

Jacobi Identity for the Poisson Bracket

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and since $\mathcal{L}_{X_f}g = i_{X_f}dg$ (by Cartan's formula) and $dg = i_{X_f}\Omega$, it follows that the Poisson bracket can be written as

$$\begin{aligned}\{f, g\} &= -\mathcal{L}_{X_f}g \\ &= \mathcal{L}_{X_g}f.\end{aligned}$$

The Poisson bracket is clearly bilinear and skew-symmetric. Hence, with the Jacobi Identity, it forms a Lie algebra.

All but one of the following proofs of Jacobi identity uses the same general method. The major step is to show that

$$X_{\{f, g\}} = -[X_f, X_g]. \quad (1)$$

The Jacobi identity then easily follows. In the proofs of Abraham and Marsden, Libermann and Marle, and Souriau, this equation is proven directly. In the proof of Arnold, a seemingly intuitive statement is made which is equivalent to Equation 1. However, Arnold makes this statement without proof. The remaining proof, that of Marsden and Ratiu, takes a completely different approach, and then Equation 1 can then be easily shown as a corollary.

2 Marsden and Ratiu

The proof of Marsden and Ratiu makes use of the identity $\varphi_t^* \{f, g\} = \{\varphi_t^* f, \varphi_t^* g\}$ where φ_t^* is the flow of X_h and $f, g, h \in \mathcal{F}(P)$, but first, the following preliminary result will be shown:

$$\left. \frac{d}{dt} \right|_{t=0} X_{\varphi_t^* f} = X_{\mathcal{L}_{X_h} f}. \quad (2)$$

This result follows from

$$\begin{aligned}
 \Omega \left(\frac{d}{dt} \Big|_{t=0} X_{\varphi_t^* f} \right) &= \frac{d}{dt} \Big|_{t=0} \Omega \left(X_{\varphi_t^* f} \right) \\
 &= \frac{d}{dt} \Big|_{t=0} \mathbf{d}(\varphi_t^* f) \\
 &= \mathbf{d} \left(\frac{d}{dt} \Big|_{t=0} \varphi_t^* f \right) \\
 &= \mathbf{d} \left(\mathcal{L}_{X_h} f \right) \\
 &= \Omega \left(X_{\mathcal{L}_{X_h} f} \right)
 \end{aligned}$$

Jacobi Proof 1

Differentiate $\varphi_t^* \{f, g\} = \{\varphi_t^* f, \varphi_t^* g\}$ with respect to t and evaluate at $t = 0$. Computation on the LHS follows from the definition of the Lie derivative:

$$\begin{aligned}
 \frac{d}{dt} \Big|_{t=0} \varphi_t^* \{f, g\} &= \mathcal{L}_{X_h} \{f, g\} \\
 &= \{\{f, g\}, h\}.
 \end{aligned}$$

Computation on the RHS uses Equation 2:

$$\begin{aligned}
 \frac{d}{dt} \Big|_{t=0} \{\varphi_t^* f, \varphi_t^* g\} &= \frac{d}{dt} \Big|_{t=0} \Omega \left(X_{\varphi_t^* f}, X_{\varphi_t^* g} \right) \\
 &= \Omega \left(X_{\mathcal{L}_{X_h} f}, X_g \right) + \Omega \left(X_f, X_{\mathcal{L}_{X_h} g} \right) \\
 &= \{\{f, h\}, g\} + \{f, \{g, h\}\}
 \end{aligned}$$

Combining the LHS and RHS yields

$$\{\{f, g\}, h\} = \{\{f, h\}, g\} + \{f, \{g, h\}\},$$

which is equivalent to the Jacobi identity by skew-symmetry.

3 Abraham and Marsden

The proof found in Abraham and Marsden proceeds by first proving Equation 1. This is achieved by defining a Poisson bracket for one-forms on a symplectic manifold P .

Some notation used throughout this section is as follows:

- α^\sharp indicates the vector field associated with the one-form α . Hence $i_{\alpha^\sharp}\Omega = \alpha$.
- X^\flat indicates the one-form associated with the vector field X . Hence $X^\flat = i_X\Omega$.

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Definition 1 Given a symplectic manifold (P, Ω) and one-forms $\alpha, \beta \in \mathcal{X}^*(P)$, the Poisson bracket of α and β is given by $\{\alpha, \beta\} = -[\alpha^\sharp, \beta^\sharp]^\flat$.

