Geometric Phase in Holonomic and Nonholonomic Systems - Two examples

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1 Introduction

For this final project I read *The Geometric structure of Nonholonomic Mechanics* by Koon and Marsden and *Symmetries in Motion: Geometric Foundations of Motion Control* by Ostrowski and Marsden. The goal was initially to learn something about the current research in geometric phases and how that might be extended to nonholonomic systems and eventually, in the case of my research, might be extended to hybrid nonholonomic systems. I read the papers with this in mind, but was unable to come up with something directly pertaining to phases to turn in, so I instead decided to work through two examples, one holonomic and one nonholonomic, and compute the phase associated with cyclic inputs in the shape variables. I chose the simple planar rigid body out of the first paper and the snakeboard out of the second. I realize, of course, that these results are already available in these papers. However, I learned a great deal from the second example about momentum equations, connections, fiber translations, and vertical lifts, just by going through the calculation. Then, in order to stay in the spirit of the original project, I did some sample calculations like those found in the Ostrowski and Marsden paper.

2 Simple Planar Rigid Body

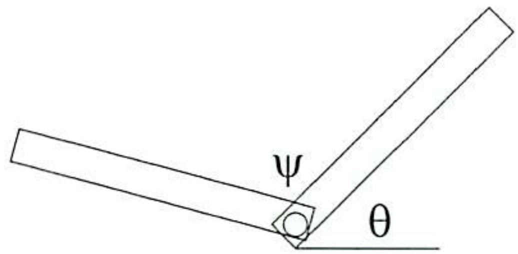

Figure 1: Simple Planar Rigid Body

The first example I take from Marsden and Koon, *The Geometric Structure of Nonholonomic Mechanics.* We start with two planar bodies connected at their centers of mass by a pin joint. The associated Lagrangian (including only the angular components) is:

$$L(\dot{\theta}, \dot{\psi}) = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 (\dot{\theta}^2 + \dot{\psi}^2)$$

so, using the fact that angular momentum is conserved, we can say that

$$I_1 \dot{\theta} + I_2 (\dot{\theta} + \dot{\psi}) = \mu$$

which implies that

$$d\theta + \frac{I_2}{I_1 + I_2} d\psi = \frac{\mu}{I_1 + I_2} dt$$

Now, if we assume $\mu = 0$ and directly integrate, we see that

$$\Delta\theta = \frac{-I_2}{I_1 + I_2} \int_0^{2\pi} d\psi = \frac{-2\pi I_2}{I_1 + I_2}$$

Moreover, if μ is nonzero, then

$$\Delta\theta = \frac{-I_2}{I_1 + I_2} \int_0^{2\pi} d\psi + \frac{-I_2}{I_1 + I_2} \mu \int_0^{T} dt = \frac{-I_2}{I_1 + I_2} (-2\pi + \mu T)$$

3 Planar Skater

Snakeboard

Now we move onto a more complicated example, the snakeboard. This mechanism is nonholonomic due to the no slip constraints on the wheels. It does, however, display some aspects of geometric phase, as I will show in simulations later on. First, we have to develop the equations of motion. Our goal is to write the equations of motion as in Marsden and Koon, in the following form:

$$\begin{aligned} g^{-1}\dot{g} &= -A(r)\dot{r} + \mathcal{I}^{-1} p && (1) \\ \dot{p} &= \frac{1}{2}\dot{r}^T \sigma_{\dot{r}\dot{r}} \dot{r} + p^T \sigma_{p\dot{r}} \dot{r} + \frac{1}{2} p^T \sigma_{pp} p && (2) \\ \ddot{r} &= u && (3) \end{aligned}$$

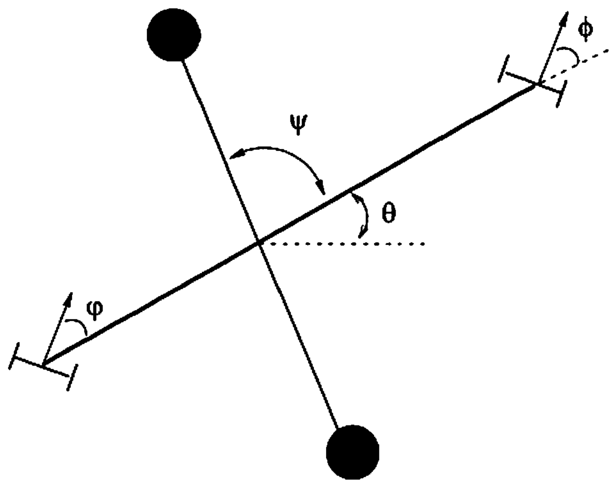

Figure 2: The Planar Skater

and then to use that form to put them in

$$\dot{z} = f(z) + h_i(z)u^i \tag{4}$$

For the Planar Skater, the Lagrangian is:

$$L(\dot{x},\dot{y},\dot{\theta},\dot{\psi},\dot{\phi},\dot{\varphi}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}J\dot{\theta}^2+\frac{1}{2}J_r(\dot{\theta}+\dot{\psi})^2+\frac{1}{2}J_w\left((\dot{\theta}+\dot{\phi})^2+(\dot{\theta}+\dot{\varphi})^2\right)$$

while the one forms corresponding to the constraints are:

$$\omega^1 = -\sin(\phi+\theta)dx + cos(\phi+\theta)dy - l\cos(\phi)d\theta$$

and

$$\omega^2 = -\sin(\varphi+\theta)dx + cos(\varphi+\theta)dy - l\cos(\phi)d\theta$$

Note that the constraints can be written as $\omega^i(\dot{q}) = 0$.

