Linear Hamiltonian Systems and the Hopf Map

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Aaron - This is an excellent paper. Full of creative ideas — For example, I did not know Thm 3.5 on p 15 - is that yours?
You got an A+ in the course - good work!
Jerry.

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1. Introduction

Let (P, Ω) be a symplectic manifold and G a Lie group acting canonically on P with equivariant momentum map $J: P \to \mathfrak{g}^*$. Suppose μ is a regular value for J. Following Smale [9], we may ask, what is the topological type of the level sets $P_\mu = J^{-1}(\mu)$? Let $G_\mu \subseteq G$ be the stabilizer subgp of μ under the Ad^* action of G. Then, by Marsden and Weinstein [6], if G_μ acts freely on P_μ, there is an additional structure: $M_\mu = P_\mu / G_\mu$ is a manifold with a symplectic structure inherited from P. Thus we can expand Smale's question to include: what is the topological type of M_μ and which principal bundles $P_\mu(M_\mu, G_\mu)$ arise in this manner?

We propose to examine this question in a simple but still interesting case: $P = \mathbb{R}^{2n}$, Ω a symplectic structure (not nec. the usual one) and G a group of linear symplectic maps that leave a quadratic form H invariant. We will show that under suitable conditions we can decompose G into two pieces: a 1-dim subgp. G_1 that acts freely on the nondegenerate hypersurfaces, H^c, of H and another subgp. G_2 that acts transitively on H^c. We will use the momentum map for G_2 to map the reduction of H^c by G_1 to a coadjoint orbit for G_2. For $n=2$ this specializes to the Hopf fibration.

The rest of this paper is organized as follows: section 2 defines linear Hamiltonian systems and discusses the questions of normal forms, invariant functions and related topics. We also explicitly display normal forms for $Sp(1)$, $Sp(2)$, using Kocak [3]. Section 3 begins with an elementary discussion of the topology and geometry of quadratic hypersurfaces and moves on to the central tool of the paper which identifies the Lie alg of the symmetry group of H with the centralizer of an associated element in $sp(n)$. We apply this to perform the reduction mentioned above. After examining one general case, section 4

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applies the table of normal forms to examine some special cases. In the last section we return to Smale's question and present some conclusions and regrets - regrets in the form of questions that are left unanswered.

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2. Linear Hamiltonian systems and normal forms

a. Let V be a real vector space of dim $2n$. Let Ω be a symplectic form on V. Let $Sp(V,\Omega) = \{g \in GL(V) : \Omega(gx, gy) = \Omega(x,y)\ \forall x,y \in V\}$ and $sp(V,\Omega) = \{A \in gl(V) : \Omega(Ax,y) + \Omega(x,Ay) = 0\}$

<u>Def</u>: A pair (A,Ω) s.t. Ω is symplectic and $A \in sp(V,\Omega)$ is called a <u>linear Hamiltonian system</u> (lHs) on V.

Let (A,Ω) be an lHs. Consider the function $\hat{H}_A(x,y) = \frac{1}{2}\Omega(Ax,y)$. $\hat{H}_A : V \times V \to \mathbb{R}$ and is symmetric and bilinear. Let $H_A(x) = \hat{H}_A(x,x)$ = associated quadratic form. Let $Q(V)$ denote all quadratic functions $V \to \mathbb{R}$, so $H_A \in Q(V)$. Note that the equation $\dot{x} = Ax$ is Hamiltonian with Hamiltonian function $H_A(x)$; we can explicitly write the flow φ_t for H_A: $\varphi_t = e^{tA}$.

Given $H, K \in Q(V)$, we may form their Poisson-bracket $\{H,K\}$ which is another member of $Q(V)$.

<u>2.1 Prop</u>: The map $J : A \mapsto H_A$ is a Lie algebra isomorphism between $sp(V,\Omega)$ and $(Q(V), \{,\})$.

<u>Pf</u>

Let $A, B \in sp(V,\Omega)$. Then $\{H_A, H_B\}(x) = \Omega(X_{H_A}(x), X_{H_B}(x))$

$= \Omega(Ax, Bx)$. On the other hand, $H_{[A,B]}(x) = \frac{1}{2}\Omega([A\,B](x), x)$

$= \frac{1}{2}\Omega((AB - BA)x, x) = \frac{1}{2}\Omega(ABx, x) - \frac{1}{2}\Omega(BAx, x)$

$= -\frac{1}{2}\Omega(Bx, Ax) + \frac{1}{2}\Omega(Ax, Bx) = \Omega(Ax, Bx) = \{H_A, H_B\}(x)$

$\therefore A \mapsto H_A$ is a Lie alg. hom. Injectivity follows from nondegeneracy of Ω. For surjectivity, if K is quadratic, let $\hat{K}$ = associated bilinear form, then $A = \Omega^{-1} \cdot \hat{K}$ will work /

Cor: $\dim sp(V,\Omega) = \dim Sp(V,\Omega) = n\cdot(2n+1)$

Pf

The dim of the space of symmetric bilinear forms is $1+2+\cdots+2n = \frac{2n\cdot(2n+1)}{2}$ /

Def: Let (A_1,Ω_1), (A_2,Ω_2) be two IHs on V. We say (A_i,Ω_i) $i=1,2$ are similar if $\exists\, T\in GL(V)$ s.t. $A_1 = T^{-1}A_2T$ and $\Omega_1 = T^t\Omega_2 T$ (i.e. $\Omega_1(x,y) = \Omega_2(Tx,Ty)$).

We compute $H_{A_1}(x) = \frac{1}{2}\Omega_1(A_1x,x) = \frac{1}{2}\Omega_2(TA_1x,Tx)$
$= \frac{1}{2}\Omega_2(TT^{-1}A_2Tx, Tx) = H_{A_2}(Tx)$.

Similarity is an equiv. relation and under it, we see that the induced action on the quadratic representation is change of variables. Here are two approaches to studying this action:

(1) Fix a symplectic structure Ω; then we are restricting $T\in Sp(V,\Omega)$ and similarity classes are adjoint orbits of $Sp(V,\Omega)$ on $sp(V,\Omega)$

(2) Alternately, consider equivalence classes of H_A or A, and look at the orbits of Ω under the associated symmetries. In other words, for each triple (p,q,r): $p+q+r=2n$ representing a quadratic form of type $(+,-,0)$, we can attempt to determine the orbits of Ω under the gp. $O(p,q,r)$

We use approach (1) mostly and make some comments later about approach #2. The basic task is to pick a representative member of each equivalence class. This is called a normal form.

b. Adjoint orbits and normal forms

Let $G\subseteq GL(n)$ be any matrix Lie gp. with Lie alg. $\mathfrak{g}$. We can generalize the discussion above to include the task of determining normal forms for the adjoint action of

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G on $\mathfrak{g}$. Suppose $A \in \mathfrak{g}$. One clear invariant of A is its (complex) eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. This set is invariant under conjugation by $GL(n)$, since the equation $\det(A - \lambda I) = 0$ is. Thus any particular conjugation has the effect of permuting the λ_i ∴ the symmetric polynomials $\sigma_1, \dots, \sigma_n$ are invariants under conjugation. Thus if $\sigma_i(A) = \prod_{j_1 < \dots < j_i} \lambda_{j_1} \cdots \lambda_{j_i}$; then the mapping

$\mathfrak{g} \to \mathbb{C}^n$ by $A \mapsto (\sigma_1(A) \dots \sigma_n(A))$ is constant on

the adjoint orbits. Notice that if G is a real matrix group then we can replace $\mathbb{C}^n$ by $\mathbb{R}^n$ because the $\sigma_i(A)$ occur as coefficients in the expansion of $\det(A - \lambda I)$ ∴ are real. For $G = Sp(n)$ we can say more:

2.2 <u>Lemma</u>: If $A \in sp(n)$ then $\sigma_i(A) = 0$ for $i = 1, \dots 2n-1$, i odd.

<u>Pf</u>

Let $p_A(\lambda) = \det(A - \lambda I)$ and let $\lambda_1, \dots, \lambda_{2n}$ be the roots of $p(A)$. If λ is a root then $-\lambda$ is a root because $p_A(T) = (-1)^{2n} p_A(-T) = p_A(-T)$, which follows from $A^T \mathbb{J} + \mathbb{J} A = 0$.

