Dynamics of Nonholonomic Systems: a Sphere Rolling on a Spinning Table

Nice report!

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Every constraint that is not holonomic is nonholonomic. One will readily understand that it is not possible to give a general discussion of nonholonomic constraints such as can be done for holonomic ones because the latter is a narrowly circumscribed class while the former is not. (Thus, bananas are readily discussed, while nonbananas are not.)

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1 Introduction

Most of the systems encountered in mechanics are in some sense constrained, that is there are certain configurations and motions that are not admissible. In fact in [1] it is stated that "In a system of two or more particles, unconstrained motion does not exist", because there is always the constraint that two particles can never occupy the same space (overlap). Even though this definition of what constitutes a constraint may be a little extreme it indicates that the study of constrained systems should be an important part of mechanics.

Treating constraints in all generality is a formidable problem. In class techniques for dealing with a particular class of constraints (holonomic equality constraints) were introduced. This report focuses on a different type of constraints, namely nonholonomic, affine, equality constraints. Let Q be the configuration space for the mechanical system in question and TQ be the corresponding tangent space. Let $(q^i, \dot{q}^i)$ be a chart on TQ and consider a number of affine equality constraints given by:

$$\sum_{i=1}^{n} A_{ki}\dot{q}^i + A_{k0} = 0 \quad \text{for } k = 1, \dots, l \tag{1}$$

where $A_{ki} : Q \longrightarrow \Re$ for $k = 1, \dots, l$ and $i = 0, \dots, n$. Note that if $A_{k0} \equiv 0$ it is straight forward to write the constraints as 1-forms on Q.

$$\sum_{i=1}^{n} A_{ki} dq^i = 0 \quad \text{for } k = 1, \dots, l$$

The form notation can be used for the general case as well. First consider extending the configuration space to $Q \times \Re$. Then, letting (q^i, t) denote coordinates on the new space, the constraints can again be written as 1-forms:

$$\sum_{i=1}^{n} A_{ki} dq^i + A_{k0} dt = 0 \quad \text{for } k = 1, \ldots, l \tag{2}$$

This last term A_{k0} makes the constraint affine (instead of linear) in the velocity space. It will provide a convenient way of including the effect of the spinning table in our example.

Without loss of generality we can assume that the constraints are linearly independent, ie. that the matrix:

$$\begin{bmatrix} A_{11} & \ldots & A_{1n} \\ & \ldots & \\ A_{l1} & \ldots & A_{ln} \end{bmatrix} \tag{3}$$

has rank l. If the constraints are were not linearly independent we could pick the maximum number of linearly independent among them and ignore the rest (linearly dependent constraints do not constrain the system further).

It should be noted that some of the above "velocity" constraints (functions on TQ) may be expressible as "position" constraints (functions on Q). In other words it may be possible to find smooth functions $c_k : Q \times \Re \longrightarrow \Re$ for $k = 1, \ldots, m \leq l$ such that:

$$\frac{d}{dt} c_k = \sum_{i=1}^{n} \frac{\partial c_k}{\partial q^i} \dot{q}^i + \frac{\partial c_k}{\partial t} = \sum_{i=1}^{n} A_{ki} \dot{q}^i + A_{k0} \tag{4}$$

or, in 1-form notation:

$$dc_k = \sum_{i=1}^{n} A_{ki} dq^i + A_{k0} dt$$

The constraints that satisfy such relations are called *holonomic*. Ways of dealing with holonomic constraints were presented in class.

If such functions, c_k do not exist the constraints are called *nonholonomic*. Similarly to the holonomic case there are a number of ways for dealing with non-holonomic constraints. For example [2] and [1] present techniques for dealing with contact constraints (such as rolling without slipping) that make use of the geometry of the surfaces to derive the equations of motion for simple systems (like a ball rolling inside a bigger ball or a disk rolling on a plane). This report will focus on an analytical technique that is quite general and leads to equations very similar to the Euler-Lagrange equations for unconstrained systems. The general technique is first presented in the next section and then illustrated in section 3 by deriving the equations of motion for the specific example of a ball rolling on a rotating table. Finally simulation results obtained for the example of 3 are given in section 4.