Now, by finding a,b,c such that

$$S_q = sp\{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\}$$

$$\begin{aligned} a &= -l(\cos(\phi)\cos(\varphi+\theta)+\cos(\varphi)\cos(\phi+\theta)) \\ b &= -l(\cos(\phi)\sin(\varphi+\theta)+\cos(\varphi)\sin(\phi+\theta)) \\ c &= \sin(\phi-\varphi) \end{aligned}$$

is a spanning set of vectors that satisfy the constraints, and such that all the vectors in the set are also tangent to G=SE(2), we have defined the constrained fiber distribution (as found in Murray and Kelly).

Now, if we let $\ll \cdot \gg$ denote the inner product defined by the kinetic energy metric for our system, we can define the momentum p as follows:

$$
\begin{aligned}
p &= \ll \dot{q}, X(q) \gg \\
&= \ll (\dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}, \dot{\phi}, \dot{\varphi}), (a, b, c, 0, 0, 0) \gg \\
&= ma\dot{x} + mb\dot{y} + (J + J_r + J_w)c\dot{\theta} + J_r c\dot{\psi} + J_w c(\dot{\phi} + \dot{\varphi})
\end{aligned}
$$

Now I will make the same assumption as in Bloch and in Ostrowski and Burdick: that $\phi = -\varphi$ and that $J + J_r + J_w = ml^2$. This simplification will make the rest of the computations simpler (I assme, since I have not done them without this simplificition!).

Now putting the above system in the form of Equations 1-3, we get that for $\dot{r} = (\dot{\psi}, \dot{\phi})$:

$$
A = \begin{pmatrix} -\frac{J_r}{2ml}\sin(2\phi) & 0 \\ 0 & 0 \\ -\frac{J_r}{2ml}\sin^2\phi & 0 \end{pmatrix}
$$

and

$$
\mathcal{I}^{-1} = \begin{pmatrix} -\frac{1}{2l} \\ 0 \\ -\frac{1}{2ml^2}\tan\phi \end{pmatrix}
$$

Now we can rewrite the shape space variables as inputs, i.e.

$$
\ddot{\psi} = u_1
$$

$$
\ddot{\phi} = u_2
$$

and get the state equations in the form of Equation 4.

$$
f = \begin{pmatrix} \frac{\cos\theta(-p+\dot{\psi}J_r\sin(2\phi))}{2ml} \\ \frac{\sin\theta(-p+\dot{\psi}J_r\sin(2\phi))}{2ml} \\ \frac{-2\dot{\psi}J_r\sin^2\phi+p\tan\phi}{2ml^2} \\ 2\dot{\phi}\dot{\psi}J_r\cos^2\phi - \dot{\phi}p\tan\phi \\ \dot{\psi} \\ \dot{\phi} \\ 0 \\ 0 \end{pmatrix}, h_\psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, h_\phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

Note that $\dot{p} = 2\dot{\phi}\dot{\psi}J_r\cos^2\phi - \dot{\phi}p\tan\phi$ is the generalized momentum equation.

Using these state equations I did the following simulations of the how phase enters into the dynamics of this nonholonomic system. These are similar to those found in the paper I was to read, *Symmetries in Motion: Geometric Foundations of Motion Control* by Marsden and Ostrowski.

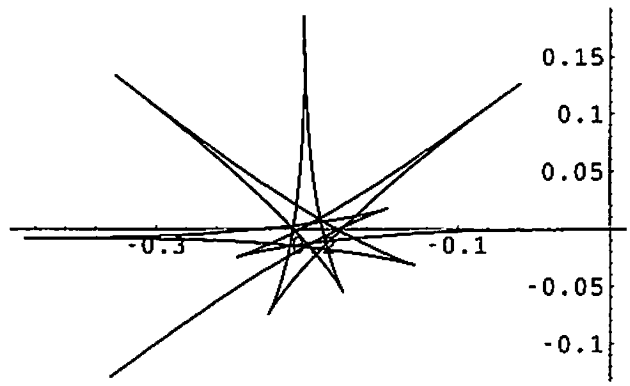

Figure 3: Star Shaped Mode

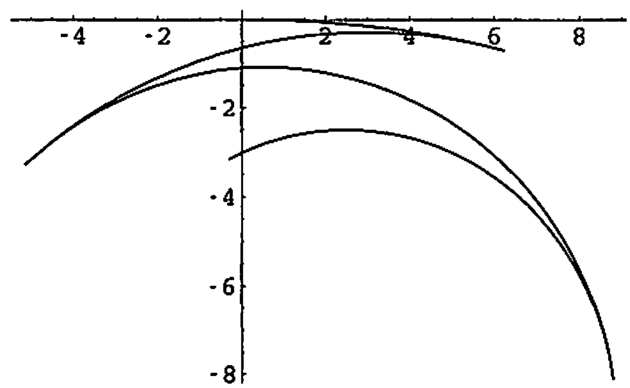

Figure 4: Parallel Parking Mode

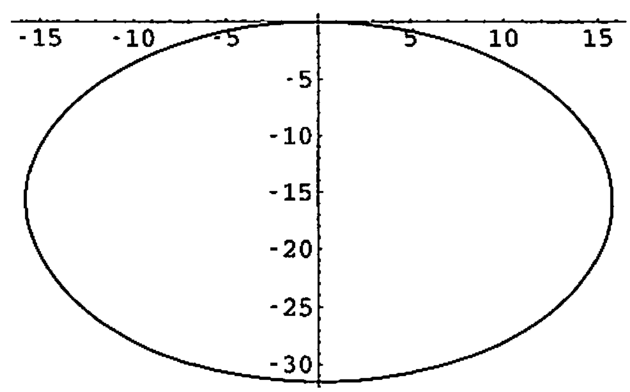

Figure 5: Yet some cyclic inputs produce no phase at all!