Thus for i odd: $\sigma_i(A) = \sigma_i(\lambda_1 \dots \lambda_{2n}) = (-1)^i \sigma_i(-\lambda_1, \dots, -\lambda_{2n})$ since σ_i is homogeneous of degree i, $= (-1)^i \sigma_i(\lambda_1 \dots \lambda_{2n})$ since $(-\lambda_1 \dots -\lambda_{2n})$ is a permutation of $(\lambda_1 \dots \lambda_{2n})$ ∴ $\sigma_i(A) = 0$ /

Consider the map $\Theta : sp(n) \to \mathbb{R}^n$ by $\Theta(A) = (\sigma_2(A), \sigma_4(A), \dots, \sigma_{2n}(A))$. Then Θ is constant on adjoint orbits; however Θ does <u>not</u> define $\mathbb{R}^n$ as a moduli space; i.e. $\exists$ distinct orbits with the same image under Θ. We shall see this shortly.

However, there is a side benefit to this discussion, it is clear that $B = \sigma_1^2 - 2\sigma_2$ is homogeneous, quadratic of degree 2 and invariant. The associated symmetric bilinear form $\hat{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

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is called the <u>Cartan-Killing form</u> To avoid explicit use of the eigenvalues : note $\lambda_1 + \cdots \pm \lambda_n = tr(A) = \sigma_1$ and $\prod_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left[tr(A)^2 - tr(A\cdot A)\right] = \sigma_2$, ie. $B(A) = tr(A\cdot A)$ so

$\hat{B}(A,B) = tr(A\cdot B)$. If $\hat{B}$ is a nondegenerate form, $\mathfrak{g}$ is called <u>semisimple</u>.* If $\mathfrak{g}$ is semisimple we may identify $\mathfrak{g}$ with $\mathfrak{g}^*$ in the usual way $A \longleftrightarrow \eta$ where $\eta(C) = \hat{B}(A,C)$. Write $\eta = A^b$ and $A = \eta^{\#}$.

2.3 <u>Prop</u>: If G is a matrix Lie gp with semisimple Lie alg $\mathfrak{g}$ then the map $A \longmapsto A^b$ is an equivariant isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ under adjoint and coadjoint actions of G, resp. In particular, orbits are mapped to each other.

<u>Pf</u>

Use $\langle\, ,\rangle$ to denote the natural pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $(\, ,)$ the inner product $\mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$. Then

$\langle Ad^*_{g^{-1}}(A^b), C\rangle = \langle A^b, Ad_{g^{-1}}(C)\rangle = (A, Ad_{g^{-1}}(C)) = (Ad_g(A), C)$

(since $(\, ,)$ is inv.) $= \langle Ad_g(A)^b, C\rangle \therefore \quad Ad^*_{g^{-1}}(A^b) = [Ad_g(A)]^b$

This proves equivariance. The isomorphism follows from nondeg. of $(\, ,)$. /

Thus far, Θ allows us to distinguish inequivalent IHs and we've gotten Θ essentially for free from the group theory. As noted above, we have not captured the complete ring of invariants.

To circumvent this problem, we appeal to Burgoyne and Cushman [1] which provides a constructive decomposition of the adjoint orbits of the classical matrix groups. These results are specialized in Kocak [3] to $Sp(n)$. We conclude this section by applying Kocak's tables to the groups $Sp(1)$ and $Sp(2)$. This will provide both a usefull list of normal forms and an illustration of the group theory.

* This is true only if $Center(\mathfrak{g}) = 0$ (see § 4.b).

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The key to using these tables is the idea of a decomposable form. (A,Ω) is <u>decomposable</u> if $V = V_1 \oplus V_2$ s.t. V_i are A-invariant Ω-orthogonal and Ω-non-degenerate. If (A,Ω) is decomposable write $\Omega_i = \Omega | V_i$ $A_i = A | V_i$ and $(A,\Omega) = (A_1,\Omega_1) \oplus (A_2,\Omega_2)$. This reduces the question of normal forms to normal forms for indecomposable IHs for decomposability is a conjugate-invariant property. This is similar to the Jordan blocks for conjugacy classes in $GL(n)$.

c. $Sp(1)$

$\dim Sp(1) = 3$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1)$ iff $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ ie. $Sp(1) = SL(2,\mathbb{R})$

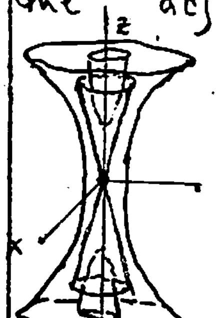
$sp(1) = sl(2;\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$. We have Cartan-Killing

form $B\left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right] = \operatorname{tr} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = 2(a^2+bc) = -2\det(A)$

and thus $\det(A) = \text{const} \Rightarrow$ adjoint orbits. We pick an

orthonormal basis: $X = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $Y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Z = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

so that wrt this basis $B(xX + yY + zZ) = x^2+y^2-z^2$ and thus the adjoint orbits are sheets of the hyperboloids $x^2+y^2-z^2 = \text{const}$



(from pg 382 Marsden and Ratiu [5])

Let $\omega^+ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\omega^- = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that

ω^+ is conjugate to ω^- using $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin Sp(1)$.

We note that since $V = \mathbb{R}^2$ is 2-dim all the normal forms are indecomposable. For an indecomposable form, we describe it using the notation $(\lambda_1, \ldots \lambda_{2n})^i$ where λ_j denotes the eigenvalues and i is an index for a symplectic structure ($i = \pm$ for ω^+, ω^-)

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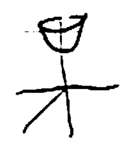
If superscript i is absent we are assuming the usual symplectic form.

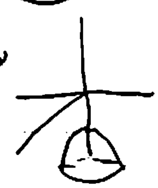
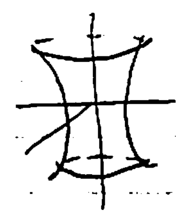
Table 1: Normal forms for $Sp(1)$

	Form	Matrix A	Det	Symplectic Structure	Hamiltonian H_A
1.	$(0,0)^+$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	0	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\frac{1}{2} x_1^2$
2.	$(0,0)^-$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	0	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$-\frac{1}{2} x_1^2$
3.	$(i\beta,-i\beta)^+$ $\beta\in\mathbb{R}, \beta>0$	$\begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$	β^2	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\frac{1}{2}\beta(x_1^2+x_2^2)$
4.	$(i\beta,-i\beta)^-$	$\begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$	β^2	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$-\frac{1}{2}\beta(x_1^2+x_2^2)$
5.	$(\alpha,-\alpha)$ $\alpha\in\mathbb{R}, \alpha>0$	$\begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$	$-\alpha^2$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$-\alpha x_1 x_2$

Note these correspond exactly to the following orbits:

$H_A = 0 \longleftarrow \{0\}$; $(0,0)^+ \longleftarrow$  , the upper piece of the light cone;

$(0,0)^- \longleftarrow$  , the lower sheet; $(\pm i\beta)^+ \longleftrightarrow$  ;

$(\pm i\beta)^- \longleftarrow$  ; $(\pm\alpha) \longleftarrow$ 

As anticipated, the invariant $\sigma_2 = \det$ fails to distinguish $(\pm i\beta)^+$ and $(\pm i\beta)^-$, for example.

Before leaving $Sp(1)$, we note that we have actually constructed a map $Sp(1) \to O(2,1)$ by $g \mapsto Ad_g$, since $B(xX+yY+zX)$ $= x^2+y^2-z^2$ is Ad_g invariant. $\dim O(2,1) = \dim Sp(1)$ $\therefore$ this is a covering of the connected component of $O(2,1)$.

d. $Sp(2)$

We have $\dim Sp(2) = 10$, Write $A \in sp(2)$ as

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$$A = \left(\begin{array}{cc|cc} a & b & x & y \\ c & d & y & z \\ \hline s & t & -a & -c \\ t & u & -b & -d \end{array}\right) \quad a,b,c,d,x,y,z,s,t,u \in \mathbb{R}$$

Computation of $tr(A \cdot A)$ gives $B(A) = 2a^2 + 2d^2 + 2xs + 2uz + 4bc + 4yt$

which has signature $(6,4)$. In addition to $B(A)$ we also have $\det(A)$ a homogeneous polynomial of degree 4. These two functions (up to scalar) form the components of $\Theta : sp(2) \to \mathbb{R}^2$. Generically, we would expect the equations $\det(A) = c_1$, $B(A) = c_2$ to define 8-dim submflds in $sp(2) \simeq \mathbb{R}^{10}$, in which the orbits must lie

Moving to the normal forms, there are now decomposable as well as indecomposables. The decomposable forms are derived from direct sums of the forms for $sp(1)$. Here is a list of the indecomposable forms from Kocak's tables:

First we define some symplectic forms:

$$\omega^{1+} = \left[\begin{array}{c|c} 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \end{array}\right], \quad \omega^{2+} = \left[\begin{array}{c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right], \quad \omega^{1-} = -\omega^{1+}, \quad \omega^{2-} = -\omega^{2+}$$

ω^{1+} is conjugate to the standard form $\mathbb{J}$ by $\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

ω^{2+} is conjugate to $\mathbb{J}$ by $\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$

