniform mesh in time is to solve for q_{i+1} such that

$$D_2 L_d(\tilde{q}, q_i, (1-\alpha)h) + D_1 L_d(q_i, q_{i+1}, h) = 0$$

Conservation of the Discrete Symplectic Form

- Define the **discrete Lagrangian map** F_{L_d} such that

$$F_{L_d}(q_k, q_{k+1}) = (q_{k+1}, q_{k+2})$$

- Denoting the restriction of $\mathfrak{G}_d$ to solution paths as $\hat{\mathfrak{G}}_d$, we have that

$$d\hat{\mathfrak{G}}_d = (F_{L_d}^N)^* \Theta_{L_d}^+ - \Theta_{L_d}^-$$

where
$$\Theta_{L_d}^+(q_k, q_{k+1}) = D_2 L_d(q_k, q_{k+1}, h) \cdot \delta q_{k+1}$$
$$\Theta_{L_d}^-(q_k, q_{k+1}) = -D_1 L_d(q_k, q_{k+1}, h) \cdot \delta q_k$$

Conservation of the Discrete Symplectic Form

- Taking an exterior derivative yields

$$0 = dd\hat{\mathfrak{G}}_d = (F_{L_d}^N)^*(d\bar{\Theta}_{L_d}^+) - d\bar{\Theta}_{L_d}^-$$

- This indicates conservation of the **discrete Lagrangian symplectic form**

$$(F_{L_d}^N)^*(\Omega_{L_d}) = \Omega_{L_d}$$

where $\Omega_{L_d} = d\bar{\Theta}_{L_d}^+ = d\bar{\Theta}_{L_d}^-$

- In coordinates $\Omega_{L_d}(q_0, q_1) = \dfrac{\partial^2 L_d}{\partial q_0^i \partial q_1^j} dq_0^i \wedge dq_1^j$

Ex. 1: Shooting with a Variational Collision Integrator

- Consider a planar rigid body in a gravitational field. Both a floor and a wall are designated as contact manifolds.
- The method of shooting is used to find initial conditions for which the body will return to a prescribed position with a prescribed angle of entry
- Both circular and elliptical bodies are tested

The Method of Shooting

- Method optimizes a trajectory by parameterizing it with initial conditions
- An objective function for a system trajectory is treated as a function of the initial conditions by incorporating numerical integration
- Essentially this is Newton's Method with an integrator in the loop

A Variational Principle for Lossful Collisions

- Define an **exterior force field** $F : TQ_e \to T^*Q_e$ and a **contact force field** $f^{con} : T(\partial C \times \mathbb{R}) \to T^*(\partial C \times \mathbb{R})$
- The **Lagrange-d'Alembert principle** applied to a non-smooth trajectory provides

$$
\begin{aligned}
0 \quad = \quad & \delta \int_0^1 L\left(c_q(\tau), \frac{c_q'(\tau)}{c_t'(\tau)}\right) c_t'(\tau) d\tau \\
& + \int_0^1 F(c(\tau), c'(\tau)) \cdot \delta c(\tau) d\tau \\
& + f^{con}(c(\tau_i), c'(\tau_i)) \cdot \delta c(\tau_i)
\end{aligned}
$$

Extension to Discrete Mechanics

- Approximating the virtual work in the same manner as the discrete Lagrangian

$$F_0^-(q_0, q_1, h)\cdot\delta q_0 + F_0^+(q_0, q_1, h)\cdot\delta q_1 \approx \int_0^h F(q,\dot{q})\cdot\delta q ds$$