2 Analysis Method

We would like to be able to derive the equations of motion of a system that includes nonholonomic constraints in a way similar to the way we derived the equations for an unconstrained system. In this section we present one such technique, which is found in [1] and [2]. The treatment of the problem in these two references is virtually identical, apart form differences in the terminology and notation. In this presentation the notation will be adapted to match the one used in class.

2.1 Forces of Constraint

The method presented here leads to equations that are essentially identical with the Euler-Lagrange equations for the unconstrained system, with external forces added to account for the constraints. All the calculations will be in coordinates. We start with a configuration manifold Q and a Lagrangian, L, on it's tangent bundle TQ:

$$L : TQ \longrightarrow \Re$$

One way of obtaining the Euler-Lagrange equations is using Hamiltons critical action principle (as in [3]):

$$\delta \int_{q_1}^{q_2} L(q, \dot{q}) dt = 0$$

which, after some manipulation, leads to:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0 \tag{5}$$

with a summation over $i = 1, \ldots, n$. In [2] this last equation is called the *D'Alembert-Lagrange* equation while in [1] it is called the *fundamental equation.* The derivation of the equation is slightly different in these references; they both present it as a version of Newtons equations in generalized coordinates, though [1] points out the connection with Hamiltons principle when he uses it as an alternative way of deriving the equations of motion. As discussed in class it is easy to extend the above analysis to include external forces F_e. The resulting D'Alembert-Lagrange equations are:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} - F_e^i \right) \delta q^i = 0 \tag{6}$$

Note that if there were no constraints the variations δq^i would be arbitrary and 6 would give the standard Euler-Lagrange equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_e^i$$

for all $i = 1, \ldots, n$. However, in the presence of constraints of the form 1 the variations δq^i are no longer independent. Indeed for every $(q, t) \in Q \times \Re$ the velocities need to satisfy:

$$\sum_{i=1}^{n} A_{ki} \dot{q}^i + A_{k0} = 0 \quad \text{for } k = 1, \ldots, l \tag{7}$$

If we identify $T_q Q$ with $\Re^n$ the above equations can be viewed as demanding that the feasible velocities lie in a hyperplane orthogonal to the l (linearly independent by 3) vectors:

$$\begin{bmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{bmatrix} \quad \text{for } k = 1, \ldots, l \tag{8}$$

Note that the A_{k0} terms only influence the distance of the hyperplane from the origin, so they are excluded from the above statement.

This restriction on the possible velocities needs to be somehow introduced into the equations of motion. An intuitive way of doing this is by assuming that the constraints provide forces F_e with exactly the right direction and magnitude to keep the system form violating them. To fully determine the direction of these forces we need to assume that they do not influence the motion tangential to the constraint (in [1] this is stated as a general principle, called the *D'Alembert principle*). If this is the case F_e is orthogonal to the constraint hyperplane for all $q \in Q$, hence we should be able to write it as a linear combination of the vectors 8:

$$F_e = \sum_{k=1}^{l} \lambda_k \begin{bmatrix} A_{k1} \\ \vdots \\ A_{kn} \end{bmatrix} \tag{9}$$

Here λ_k are real numbers and they determine the magnitude and direction of the constraint forces in the subspace spanned by the vectors 8; they will have to be calculated together with the equations of motion. Substituting 9 in equation 6 we obtain:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} - \sum_{k=1}^{l} \lambda_k A_{ki} \right) \delta q^i = 0 \tag{10}$$