(1) $(0,0,0,0)^{1I}$

Matrix $A = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\quad \det(A) = 0$, $\sigma_2(A) = 0$ $\quad$ Symplectic forms $= \omega^{1+}, \omega^{1-}$

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Notice that A does not look a member of $sp(2)$. Conjugating by the matrices given above for $\omega^{1\pm}$ will put A back into a standard form. However none of the quantities we are computing require A in this form. For example

$$H_A^+ = \frac{1}{2}(x_1\ x_2\ x_3\ x_4)\left[\left(\begin{array}{cc|cc} & & 0 & 1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \\ \hline 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \end{array}\right]\left[\begin{array}{cc|cc} 0 & 0 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \frac{1}{2}(x_1\ x_2\ x_3\ x_4)\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 1 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{0} \end{array}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1x_3 - \frac{1}{2}x_2^2.$$

$$H_A^- = -H_A^+ = \frac{1}{2}x_2^2 - x_1x_3$$

(2) $(\iota\beta, \iota\beta, -\iota\beta, -\iota\beta)^{2\pm}$. Matrix $A = \left[\begin{array}{cc|cc} 0 & 0 & -\beta & 0 \\ 1 & 0 & 0 & -\beta \\ \hline \beta & 0 & 0 & 0 \\ 0 & \beta & 1 & 0 \end{array}\right]$

$\beta > 0,\ \beta \in \mathbb{R}$

$\det(A) = \beta^4$

$\sigma_2(A) = 2\beta^2$

$$H_A^+ = \frac{1}{2}\left[x_1^2 + x_3^2 - 2\beta x_1x_4 + 2\beta x_2x_3\right]$$

$$H_A^- = -H_A^+$$

(3) $(+\alpha, -\alpha, +\alpha, -\alpha)^{1+}$. $A = \left[\begin{array}{cc|cc} \alpha & 0 & & \\ 1 & \alpha & \multicolumn{2}{c}{0} \\ \hline & & -\alpha & 0 \\ \multicolumn{2}{c|}{0} & 1 & -\alpha \end{array}\right]$

$\det(A) = \alpha^4$

$\sigma_2(A) = -2\alpha^2$

$$H_A = x_1x_3 - \alpha(x_1x_4 + x_2x_3)$$

(4) $(\alpha+\iota\beta, \alpha-\iota\beta, -\alpha+\iota\beta, -\alpha-\iota\beta)^{2+}$ $A = \left[\begin{array}{cc|cc} \alpha & 0 & -\beta & 0 \\ 0 & -\alpha & 0 & -\beta \\ \hline \beta & 0 & \alpha & 0 \\ 0 & \beta & 0 & -\alpha \end{array}\right]$

$\det(A) = (\alpha^2 + \beta^2)^2$

$\sigma_2(A) = -2(\alpha^2 - \beta^2)$

$$H_A = -\alpha x_1x_2 - \beta x_1x_4 + \beta x_2x_3$$

Here is a complete chart, including decomposables, in what is hopefully obvious notation:

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Table 2: Normal forms for $Sp(2)$

DECOMPOSABLE (1–6)

	Form	Det A	$\sigma_2(A)$	H_A
1.	$(0,0)^\pm + (0,0)^\pm$	0	0	$\pm\frac{1}{2}x_1^2 \pm \frac{1}{2}x_3^2$
2.	$(0,0)^\pm + (\pm i\beta)^\pm$	0	$-\beta^2$	$\pm\frac{1}{2}x_1^2 \pm \frac{1}{2}\beta(x_3^2+x_4^2)$
3.	$(0,0)^\pm + (\pm\alpha)$	0	$-\alpha^2$	$\pm\frac{1}{2}x_1^2 - \alpha x_3 x_4$
4.	$(\pm i\beta_1)^\pm + (\pm i\beta_2)^\pm$	$\beta_1^2\beta_2^2$	$\beta_1^2+\beta_2^2$	$\pm\frac{1}{2}\beta_1(x_1^2+x_2^2) \pm \frac{1}{2}\beta_2(x_3^2+x_4^2)$
5.	$(\pm i\beta)^\pm + (\pm\alpha)$	$\alpha^2\beta^2$	$\beta^2-\alpha^2$	$\pm\frac{1}{2}\beta(x_1^2+x_2^2) - \alpha x_3 x_4$
6.	$(\pm\alpha_1) + (\pm\alpha_2)$	$\alpha_1^2\alpha_2^2$	$-(\alpha_1^2+\alpha_2^2)$	$-\alpha_1 x_1 x_2 - \alpha_2 x_3 x_4$
7.	$(0,0,0,0)^{1\pm}$	0	0	$\pm(x_1 x_3 - \frac{1}{2}x_2^2)$
8.	$(\pm i\beta, \pm i\beta)^{2\pm}$	β^4	$2\beta^2$	$\pm\frac{1}{2}[x_1^2+x_3^2 - 2\beta(x_1x_4 - x_2x_3)]$
9.	$(\pm\alpha, \pm\alpha)^{1+}$	α^4	$-2\alpha^2$	$x_1x_3 - \alpha(x_1x_4 + x_2x_3)$
10.	$(\pm\alpha \pm\beta)^{2+}$	$(\alpha^2+\beta^2)^2$	$-2(\alpha^2-\beta^2)$	$-\alpha x_1 x_2 - \beta x_1 x_4 + \beta x_2 x_3$

(1.2)

3. Momentum maps and isotropy subgroups.

a. Quadratic hypersurfaces

Let H be a quadratic form on V. We use the notation H is type (p,q,r) where $p+q+r = \dim V$, r = nullity, q = index. Let $O(H) = \{g \in GL(V) : H(gx) = H(x)\}$, $G(H) = O(H) \cap Sp(V,\Omega)$, $\mathfrak{g}(H)$ = Lie algebra of $G(H)$ and $\mathfrak{o}(H)$ = Lie algebra of $O(H)$. For $c \in \mathbb{R}$, let $H^c = \{x \in V : H(x) = c\}$. We focus initially on the case when H is nondegenerate, so assume for now that $r=0$.

3.1 Prop: Let H be type (p,q) and $\lambda > 0$. Assume $p \geq q$. Then

(1) $O(H)$ acts transitively on H^c $\forall c \in \mathbb{R}$.

(2) H^λ is diffeomorphic to H^1, $H^{-\lambda}$ is diffeomorphic to H^{-1}.

(3) (a) If $q \geq 1$ then $H^1 \approx \mathbb{R}^q \times S^{p-1}$, $H^0 \approx \mathbb{R} \times S^{p-1}$, $H^{-1} \approx \mathbb{R}^p \times S^{q-1}$

(b) If $q = 0$ then $H^1 \approx S^{p-1}$, $H_0 = \{0\}$, $H_{-1} = \emptyset$

Pf

(1) This is follows from Witt's Thm. on quadratic forms. See Serre [8], (pg 31).

(2) $H(\lambda x) = \lambda^2 H(x)$ $\therefore$ this is clear.

(3) Since we are only concerned with diffeomorphism classes, we may assume H has the form $H(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$.

(a) Let $q \geq 1$ Consider H_{-1} first: Let x: $x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2) = -1$ or $(x_1^2 + \cdots + x_p^2) + 1 = x_{p+1}^2 + \cdots + x_{p+q}^2 = \lambda^2$. Then $\lambda^2 \geq 1$

Define the map $f: (x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}) \mapsto \left(\left(\frac{x_1}{1+\lambda}, \ldots, \frac{x_p}{1+\lambda}\right), \left(\frac{x_{p+1}}{\lambda}, \ldots, \frac{x_{p+q}}{\lambda}\right)\right)$

We claim f maps H_{-1} to the unit ball B^p times the unit sphere S^{q-1}:

$$\left(\frac{x_1}{1+\lambda}\right)^2 + \cdots + \left(\frac{x_p}{1+\lambda}\right)^2 = \frac{x_1^2 + \cdots + x_p^2}{(1+\lambda)^2} = \frac{\lambda^2 - 1}{(\lambda+1)^2} = \frac{\lambda - 1}{\lambda + 1}$$

Now $\lambda \geq 1$ and the transformation $\lambda \mapsto \frac{\lambda - 1}{\lambda + 1}$ maps $[1, \infty)$ into $[0,1)$ $\therefore$

$$0 \leq \left(\frac{x_1}{1+\lambda}\right)^2 + \cdots + \left(\frac{x_p}{1+\lambda}\right)^2 < 1, \text{ so } \left(\left(\frac{x_1}{1+\lambda}\right), \ldots, \left(\frac{x_p}{1+\lambda}\right)\right) \in B^p \approx \mathbb{R}^p$$

Next, note $\left(\frac{x_{p+1}}{\lambda}, \ldots, \frac{x_{p+q}}{\lambda}\right) \in S^{q-1}$ since $\left(\frac{x_{p+1}}{\lambda}\right)^2 + \cdots + \left(\frac{x_{p+q}}{\lambda}\right)^2 = \frac{\lambda^2}{\lambda^2} = 1$

(13)

Finally, note that f is invertible because given $(b,s) \in B^p \times S^{q-1}$ we recover $\frac{\lambda-1}{\lambda+1} = \|b\|$ and $\therefore \lambda \therefore$ the components $(x_1, \dots x_p)$ and $(x_{p+1}, \dots x_{p+q})$ from b, s respectively.