- The **Lagrange-d'Alembert principle** discretizes as

$$\begin{aligned}
0 = \delta \sum_0^{i-2} L_d(q_k, q_{k+1}, h) + F_k^-\cdot\delta q_k + F_k^+\cdot\delta q_{k+1} \\
+ \delta \sum_{k=i}^{N-1} L_d(q_k, q_{k+1}, h) + F_k^-\cdot\delta q_k + F_k^+\cdot\delta q_{k+1} \\
+ \delta L_d(q_{i-1}, \tilde{q}, \alpha h) + F^-(q_{i-1}, \tilde{q}, \alpha h)\cdot\delta q_{i-1} + F^+(q_{i-1}, \tilde{q}, \alpha h)\cdot\delta\tilde{q} \\
+ \delta L_d(\tilde{q}, q_i, (1-\alpha)h) + F^-(\tilde{q}, q_i, (1-\alpha)h)\cdot\delta\tilde{q} + F^+(\tilde{q}, q_i, (1-\alpha)h)\cdot\delta q_i \\
+ f^{con}(\tilde{q}, q_i, (1-\alpha)h)\cdot\delta\tilde{q} + f^{con}_{pow}(\tilde{q}, q_i, (1-\alpha)h)\cdot\delta\alpha
\end{aligned}$$

Implications of the Discrete Variational Principle

- The **forced Discrete Euler-Lagrange equations** hold "away" from the collision. That is,

$$D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) + F^+_{k-1} + F^-_k = 0$$
$$\forall k \in \{0, \ldots, i-2, i+1, \ldots, N\}$$

- The collision time and configuration, as defined by α and $\tilde{q}$, must satisfy

$$D_2 L_d(q_{i-2}, q_{i-1}, h) + D_1 L_d(q_{i-1}, \tilde{q}, \alpha h)$$
$$+F^+(q_{i-2}, q_{i-1}, h) + F^-(q_{i-1}, \tilde{q}, \alpha h) = 0$$
$$\tilde{q} \in \partial C$$

Implications of the Discrete Variational Principle

- The post-collision configuration q_i must satisfy

$$D_3 L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3 L_d(\tilde{q}, q_i, (1-\alpha)h) - f_{pow}^{con} = 0$$

$$i^* \Big(D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1-\alpha)h)$$
$$+ F^+(q_{i-1}, \tilde{q}, \alpha h) + F^-(\tilde{q}, q_i, (1-\alpha)h) + f^{con}(\tilde{q}, q_i, (1-\alpha)h) \Big) = 0$$

- The final step in returning to the uniform mesh in time is to solve for q_{i+1} such that

$$D_2 L_d(\tilde{q}, q_i, (1-\alpha)h) + D_1 L_d(q_i, q_{i+1}, h)$$
$$+ F^+(\tilde{q}, q_i, (1-\alpha)h) + F^-(q_i, q_{i+1}, h) = 0$$

Implications of the Discrete Variational Principle

- The post-collision configuration q_i must satisfy

$$\overrightarrow{n} \cdot \Big(D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1-\alpha)h)$$
$$+ F^+(q_{i-1}, \tilde{q}, \alpha h) + F^-(\tilde{q}, q_i, (1-\alpha)h) + f^{con}(\tilde{q}, q_i, (1-\alpha)h) \Big) = 0$$

$$i^* \Big(D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1-\alpha)h)$$
$$+ F^+(q_{i-1}, \tilde{q}, \alpha h) + F^-(\tilde{q}, q_i, (1-\alpha)h) + f^{con}(\tilde{q}, q_i, (1-\alpha)h) \Big) = 0$$

Ex. 2: Collision Model Comparison

- Consider a planar wedge-shaped rigid body in a gravitational field. The floor is the only contact manifold considered.
- The exterior force field $F(q, \dot{q})$ is set to zero.
- Four plausible ways of defining $f^{con}(\tilde{q}, \dot{\tilde{q}})$ are considered.

Collision Models

- $f^{con} = 0$, the fully elastic case
- $f^{con} = -\beta\overrightarrow{n}$, for a dissipative but frictionless collision
- $f^{con} = -\beta\overrightarrow{n} + f^{tan}$, where $f^{tan} \in T^*\partial C$
- f^{con} as in case 3, but scaled to provide a perfectly plastic impact (instantaneously stops the point of contact)

Future Directions

- Explore other methods for determining f_{pow}^{con} from a given contact model
- Test the integration scheme on an application with non-zero exterior forces F
- Incorporate the variational integrator into Discrete Mechanics and Optimal Control (DMOC) theory

Thank You

Any Questions?