Even though this seems like a small progress from 6 there is a major difference. By adding external forces such that the system is forced to satisfy the constraints we no longer need to explicitly restrict the variations δq^i; the constraint forces, with appropriate magnitude and direction, carry out the restriction for us. Hence, for arbitrary δq^i, we can write:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} - \sum_{k=1}^{l} \lambda_k A_{ki} = 0 \tag{11}$$

for all $i = 1, \ldots, n$. It should be noted that the above equations are no longer sufficient to fully determine the dynamics of the system, as there are only n equations for the $n + l$ unknowns q^i and λ_k. However we also know that the constraint forces need to be such that the velocities $\dot{q}^i$ satisfy the (linearly independent) equations 1. Hence the motion of the system needs to satisfy the equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} - \sum_{k=1}^{l} \lambda_k A_{ki} = 0 \quad \text{for } i = 1, \ldots, n \tag{12}$$

$$\sum_{i=1}^{n} A_{ki}\dot{q}^i + A_{k0} = 0 \quad \text{for } k = 1, \ldots, l \tag{13}$$

There are now $n + l$ equations for the $n + l$ unknowns and the complete system can be solved to determine the motion of the system q^i and the forces of constraint (through λ_k).

Comments: The technique outlined above is based on the analysis found in [1]. A similar derivation is given in [2]. The main difference between the two is that the authors of [2] rely more on algebraic manipulation and less on geometry; as a result from their derivation it is slightly more difficult to realize the relation between Lagrange multipliers and forces of constraint.

From the above analysis it should be apparent that D'Alembert's principle implies that the forces of constraint do no work, as they are always orthogonal to the velocity. Analytically it is easy to see that the variations δq^i always have to be tangential to the constraint, and therefore orthogonal to the vectors 8. Hence:

$$\sum_{i=1}^{n} A_{ki}\delta q^i = 0 \quad \text{for } k = 1, \ldots, l \tag{14}$$

$$\Rightarrow \quad \sum_{k=1}^{l} \lambda_k \sum_{i=1}^{n} A_{ki}\delta q^i = 0 \tag{15}$$

$$\Rightarrow \quad \sum_{i=1}^{n} (\sum_{k=1}^{l} \lambda_k A_{ki})\delta q^i = 0 \tag{16}$$

$$\Rightarrow \quad \sum_{i=1}^{n} F_{ei}\delta q^i = 0 \tag{17}$$

$$\Rightarrow \quad < F_e, \delta q >= 0 \tag{18}$$

$$\Rightarrow \quad \int_{q_1}^{q_2} < F_e, \delta q > dt = 0 \tag{19}$$

where the standard inner product is used in the last two equations. The last equation states that the virtual work done by the constraint forces is zero.

It should be noted that, even though the above technique was derived with nonholonomic constraints in mind, it is perfectly valid for holonomic constraints as well. Equation 4 indicates how one can write holonomic constraints in the form of equation 1, so that they look nonholonomic. Once this is done the above technique can be used to derive the equations of motion. However it may be preferable to start by using any holonomic constraints to reduce the dimension of Q before applying the above technique, as this would lead to fewer equations of motion.

Finally the above analysis depends heavily on coordinates and the fact that the configuration space Q is finite dimensional. It is not clear (at least not from [1] and [2]) if any of the calculations are expressions of some fundamental intrinsic principle or if and how they can be extended to infinite dimensions.

2.2 The Vakonomic Approach:

An alternative approach for dealing with constraints, the vakonomic approach is described in [4]. A technique very similar to the Lagrange multiplier method introduced in class in relation to holonomic constraints is used. The equations of motion are obtained as the standard Euler-Lagrange equations for the augmented Lagrangian $\hat{L}$ given by:

$$\hat{L}(q, \lambda, \dot{q}, \dot{\lambda}, t) = L(q, \dot{q}, t) - \sum_{k=1}^{l} \lambda_k \left(\sum_{i=1}^{n} A_{ki}\dot{q}^i + A_{k0} \right) = 0 \tag{20}$$

The resulting Euler-Lagrange equations together with the constraint equations 1 can be solved to obtain equations of motion for the system. It should be noted that, in order to obtain the multipliers λ in this case one needs to solve differential equations and therefore should decide on appropriate initial conditions.