For H^0, if $x \in H^0$ then $\lambda x \in H^0$ $\therefore$ it suffices to characterize those $x \in H^0$ s.t. $\|x\|^2 = 1$. But:

$x \in H^0 \iff x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 = 0$

$\|x\|^2 = 1 \iff x_1^2 + \dots + x_p^2 + x_{p+1}^2 + \dots + x_{p+q}^2 = 1$

$\iff x_1^2 + \dots + x_p^2 = \frac{1}{2}$ $\therefore$ $H^0 \approx \mathbb{R} \times S^{p-1}$

Finally, for H^1, multiply the equation $H(x) = 1$ by -1, interchange p and q and use the result for H^{-1}

(b) is obvious /

Comment: The map f used in the proof is closed related to stereographic projection; for $q=1$, it defines an isometry between hyperboloid and disc models of hyperbolic space.

Next we explore a little of the geometry of these hypersurfaces. Let $\hat{H}(x,y)$ = bilinear form associated to H. We define a left invariant metric g on V by $g \mid T_0 V = \hat{H}$. Then $O(H)$ becomes a group of isometries for g because of:

3.2 Lemma: The symmetries of $\hat{H}$ coincide with $O(H)$.

Pf

If $g \in GL(V)$ $\hat{H}(gx, gy) = \hat{H}(x,y)$, then $g \in O(H)$ because $H(x) = \hat{H}(x,x)$. Conversely $\hat{H}(x,y) = \frac{1}{4}(H(x+y) - H(x-y))$ /

For any set S, let $S^\perp = \{v \in V : H(v,s) = 0 \ \forall s \in S\}$.

3.3 Prop: Let $c \neq 0$. Then $g \mid H^c$ is nondegenerate. In fact, for $x \in H^c$, $T_x H^c \simeq \{x\}^\perp$

Pf

It suffices to show the 2nd statement, since we can show that $\hat{H}$ nondegenerate on a subspace $W \Rightarrow$ it is nondegenerate

on $W^\perp$, and by hypothesis $\hat{H}(x,x) = c \neq 0$. Let $\gamma(t)$ lie on H^c with $\gamma(0) = x$. Then $g(\gamma(t),\gamma(t)) = c$ $\therefore$ $\frac{d}{dt}\Big|_{t=0} g(\gamma(t),\gamma(t)) = 2g(x,\dot\gamma(0)) = 0$ $\therefore$ $\dot\gamma(0) \in \{x\}^\perp$.
Since $\dim \{x\}^\perp = \dim T_x H^c$, these must coincide /

Fix $c \neq 0$. We now give a characterizations for the geodesics on H^c; these are simple generalizations from the case $H^c = S^n$.
Let $x \in H^c$ and define, for $y \in H^c$, $R_x(y) = y - 2\frac{\hat{H}(x,y)}{c}x$.
An easy calculation shows $R_x(y) \in H^c$ and another shows $\hat{H}(R_x(y_1), R_x(y_2)) = \hat{H}(y_1,y_2)$. Thus R_x is an isometry; its fixed point set $\{y : R_x(y) = y\} = \{x\}^\perp$. As in Milnor [7] (p65), we can conclude that if $y_1, y_2 \in \{x\}^\perp$ then the geodesics joining y_1 to y_2 also lie in $\{x\}^\perp$.

Here is another description. Let $B \in \mathfrak{o}(H)$. Then B is an infinitesimal isometry for H^c since the flow for B is given explicitly by $t \mapsto e^{tB} \in O(H)$. Let $x \in H^c$. We ask, when is $\varphi(t) = e^{tB}\cdot x$ a geodesic on H^c (wrt metric g)?

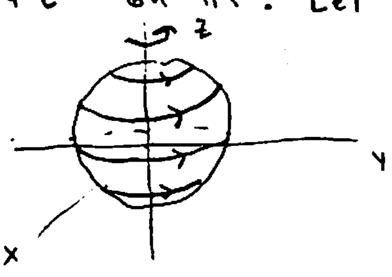
<u>3.4 Prop</u>: For $x \in H^c$, $\varphi(t) = e^{tB}\cdot x$ is geodesic iff x is a non-zero real eigenvector for B^2.

<u>Pf</u>

We use the fact that geodesics on hypersurfaces in euclidean space have acceleration normal to the surface. This is still true in our case. The acceleration $\ddot\varphi(t) = B^2 e^{tB}\cdot x$
$= \lambda(t)\varphi(t)$ since $g|H_c$ is nondegenerate and $T_x H^c = \{x\}^\perp$.
In particular $\ddot\varphi(0) = B^2 x = \lambda(0)x$ /

<u>Example</u>: Let $H = x^2+y^2+z^2$ on $\mathbb{R}^3$. Let $B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

$$e^{tB} = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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and $B^2 = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 0 \end{pmatrix}$ has nonzero eigenvectors in the xy-plane which is precisely where the geodesics of the form $e^{tB} \cdot x$ lie. /

b. The momentum map $\vec{J} : V \to sp(n)^*$

Fix V and Ω a symplectic form on V.

3.5 Thm: Let $J : sp(V) \to Q(V)$ by $A \mapsto H_A$. J defines an equivariant momentum map $\vec{J} : V \to sp(n)^*$ such that:

(1) If $X_1, \dots, X_N$ is a basis for $sp(V)$ $(N = 2n^2+n)$ and $\eta_1, \dots, \eta_N$ is the dual basis, then wrt this dual basis $\vec{J}$ has components $\vec{J}(v) = (H_{X_1}(v), \dots H_{X_n}(v))$

(2) If $\mu = a^i \eta_i$, then $\vec{J}^{-1}(\mu) = H_{X_1}^{a_1} \cap \dots \cap H_{X_N}^{a_N}$

Pf

By definition $\vec{J} : V \to sp(n)^*$ by $\langle \vec{J}(v), A \rangle = J(A)(v) = H_A(v)$ for $A \in sp(n)$. We verify equivariance: let $g \in Sp(V)$, $A \in sp(n)$, $v \in V$:

$$\langle \vec{J}(g \cdot v), A \rangle = H_A(g \cdot v) = \tfrac{1}{2} \Omega(Agv, gv) = \tfrac{1}{2} \Omega(g^{-1}Ag\, v, v)$$
$$= H_{Ad_{g^{-1}}(A)}(v) = \langle \vec{J}(v), Ad_{g^{-1}}(A) \rangle = \langle Ad_g^* \vec{J}(v), A \rangle$$

$\therefore \vec{J}$ is equivariant. statements (1) and (2) are now clear /

Notice that $\vec{J}(\lambda v) = \lambda^2 \vec{J}(v)$ since each component is quadratic. The principal application of $\vec{J}$ is to subgroups of $Sp(V)$ or subalgebras of $sp(n)$. Suppose $G \subseteq Sp(V)$ is a Lie subgp with algebra $\mathfrak{g}$. Then we may consider $J|\mathfrak{g} : \mathfrak{g} \to Q(V)$ and $\vec{J}_{\mathfrak{g}} : V \to sp(V)^* \xrightarrow{\text{restrict}} \mathfrak{g}^*$, where $\vec{J}_{\mathfrak{g}}$ = momentum map for $J|\mathfrak{g}$. This is proved on pg 301 of Marsden and Ratiu [5].

Choosing a basis as in Thm 3.5 (1) so that $X_1 \dots X_K$ form a basis for $\mathfrak{g}$ then $\vec{J}_{\mathfrak{g}}$ = (projection onto 1st k-components) $\circ \vec{J}$.

To apply these ideas, we present the central tool for the rest of this paper. We present two proofs, one based on mechanics and the other, more detailed one, based on Lie gps. Recall that $sp(V)$ is

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semisimple and $A \in sp(V) \longmapsto A^b \in sp(V)^*$ under the Cartan-Killing form isomorphism.

<u>3.6 Thrm</u>: Let $A \in sp(V)$, $H = H_A \in Q(V)$, $G(H) = O(H) \cap Sp(V)$, $\mathfrak{g}(H) =$ Lie algebra of $G(H)$, $e(A) =$ centralizer of A. Then:

(i) $\mathfrak{g}(H) = e(A)$

(ii) If $\mu = A^b$, then $e(A) = \mathfrak{g}_\mu =$ Lie alg of stabilizer G_μ of μ

<u>Pf</u>

(i): <u>Mechanics proof</u>: Let $B \in e(A)$. Then $[B, A] = 0$ $\therefore$ $\{H_B, H_A\} = 0$ by Prop 2.1.; $\Rightarrow$ H_A is constant on the flows φ_t of B. I.e $H_A(\varphi_t(x)) = H_A(x)$. But $\varphi_t = e^{tB}$ $\therefore$ $e^{tB} \in G(H_A)$ $\therefore$ $B \in \mathfrak{g}(H_A)$. The argument is reversable $\therefore$ $e(A) = \mathfrak{g}(H_A)$.

<u>Lie gp proof</u>: The fact we used in the mechanics proof is that $B \in \mathfrak{g}$ iff $e^{tB} \in G$. Consider $\hat{H}_A(x, y) =$ associated bilinear form for H_A. We've shown (lemma 3.2) the symmetry gp's for $\hat{H}_A$ and H_A coincide. Suppose $e^{tB} \in G(\hat{H}_A)$. Then $\hat{H}_A(e^{tB}x, e^{tB}y) = \hat{H}_A(x, y)$ reads $\frac{1}{2}\Omega(Ae^{tB}x, e^{tB}y) = \frac{1}{2}\Omega(Ax, y)$ or

$\Omega(e^{-tB}Ae^{tB}x, y) = \Omega(Ax, y)$ $\forall x, y$ $\Rightarrow$ $e^{-tB}Ae^{tB} = A$

Differentiating @ $t = 0$ yields $[B, A] = 0$ $\therefore$ $\mathfrak{g}(H_A) \subset e(A)$.

Conversely, if $B \in e(A)$ then $BA = AB$ $\Rightarrow$ $\exp tB \cdot A = A \cdot \exp tB$.

Then $H_A(\exp tB x) = \frac{1}{2}\Omega(A \exp tB x, \exp tB x) = \frac{1}{2}\Omega(\exp tB Ax, \exp tBx)$

$= \frac{1}{2}\Omega(Ax, x) = H_A(x)$ (since $\exp tB \in Sp(V)$) $\therefore$ $\exp tB \in G(H_A)$

$\therefore$ $B \in \mathfrak{g}(H_A)$ $\therefore$ $e(A) = \mathfrak{g}(H_A)$.

(ii) Let $\mathfrak{g}_\mu =$ Lie algebra of G_μ. Then, by Marsden and Ratiu [5] p 366

$\mathfrak{g}_\mu = \{B \in \mathfrak{g} : ad^*_B(\mu) = 0\}$ Let $C \in \mathfrak{g}$, $B \in \mathfrak{g}_\mu$. Then

$$0 = \langle ad^*_B(\mu), C \rangle = \langle \mu, [B, C] \rangle = (A, [B, C]) = ([B, A], C)$$

by invariance of CK-form $(,)$ $\forall C$ $\Rightarrow$ $[B, A] = 0$ $\therefore$ $B \in e(A)$

The argument is reversible /

<u>3.7 Lemma</u>: $Sp(V)$ acts irreducible on V.

<u>Pf</u>

In some sense this is obvious; if not, then the matrices would break up

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into block form $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$. Here is a precise proof:

Suppose $0 \neq W \subset V$ is invariant. Then it is also $sp(V)$ invariant, since if $A \in sp(V)$ $Aw = \frac{d}{dt} \exp(tA) \cdot w \in W$ if $w \in W$.
Let $v \in V$. We show $\exists A \in sp(V)$ s.t. $Aw = v$. This clearly proves irreducibility.

Consider the map $Q(V) \to V^*$ by $H_A \mapsto \iota_w(\hat{H}_A)$ ie. $\langle \iota_w(\hat{H}_A), x \rangle = \hat{H}_A(w, x)$. This is obviously linear and Kernel $= \{ H_A : \hat{H}_A(w,x) = 0 \;\; \forall x \in V \}$ ie. w is a null-vector for $\hat{H}_A$. If we pick a basis for V starting with w then $H_A \in$ Kernel iff $\hat{H}_A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & * & \\ 0 & & \end{bmatrix} \Rightarrow \dim \text{Kernel} = 2n$

$\therefore$ $\dim \text{Image} = \frac{1}{2}(2n)(2n+1) - 2n = 2n^2 - n$. But $\dim V^* = 2n$
$\therefore$ map surjects whenever $2n^2 - n \geq 2n$ or $2n^2 - 3n \geq 0$
$n(2n-3) \geq 0$ ie if $n \geq 3/2$ or $n > 1$. The case $n = 1$ is $SL(2, \mathbb{R})$ which is easy to see directly. So assume $n > 1$.
Then $\exists A$ s.t. $\hat{H}_A(x, w) = \frac{1}{2}\Omega(x, v)$ (since $-\iota_v(\frac{1}{2}\Omega) \in V^*$) $\forall x \in V$. $\therefore$, letting $\mathbb{J}$ = matrix for Ω we have

$$x^t \mathbb{J} A w = x^t \mathbb{J} v \quad \forall x \Rightarrow \mathbb{J} A w = \mathbb{J} v \quad \text{or} \quad Aw = v \; /$$

We are in the position to apply the Coadjoint Orbit Covering thrm (Marsden and Ratiu [5] p 378), to $\vec{J} : V \to sp(V)^*$
In particular, $\vec{J}(V)$ is a coadjoint orbit
<u>Question</u>: Identify this orbit, ie, find $A \in sp(V)$ s.t. $\vec{J}(V) = Sp(V) \cdot A^b$. If not A, what is $\sigma_i(A)$ $i = 0, \ldots, 2n$?
We provide some clues to this in section 4.

c. Reduction

It is with reduction that the seperate themes of parts (a) and (b) of this section come together. Roughly the idea is to use the momentum map twice: 1^{st} to reduce the hypersurface by some subgp of its symmetry group and 2^{nd} to understand the resulting object in terms of a coadjoint orbit.

(18)

Fix $H = H_A \in Q(V)$ and set $G = G(H)$, $\mathfrak{g} = \mathfrak{g}(H)$. Suppose $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{m}$ where $\mathfrak{a}$ is an ideal $\ni A$, and $\mathfrak{m}$ is a subalgebra. Suppose $\mathfrak{a} \leftrightarrow$ subgp. $K \subset G$. Then the goal is to construct a diagram

$$\begin{tikzcd} H^c \arrow[r] \arrow[d] & \mathfrak{g}^* \arrow[r] & \mathfrak{m}^* \\ H^c/K_\mu \arrow[urr, dashed] & & \end{tikzcd}$$

It would seem that there are 3 obvious choices for $\mathfrak{a}$:
(1) $\mathfrak{a} = \langle A \rangle$ (2) $\mathfrak{a} = Z(\mathfrak{g}) =$ center of $\mathfrak{g}$ or
(3) $\mathfrak{a} =$ radical of $\mathfrak{g}$ = maximal solvable ideal of $\mathfrak{g}$.
In case (3), the decomposition $\mathfrak{a} \dot{+} \mathfrak{m}$ is then a Levi decomposition and $\mathfrak{m}$ is semisimple. However we shall only consider case (1).
Let $G \subset Sp(V)$ with algebra $\mathfrak{g}$. To apply reduction we need to (a) Find regular values μ for $\vec{J}$ in $\mathfrak{g}^*$ and
(b) Determine that the action of G_μ is free and proper.

3.8 Prop: Let $\mu \in \mathfrak{g}^*$. Then μ is a regular value for $\vec{J} : V \to \mathfrak{g}^*$ iff for all $x \in J^{-1}(\mu)$ $\dim(\mathfrak{g}\cdot x) = \dim(\mathfrak{g})$.