It turns out that, even though for the case of holonomic constraints this technique leads to the same results as the one presented previously, the equations of motion are generally very different if there are any nonholonomic constraints involved. This fact is explicitly stated in [4] and [5] where it is also demonstrated by means of the example presented in the next section. The author of [1] also touches on this issue from a different stand point when he considers embedding the constraints by simply solving them for some coordinates and substituting in the Lagrangian (obtaining a system of reduced dimension). It is stated that this technique is valid only for holonomic constraints and a counter example is given for the nonholonomic case. The counter example is based on the fact that the equations obtained when nonholonomic constraints are embedded are not reducible to the equations expected from Newtonian mechanics whereas this is readily done for the equations obtained with the technique outlined in 2.1.

3 Equations of Motion for a Ball on a Spinning Table

As an example of the application of the above technique consider the system shown in figure 1. A ball of mass M radius r and moment of inertia I is rolling without slipping on a level table

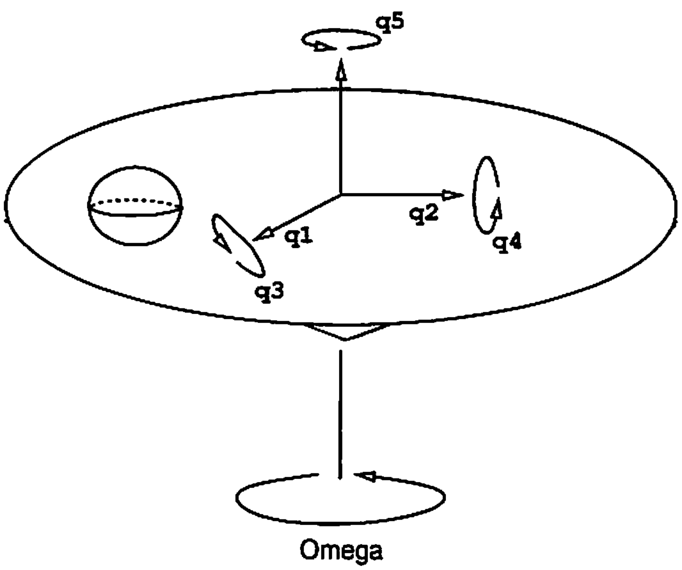

Figure 1: Ball on a Rotating Table

which is rotating with angular velocity Ω. The state of the system can be completely characterized by the orientation of the ball and it's position on the plane. Hence the configuration space is $\Re^2 \times SO(3)$. As the previous analysis was all done in coordinates we have to choose a parametrization for this 5 dimensional space. To do this consider a stationary frame in the center about which the table is rotating and let (q^1, q^2) be the position of the point of contact of the ball with the table and q^3, q^4, q^5 be the fixed axis rotation angles that characterize the orientation of the ball. Then $\dot{q}^i, i = 1, \ldots, 5$ are the corresponding linear and angular velocities. Of course other parametrizations are possible but this choice leads to simple equations of motions.

As the table is level the potential energy of the system is the same for all configurations, therefore the equations of motion can be obtained using the kinetic energy as the Lagrangian:

$$L(q, \dot{q}) = \frac{1}{2}M((\dot{q}^1)^2 + (\dot{q}^2)^2) + \frac{1}{2}I((\dot{q}^3)^2 + (\dot{q}^4)^2 + (\dot{q}^5)^2) \quad (21)$$

The rolling without slipping assumption imposes nonholonomic constraints on the system. In particular the velocity of the point of contact should be the same for the ball and the table, hence the linear and angular velocities must be related by:

$$\dot{q}^1 - r\dot{q}^4 + \Omega q^2 = 0 \quad (22)$$

$$\dot{q}^2 + r\dot{q}^3 - \Omega q^1 = 0 \quad (23)$$

Substituting the above in the equations derived in the previous section we obtain a system of 5 equations.