Pf

Choose a basis $A^1, \dots, A^k$ for $\mathfrak{g}$. Let $\hat{H}^i$ be the quadratic form associated to H_{A^i}. We have $\vec{J}(v) = (H_{A^1}(v), \dots, H_{A^k}(v))$
Consider a single $H = H_A$ with form $\hat{H}$. Let $e_1 \dots e_{2n}$ be a basis on V and $x = x^i e_i$. Then $\frac{\partial}{\partial x^i} H(x) = \frac{\partial}{\partial x^i} \hat{H}(x,x) = 2\hat{H}_{ij} x^j$
where $\hat{H} = (\hat{H}_{ij})$ $\therefore$

$$\frac{1}{2}(T_x \vec{J})(v) = \begin{bmatrix} \hat{H}^1(x,v) \\ \vdots \\ \hat{H}^k(x,v) \end{bmatrix}$$

. But $\hat{H}(x,v) = v^t \mathbb{J} A x$ $\therefore$ $T_x \vec{J}$ is surjective $\iff$ $\dim \operatorname{span}\{\mathbb{J}A^1 x, \dots, \mathbb{J}A^k x\} = \dim(\mathfrak{g}\cdot x) = \dim \mathfrak{g}$ /

3.9 Thm: Let $H = H_A$ be nondegenerate and $c \in \mathbb{R}$ a regular value of H. Assume $\mathfrak{g}(H) = \langle H \rangle \dot{+} \mathfrak{m}$ (semi-direct) where $\mathfrak{m}$ is a subalgebra. Let G_1 = 1-dim gp with alg. $\langle H \rangle$ and G_2 = Lie gp corresponding to $\mathfrak{m}$. Assume G_1 acts properly on H^c. (G_1 acts freely because H is nondegenerate)

Then we have the following diagram: α is canonical. If G_2 acts transitive on H^c then the image of α is a coadjoint orbit in m^*.

$$\begin{tikzcd}
V \arrow[r, "\vec{J}"] & \mathfrak{g}^* \arrow[r, "\text{restriction}"] & (m^*, \{\ \}_+) \\
H^c \arrow[u, phantom, "\cup" description] \arrow[d, "\text{reduction}"'] & & \\
H^c/G_1 \arrow[uurr, "\alpha"'] & &
\end{tikzcd}$$

pf

The only point left to prove is that α is canonical. Observe that the symplectic structure on H^c/G_1 is induced from V because $H^c = \vec{J}_1^{-1}(c)$ where $\vec{J}_1 : V \to \langle A \rangle^*$ is a momentum map. But; the mapping $\vec{J} : V \to (m^*, \{\ \}_+)$ is Poisson because it is equivariant $\therefore$ Thrm 12.5.1 of Marsden and Ratiu [5] p 322 applies. /

<u>Comment</u>: Thrm 3.9 is a direct generalization of Cushman and Rod [2] section 2. We use it to recapture the result on the Hopf fibration as well as some other fibrations.

4. Examples. The Hopf fibration.

a. $O(2n)$

Let $H = \frac{1}{2}(x_1^2 + \cdots + x_{2n}^2)$, or $\hat{H} = I$. Then $O(H) = O(2n)$. We calculate $G(H)$:

4.1 **Lemma**: $G(H) = O(2n) \cap Sp(n) \simeq U(n)$

Pf

Let $\mathbb{J}$ = matrix of the standard symplectic form on $\mathbb{R}^{2n}$. $g \in O(2n)$
$\Leftrightarrow$ $g^t g = I$ and $g \in Sp(n) \Leftrightarrow g^t \mathbb{J} g = \mathbb{J}$ $\therefore$
$g^{-1}\mathbb{J} g = \mathbb{J}$ or $\mathbb{J} g = g \mathbb{J}$ $\therefore$ g commutes with $\mathbb{J}$.
Use $\mathbb{J}$ to define a complex structure on $V \simeq \mathbb{R}^{2n} \simeq \mathbb{C}^n$.
Then we may view g as a complex linear transformation
i.e. we choose $v_1, \ldots, v_n$ linear ind. in V s.t. $\{v_1, \ldots, v_n, \mathbb{J}(v_1), \ldots \mathbb{J}(v_n)\}$ form a real basis for V; then $\{v_1 \ldots v_n\}$ form a complex basis where $i \cdot v_j = \mathbb{J}(v_j)$ and $i \cdot g(v_j) = \mathbb{J} g(v_j) = g(\mathbb{J} v_j) = g(i v_j)$.
Thus the matrix entries for g wrt this basis are real
and the condition $g^t g = I$ becomes $g^* g = I$ i.e. $g \in U(n)$.
$\therefore$ $G(H) \subset U(n)$.

For the converse inclusion, a Lie alg calculation shows that $\dim(\mathcal{O}(2n) \cap \mathfrak{sp}(n)) = \dim \mathfrak{u}(n) = n^2$ $\therefore$ $U(n)$ being connected $\Rightarrow$ the result. We illustrate this calculation for $n = 2$:

$\mathfrak{u}(2) \ni A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{C})$; $A^* = -A$:

$$A^* = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} = -\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Rightarrow \begin{matrix} \bar{\alpha} = -\alpha & \Rightarrow \alpha = ia & a \in \mathbb{R} \\ \bar{\delta} = -\delta & \Rightarrow \delta = id & d \in \mathbb{R} \\ \bar{\gamma} = -\beta & & \end{matrix}$$

$$\therefore \quad A = \begin{pmatrix} ia & \beta \\ -\bar{\beta} & id \end{pmatrix} \quad \therefore \quad \dim \mathfrak{u}(2) = 4$$

$$\mathcal{O}(4) \cap \mathfrak{sp}(2) \ni A \Rightarrow A = \left(\begin{array}{cc|cc} a & b & x & y \\ c & d & y & z \\ \hline s & t & -a & -c \\ t & u & -b & -d \end{array}\right) \text{ and } A^t = -A$$

(21)

$$-A^t = \left(\begin{array}{cc|cc} -a & -c & -s & -t \\ -b & -d & -t & -u \\ \hline -x & -y & a & b \\ -y & -z & c & d \end{array}\right) = \left(\begin{array}{cc|cc} a & b & x & y \\ c & d & y & z \\ \hline s & t & -a & -c \\ t & u & -b & -d \end{array}\right) \Rightarrow$$

$a = -a$, $a = 0$, $s = -x$ or $A = \left(\begin{array}{cc|cc} 0 & b & x & y \\ -b & 0 & y & z \\ \hline -x & -y & 0 & b \\ -y & -z & -b & 0 \end{array}\right)$

$d = -d$ $d = 0$ $t = -y$

$c = -b$ $u = -z$

$\therefore$ $\dim \mathcal{O}(4) \cap sp(2) = 4$.

(in general $A \in \mathcal{O}(2n) \cap sp(n) \Rightarrow A = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$ where X is $n \times n$ skew symmetric, $Y = n \times n$ symmetric)

Next, observe that $\begin{pmatrix} -i & & \\ & \ddots & \\ & & -i \end{pmatrix} \in$ center $\mathfrak{u}(n)$, or in the notation of the above proof $-\mathbb{J} \in$ Center $\mathfrak{u}(n)$. But, $-\mathbb{J} =$ matrix A s.t. $H_A = H \doteq \frac{1}{2}(x_1^2 + \cdots + x_{2n}^2)$, since $-\mathbb{J} = \mathbb{J}^{-1}$ and $I = \hat{H} = \mathbb{J}A$. Thus we can write $\mathfrak{u}(n) = \langle A \rangle \dot{+} \mathfrak{m}$ and it is known that $\mathfrak{m} = \mathfrak{su}(n) = \{A \in \mathfrak{u}(n) : \operatorname{trace}(A) = 0\}$. Furthermore, it is known that $SU(n)$ acts transitively on $H^c \approx S^{2n-1}$, in fact the isotropy subgp is $SU(n-1)$. Finally, by direct calculation e^{At} are closed and bounded orbits $\therefore$ $G_1 = \{e^{At} : t \in \mathbb{R}\}$ acts properly on H^c $c > 0$. $\therefore$ the conditions of Thm 3.9 are met and we conclude $\exists$ circle bundle S^{2n-1} s.t. M is a symplectic manifold

$$\begin{array}{c} S^{2n-1} \\ S^1 \downarrow p \\ M \end{array}$$

<u>and</u> $\exists$ canonical map $\alpha : M \to \mathfrak{su}(n)^*$ s.t. $\alpha(M)$ is a coadjoint orbit. In this case, we can conclude even more: M is compact (since $M = p(S^{2n-1})$) $\therefore$ α is proper and M is a covering space over the orbit $\alpha(M)$ by the Coadjoint Orbit Covering Thrm. If $n \geq 2$ we get some information on π_1 and π_2 of M: from the homotopy seq. of the fibration:

(22)

$0 \to \pi_2(M) \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(M) \to 0$. For $n=2$ we know that $M \cong S^2$ and it would be natural to guess for $n=3$ say, $M = S^6$. But M is symplectic and S^6 is not! (for $H^2(S^6) = 0$). In fact,

Claim: $M \simeq \mathbb{C}P(n-1)$

Pf

$A^2 = (-J)^2 = -I$ $\therefore$ the orbits $e^{At}x$ are geodesic on S^{2n-1} $\forall x \in S^{2n-1}$ by Prop 3.4. Furthermore, the metric is the usual one because $\hat{A} = I$. $\therefore$ the orbits are great circles.