$$M\frac{d}{dt}\dot{q}^1 - \lambda_1 = 0 \quad \Rightarrow \quad M\ddot{q}^1 = \lambda_1 \quad (24)$$

$$M\frac{d}{dt}\dot{q}^2 - \lambda_2 = 0 \quad \Rightarrow \quad M\ddot{q}^2 = \lambda_2 \quad (25)$$

$$I\frac{d}{dt}\dot{q}^3 - r\lambda_2 = 0 \quad \Rightarrow \quad I\ddot{q}^3 = r\lambda_2 \quad (26)$$

$$I\frac{d}{dt}\dot{q}^4 + r\lambda_1 = 0 \quad \Rightarrow \quad I\ddot{q}^4 = -r\lambda_1 \quad (27)$$

$$I\frac{d}{dt}\dot{q}^5 = 0 \quad \Rightarrow \quad I\ddot{q}^5 = 0 \quad (28)$$

The above together with the constraint equations 22 & 23 have to be solved for the 5 configuration variables and the two Lagrange multipliers.

As expected the equations of motion are second order. It should also be noted that because of the simplicity of the problem and the choice of coordinates the equations are easily related to Newtons law $F = ma$, which probably means that the we are on the right track. To do this the Lagrange multipliers have to be interpreted as representing the magnitudes of the forces of constraint as described in the previous section.

To solve the equations of motion one can start by eliminating the λ. From 24 and 27:

$$rM\ddot{q}^1 = -I\ddot{q}^4 \quad (29)$$

Also from 22:

$$\ddot{q}^4 = \frac{1}{r}(\ddot{q}^1 + \Omega\dot{q}^2)$$

Hence:

$$(Mr^2 + I)\ddot{q}^1 = -I\Omega\dot{q}^2 \quad (30)$$

Similarly:

$$(Mr^2 + I)\ddot{q}^2 = I\Omega\dot{q}^1 \quad (31)$$

The equations for q^3 and q^4 can be obtained through 29 and it's counterpart. Finally the equation for q^5 is 28. From 30 and 31 it is apparent that the equations of motion are linear and very similar to the equations for a simple harmonic oscillator. All the eigenvalues have zero real parts so we expect the system to be stable (in the sense of Liapunov) and display periodic orbits.

Other Effects: As the above equations are variants of the standard Euler-Lagrange equations it is relatively straight forward to modify them to add the effect of external forces. For example in [5] this fact is used to add the effect of frictional dissipation (which actually seems to introduce instability, a rather surprising effect). Similarly we can modify the equation of motion for the case where the rotating table is not flat but inclined, for example at an angle α with respect to he horizontal. Choosing the coordinates so that the q^1 and q^2 axes are flat on the table with q^1 pointing "down hill" we only need to add an external force $Mg\sin(\alpha)$ (g being the acceleration due to gravity) to the right hand side of equation 24. Alternatively the same effect can be introduced by using a different Lagrangian that contains a potential energy term $Mgq^1\sin(\alpha)$.

4 Simulation Results

The equation derived in the previous section were simulated using a C program and a fourth order, variable step, Runge-Kutta integration routine (a bit of an overkill for such a simple system). The results were plotted using Matlab. The parameters used were:

- M=0.01 kg
- r=0.02 m
- g=10 ms^{-1}
- $\Omega = 3$ radians s^{-1}

The resulting trajectories for two different values of inclination of the table are shown in figures 2 and 3.

5 Concluding Remarks

In this report the standard method for dealing with nonholonomic, affine equality constraints was outlined and it's application was demonstrated by means of an example. The information presented here came mainly from the two sources [1] and [2] and it indicates that this technique is very convenient as it is easily applicable, quite general and the resulting equations are in the standard Euler-Lagrange form for the unconstrained system with the addition of external forces of constraint, hence make sense intuitively.

There are a number of related issues that were not addressed here. For example, as mentioned in section 2.1, it is not clear from [1] and [2] whether the equations make intrinsic sense. Moreover both of the references concentrate to the Lagrangian formalism and do not attempt to duplicate the derivation of the equations of motion using the Hamiltonian approach. Work in this direction is described in [6].