$\mathbb{C}P(n-1) = \{[z_1, \cdots z_n] : z_i \in \mathbb{C}$, not all zero and $(z_1' \cdots z_n') \in [z_1 \cdots z_n]$ iff $\exists \lambda \in \mathbb{C}^*$ s.t. $z_i' = \lambda z_i\} = S^{2n-1}/\sim$

where $x, y \in S^{2n-1}$ $x \sim y$ iff $\exists \lambda \in \mathbb{C}$ $|\lambda| = 1$: $x = \lambda y$.

Writing $\lambda = e^{it}$ and recalling the identification of A with $\begin{pmatrix} -i & & \\ & \ddots & \\ & & -i \end{pmatrix}$ gives the result /

b. $n=2$ - the Hopf map.

In $sp(2)$, consider the four matrices:

$$E_1 = \frac{1}{2}\left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \qquad E_2 = \frac{1}{2}\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$E_3 = \frac{1}{2}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \qquad E_4 = \frac{1}{2}\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

These will be shown to form an orthonormal basis w.r.t. the trace form $\hat{B}(A, B) = tr(AB)$ on the subalgebra $u(2) \subset sp(2)$. (However $Z(u(2)) = \langle E_1 \rangle$ $\therefore$ $u(2)$ is not semisimple. But $u(2)/\langle E_1 \rangle = su(2) = \mathrm{span}\langle E_2, E_3, E_4 \rangle$ is)

In fact let $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(13)

Then $a^2 = c^2 = d^2 = a = -b^2$; a acts like a unit.

$bc = d \qquad bd = -c \qquad cd = -b$

$cb = -d \qquad db = c \qquad dc = b$

(i.e. $a = 1$, $b = \vec{i}$, $c = \vec{j}$, $d = \vec{k}$)

$$E_1 = \frac{1}{2}\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \text{"="} \frac{1}{2} a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad E_3 = \frac{1}{2}\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} = \frac{1}{2} d \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$E_2 = \frac{1}{2}\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \frac{1}{2} b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E_4 = \frac{1}{2}\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} = \frac{1}{2} c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Now $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\therefore$

$$E_1^2 = \frac{1}{4} a^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix} \qquad E_3^2 = \frac{1}{4} d^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix}$$

$$E_2^2 = \frac{1}{4} b^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix} \qquad E_4^2 = \frac{1}{4} c^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix}$$

In particular $tr(E_1^2) = tr(E_2^2) = tr(E_3^2) = tr(E_4^2) = -1$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so we can write:

$$E_1 = -\frac{1}{2} a B \; ; \; E_2 = \frac{1}{2} b A \; ; \; E_3 = \frac{1}{2} d B \; ; \; E_4 = -\frac{1}{2} c B$$

Then: $E_1E_2 = -\frac{1}{4} ab\, B = -\frac{1}{4} bB$, $E_1E_3 = -\frac{1}{4} d\, B^2 = \frac{1}{4} d A$, $E_1E_4 = \frac{1}{4} c A$

$E_2E_3 = \frac{1}{4} bd\, B = -\frac{1}{4} cB$, $E_2E_4 = -\frac{1}{4} bc\, AB = -\frac{1}{4} dB$

$E_3E_4 = -\frac{1}{4} dc\, B^2 = \frac{1}{4} b A$

Now anything of the form xB has trace 0 and anything of the form xA has trace 0 if $tr(x) = 0$. Since $tr(b) = tr(c) = 0$ this verifies that $E_i \perp E_j$ for $i \neq j$.

It is also clear from this notation since $[A, B] = 0$ and $ax = xa$ that $[E_1, E_i] = 0$ $i = 1..4$. We verify directly that $\{E_2, E_3, E_4\}$ form a subalgebra:

$E_3E_2 = (\frac{1}{2} dB)(\frac{1}{2} bA) = \frac{1}{4} cB \quad \therefore [E_2, E_3] = -\frac{1}{4} cB - \frac{1}{4} cB = -\frac{1}{2} cB = E_4$

$E_4E_2 = -\frac{1}{4}(cB)(bA) = \frac{1}{4} dB \quad \therefore [E_2, E_4] = -\frac{1}{2} dB = -E_3$

$E_4E_3 = -\frac{1}{4}(cB)(dB) = -\frac{1}{4}(-b)B^2 = -\frac{1}{4} bA \quad \therefore [E_3\, E_4] = \frac{1}{2} bA = E_2$

(24)

In particular, $\hat{B}$ is negative definite on $su(2) \cong \mathbb{R}^3$
$\therefore$ coadjoint orbits are contained in spheres.

Next, we compute the momentum map:
$\hat{H}_{E_1} = \frac{1}{2}\mathbb{J}\cdot E_1$; in our notation $\mathbb{J} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = -2E_1$ $\therefore$

$\hat{H}_{E_1} = -E_1^2 = \frac{1}{4}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \frac{1}{4}\left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right]$, so

$H_{E_1} = \hat{H}_{E_1}(x,x) = \frac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$

$\hat{H}_{E_2} = -E_1E_2 = \frac{1}{4}bB = \frac{1}{4}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \frac{1}{4}\left[\begin{array}{cc|cc} & & 0 & 1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \\ \hline 0 & -1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \end{array}\right]$ $\therefore$

$H_{E_2} = \frac{1}{2}(x_1x_4 - x_2x_3)$ $\quad$ Illy

$H_{E_3} = \frac{1}{2}(x_1x_2 + x_3x_4)$ and $H_{E_4} = \frac{1}{4}(x_2^2 + x_4^2 - (x_1^2 + x_3^2))$

In Cushman and Rod [2] $W_1 = 4H_{E_3}$, $W_2 = 4H_{E_2}$, $W_3 = 4H_{E_4}$, $W_4 = 4H_{E_1}$

are the Hopf variables. Using a basis for $su(2)^*$ dual to $\{E_1, \dots E_4\}$ we express the momentum map $\vec{J}$ in co-ordinates:
$\vec{J}(x_1, x_2, x_3, x_4) = \left(\frac{1}{4}(x_1^2 + x_2^2 + x_3^2 \pm x_4^2), \frac{1}{2}(x_1x_4 - x_2x_3), \frac{1}{2}(x_1x_2 + x_3x_4),\right.$

$\left.\frac{1}{4}((x_2^2 + x_4^2) - (x_1^2 + x_3^2))\right)$

In particular $H^c = \frac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2) = c$ and
$\vec{J} \mid H^c \longrightarrow su(2)^*$; it is clear that the coadjoint orbits must be spheres in $su(2)^*$ (they either have dim 0 or 2 - they are contained in spheres and the $su(2)$ adjoint action $= SO(3)$ action on $\mathbb{R}^3$ is transitive on spheres)
From the algebraic identity $W_1^2 + W_2^2 + W_3^2 = W_4^2$
we see that H^c is mapped to a sphere of radius $4\sqrt{c}$.

Thus H^c / G_1 is identified with this sphere. /

Comment: Let $\eta_i \in u(2)^*$ be dual to E_i.

We have $E_i^b(E_j) = (E_i, E_j) = \begin{cases} 0 & i \neq j \\ -1 & i = j \end{cases}$ $\therefore$ $\eta_i = -E_i^b$

In particular the inner product on $u(2)^*$ induced from the trace form is negative definite, also. If $y = y^i \eta_i \in u(2)^*$ then $(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = \text{const}$ on a coadjoint orbit.
In particular for $y = \vec{J}(x)$ we get

$$W_1^2 + W_2^2 + W_3^2 + W_4^2 = \text{const}$$

Conclusion: the identity $W_1^2 + W_2^2 + W_3^2 - W_4^2 = 0$ doesn't come from the group theory, at least in this way.

c. The generic case.

By the thm of Duflo and Vergne (Marsden and Ratiu [5] p 259) we know that the above example is not generic.
In the generic case, $\mathfrak{g}_\mu$ is of minimal dimension, among stabilizing subalgebras and abelian. In this section we provide such an example.

Let $H = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + 2x_4^2)$. Then $H = H_A$ where

$$A = \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

We compute $e(A)$: let $B = \left[\begin{array}{cc|cc} a & b & x & y \\ c & d & y & z \\ \hline s & t & -a & -c \\ t & u & -b & -d \end{array}\right]$. Then

$$AB - BA = \left[\begin{array}{cc|cc} -s & -t & a & c \\ -2t & -2u & 2b & 2d \\ \hline a & b & x & y \\ c & d & y & z \end{array}\right] - \left[\begin{array}{cc|cc} x & y & -a & -2b \\ y & z & -c & -2d \\ \hline -a & -c & -s & -2t \\ -b & -d & -t & -2u \end{array}\right]$$

(26)

which yield equations:

$$\begin{array}{lll} 2a = 0 & x+s=0 & z+2u=0 \\ 2d=0 & y+t=0 & \\ b+c=0 & y+2t=0 & \\ c+2b=0 & & \end{array} \Bigg\} \Rightarrow \begin{array}{l} a=b=c=d=y=t=0 \\ s=-x \\ u=-\frac{1}{2}z \end{array}$$

$$\therefore \quad B = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & x & 0 \\ & & 0 & z \\ \hline -x & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -\frac{1}{2}z & & \end{array}\right] \qquad \therefore \dim e(A) = 2.$$

this is minimal because if $p[T]$ is any polynomial then $p[A] \in e(A)$ $\therefore$ $\dim e(A) \geq$ degree of the minimal polynomial for $A \geq 2$ because $cI \notin sp(2)$ and $\therefore \neq A$

This case, although generic, is uninteresting from the point of view of this paper because coadjoint orbits in $e(A)^*$ are just points.

d. $sp(1)$ and $sp(2)$ again.