Finally an interesting problem would be the rapproachment of the technique presented here with the vakonomic approach. In section 2.2 it was stated that the equations of motion obtained by these two techniques are different if nonholonomic constraints are present. In fact the example in [5] indicates that there is no straight forward way (eg. choice of appropriate initial conditions for the Lagrange multipliers) of resolving this ambiguity, the equations are fundamentally different. It would be interesting to find out what causes this difference and hence figure out a way to resolve it. It was pointed out that for the vakonomic approach the constraint forces are not always normal to the constraint and therefore end up doing work, thus violating D'Alembert's principle. This can never be a problem for the technique in [1] and [2] as the constraint forces are apriori forced to satisfy the D'Alembert principle (and therefore to have no effect on the motion along the constraints). This observation suggests that, by introducing additional constraints on the forces it may be possible to improve on the agreement between the two techniques.

References

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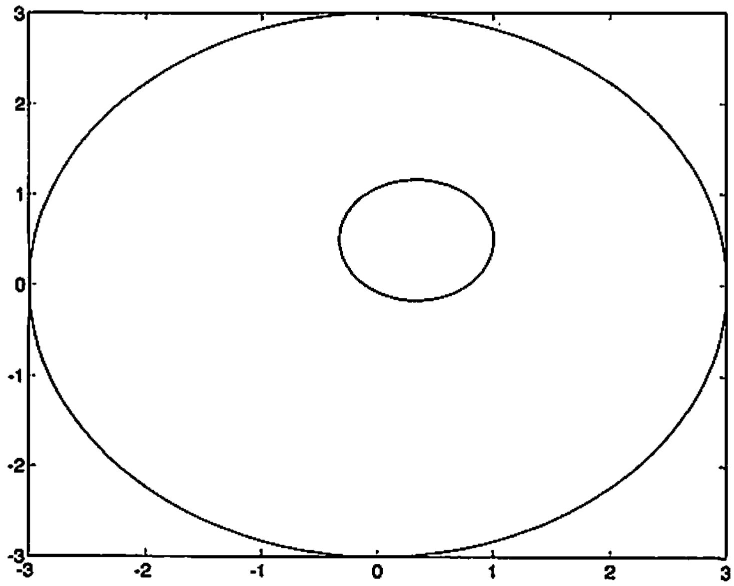

Figure 2: a=0

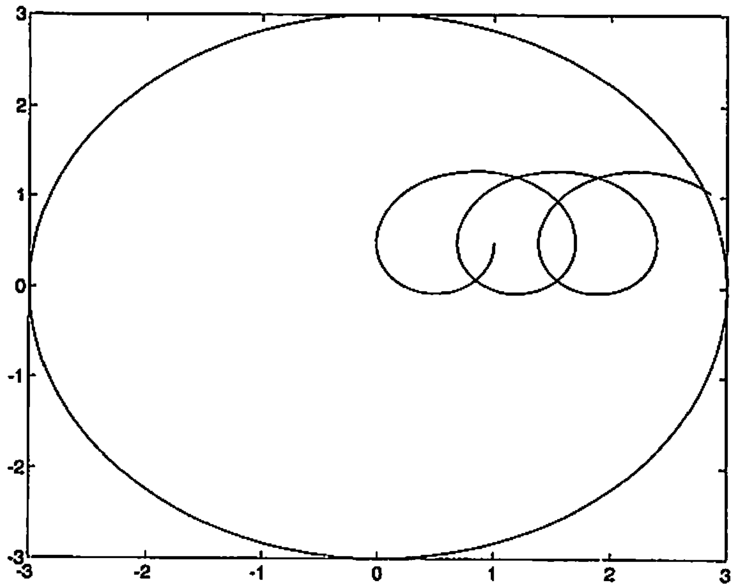

Figure 3: a=0.05