We make a few miscellaneous comments. For $sp(1)$, let

$$Y_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Y_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then Y_i are orthonormal wrt the CK-form and if $Y = y^i Y_i$, then $\|Y\|^2 = y_1^2 + y_2^2 - y_3^2$.

Let $\eta_i = Y_i^b$; then η_i form a basis for $sp(1)^*$

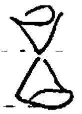
If Y_i^* is dual to Y_i then $\eta_1 = Y_1^*$, $\eta_2 = Y_2^*$, $\eta_3 = -Y_3^*$

We calculate $\vec{J} : \mathbb{R}^2 \to sp(2)^*$

$$\vec{J}(x_1, x_2) = (H_{Y_1}(x_1,x_2),\ H_{Y_2}(x_1,x_2),\ H_{Y_3}(x_1,x_2))$$
$$= \frac{1}{\sqrt{2}}(-2x_1x_2,\ x_1^2 - x_2^2,\ x_1^2 + x_2^2)$$

To determine which orbit $\vec{J}(\mathbb{R}^2)$ is, we compute

$$\|\vec{J}(x_1,x_2)\|^2 = \frac{1}{2}\left[(-2x_1x_2)^2 + (x_1^2-x_2^2)^2 - (x_1^2+x_2^2)^2\right] = 0$$

$\therefore$ $\mathcal{O} = \vec{J}(\mathbb{R}^2) \subset$ degenerate orbit $\subset$ 

To determine which sheet; pick a value say $x_1 = 0$ $x_2 = 1$

$\vec{J}(0,1) = \frac{1}{\sqrt{2}}(0,-1,1) \Rightarrow \frac{\sqrt{2}}{2}(Y_3 - Y_2)^b \in \mathcal{O}$, $\frac{\sqrt{2}}{2}(Y_3 - Y_2) =$

$$\frac{1}{2}\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix} - \begin{pmatrix}0&1\\1&0\end{pmatrix}\right] = \begin{pmatrix}0&0\\-1&0\end{pmatrix} \Rightarrow z = \frac{1}{2}(0-(-1)) = \frac{1}{2} > 0$$

$\therefore$ the upper sheet i.e. $\vec{J}(\mathbb{R}^2) = (0,0)^+$.

Consider again the question what is $\mathcal{O} = \vec{J}(\mathbb{R}^{2n}) \subset sp^*(n)$.
Since $\vec{J}$ is quadratic, we have $\vec{J}(\lambda v) = \lambda^2 \vec{J}(v) \in \mathcal{O}$ $\therefore$
$\eta \in \mathcal{O} \Rightarrow \lambda\eta \in \mathcal{O}$ for $\lambda > 0$. Also $\sigma_i(\lambda\eta) = \lambda^i \sigma_i(\eta)$
$\therefore$ we conclude $\sigma_2(\mathcal{O}) = \sigma_4(\mathcal{O}) = \cdots = \sigma_{2n}(\mathcal{O}) = 0$
$\Rightarrow$ if $A^b \in \mathcal{O}$ all eigenvalues of A are 0.
We guess that the orbit $\mathcal{O}$ represents an indecomposable form and that $\mathcal{O}$ is the $+$-sheet.

Consider one last example from $sp(2)$. Let $H = \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2}(x_3^2 + x_4^2)$
$= (\pm c)^+ + (\pm c)^-$

Then $H = H_A$ where $A = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$.

$$e(A) = \left\{ B = \begin{pmatrix}a & b\\ b & a\end{pmatrix} : \begin{array}{l} a = -a^t \\ b = b^t \end{array} \quad a, b \in gl(2,\mathbb{R}) \right\}$$

If $B = \left(\begin{array}{cc|cc} 0 & a & x & y \\ -a & 0 & y & z \\ \hline x & y & 0 & a \\ y & z & -a & 0 \end{array}\right)$ we calculate $tr(B^2) = -4a^2 + 2x^2 + 2y^2 + 2z^2$

and we are in a situation similar to the Hopf fibration.
Rather than go through the computations, we cite Kocak et al [3], to say that H^c is a quadratic hypersurface

$\frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2}(x_3^2 + x_4^2) = c$ and H^c is a circle bundle over a hyperboloid sheet of the form $x^2 + y^2 - z^2 = \text{const}$.

(28)

5. Conclusions and regrets

Let M be a compact Riemann surface of genus > 1. It is known that M admits the unit disc or upper half plane as a universal covering space via a representation of $\pi_1(M)$ in $SL(2,\mathbb{R}) = Sp(1)$. Let $H = \frac{1}{2}(x_1^2+y_2^2) - \frac{1}{2}(x_3^2+x_4^2)$ as in the last example. If this representation extends to $G(H)$ as a properly discontinuous action then we can quotient out and obtain $H^c/\pi_1(M) \to M$. In other words, we could show that, as a partial answer to Smale's question, any symplectic 2-mfd occurrs as a reduction of a circle bundle.

Unfortunately, there is neither time nor space to pursue this question. There are many other topics and questions that should have been mentioned or would have been interesting to explore. Here is a partial list:

(1) Connection to rigid body motion

(2) Explore the geometry of the hypersurfaces H^c and their reductions in terms of homogeneous spaces

(3) Study degenerate hyper-surfaces. Here is one interesting result:

Let $D = \dim(O(m,0,r) \cap Sp(n))$ $(m+r=2n)$. Then:

(a) $r<n$ $\quad D = \frac{1}{2}n(n-1) + \frac{1}{2}(m-n)(m-n+1) + \frac{1}{2}r(r+1)$

(b) $r=n$ $\quad D = n^2$

(c) $r>n$ $\quad D = \frac{1}{2}n(n+1) + \frac{1}{2}m(m-1) + n(r-n)$

The minimum value occurs for $r = \frac{n}{2}$ n even, $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ n odd with $D = \frac{3}{4}n^2$ or $\frac{1}{4}(3n^2+1)$, resp.

In particular $D < n^2 \Rightarrow$ orbits corresponding to these cases in $sp(n)^*$ have higher codimension than the nondegenerate $O(2n)$ or $O(p,q,0)$ cases we have looked at.

(4) Study fibrations corresponding to $\mathfrak{g} = \mathfrak{a} \dotplus \mathfrak{m}$ where $\dim \mathfrak{a} > 1$. The total space is then an intersection of quadratic hyper-surfaces.

The only real conclusions I can come to is that there is a surprisingly rich interaction between Lie group theory, geometry and symplectic reductions. The Hopf fibration is a beautiful illustration of this fact.

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6. Appendix: Visualizing the Hopf Fibration

The digram on the opposing page explains how to view S^3 as a union of two solid tori with their boundaries identified, starting with S^3 as a union of two solid balls with their boundaries identified (i.e. upper / lower hemispheres of S^3). View each solid tori minus its central core circle as foliated in the obvious way by standard tori. These each, in turn, are foliated; on one solid torus use longitudinal circles on the other use latitudinal. When correctly glued together this yields a fibration of S^3 that is topologically equivalent to the Hopf fibration.

One can use this diagram to check various topological features, e.g. that the core circles are linked.

Each step in the diagram represents either a cut or paste, with the understanding that the red and blue surfaces are eventually identified and surfaces of the same color are identified.

This chart was inspired by Kocak et al [4]. Many thanks to my wife, Karen, who had the patience to listen, the intelligence to understand, and the skill to render this illustration. Here is a detailed description of each step:

1. $S^3 = B^3 \cup B^3$ with boundaries identified
2. Deform each B^3 into a solid cylinder. Remove two smaller cylinders from the center and identify tops and bottoms. Yields 1st solid torus.
3. Slice remaining objects to identify outer surfaces
4. Unrow to yield two cubes
5. a. Glue the fronts of the cubes together
   b. Collapse the tops and bottoms
6. Re-glue the slices by identifying the ends of the cylinder. This yields 2nd solid torus.

Visualizing the Hopf Fibration