Lemma 1 Given a symplectic manifold (P, Ω) and one-forms $\alpha, \beta \in \mathcal{X}^*(P)$, $\{\alpha, \beta\} = -\mathcal{L}_{\alpha^\sharp}\beta + \mathcal{L}_{\beta^\sharp}\alpha + d(i_{\alpha^\sharp}i_{\beta^\sharp}\Omega)$.

PROOF.

The proof uses the following formula for a 2-form Ω :

$$d\Omega(X, Y, Z) = \mathcal{L}_X(\Omega(Y, Z)) - \mathcal{L}_Y(\Omega(X, Z)) + \mathcal{L}_Z(\Omega(X, Y)) - \Omega([X, Y], Z) - \Omega([Y, Z], X) + \Omega([X, Z], Y).$$

Applying this formula to $X = \alpha^\sharp$ and $Y = \beta^\sharp$, and using the fact the Ω is closed, gives

$$0 = \mathcal{L}_{\alpha^\sharp}(\Omega(\beta^\sharp, Z)) - \mathcal{L}_{\beta^\sharp}(\Omega(\alpha^\sharp, Z)) + \mathcal{L}_Z(\Omega(\alpha^\sharp, \beta^\sharp)) - \Omega([\alpha^\sharp, \beta^\sharp], Z) + \Omega(\alpha^\sharp, [\beta^\sharp, Z]) - \Omega(\beta^\sharp, [\alpha^\sharp, Z]).$$

Now using the fact that $i_{\alpha^\sharp}\Omega = \alpha$ and $\Omega([\alpha^\sharp, \beta^\sharp], Z) = \Omega(\{\alpha, \beta\}^\sharp, Z) = \{\alpha, \beta\}(Z)$, then

$$0 = \mathcal{L}_{\alpha^\sharp}(\beta(Z)) - \mathcal{L}_{\beta^\sharp}(\alpha(Z)) - \mathcal{L}_Z(i_{\alpha^\sharp}i_{\beta^\sharp}\Omega) + \{\alpha, \beta\}(Z) + \alpha(\mathcal{L}_{\beta^\sharp}Z) - \beta(\mathcal{L}_{\alpha^\sharp}Z),$$

or simplified, this reads

$$0 = \mathcal{L}_{\alpha^\sharp}(\beta(Z)) - \mathcal{L}_{\beta^\sharp}(\alpha(Z)) + \{\alpha, \beta\}(Z) - d(i_{\alpha^\sharp}i_{\beta^\sharp}\Omega)(Z).$$

Lemma 2 Given a symplectic manifold (P, Ω) and $f, g \in \mathcal{F}(P)$, then $d\{f, g\} = \{df, dg\}$.

PROOF.

Apply Lemma 1 to $\{df, dg\} \cdot (df)^\sharp = X_f$, so

$$\begin{aligned}
 \{df, dg\} &= -\mathcal{L}_{X_f} dg + \mathcal{L}_{X_g} df + d(i_{X_f} i_{X_g} \Omega) \\
 &= -d(\mathcal{L}_{X_f} g) + d(\mathcal{L}_{X_g} f) + d(i_{X_f} i_{X_g} \Omega) \\
 &= d\{f, g\} + d\{f, g\} - d\{f, g\} \\
 &= d\{f, g\}.
 \end{aligned}$$

Lemma 3 Given a symplectic manifold (P, Ω) and $f, g \in \mathcal{F}(P)$, then $X_{\{f, g\}} = -[X_f, X_g]$.

PROOF.

$$\begin{aligned}
 X_{\{f, g\}} &= (d\{f, g\})^\sharp = \{df, dg\}^\sharp \\
 &= -\left([(df)^\sharp, (dg)^\sharp]^\flat \right)^\sharp = -[X_f, X_g]
 \end{aligned}$$

Jacobi Proof 2

Jacobi's Identity is now a simple computation.

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} = \mathcal{L}_{X_h} \{f, g\} + \mathcal{L}_{X_g} \{h, f\} \quad (3)$$

$$= \mathcal{L}_{X_h} \mathcal{L}_{X_g} f - \mathcal{L}_{X_g} \mathcal{L}_{X_h} f \quad (4)$$

$$= \mathcal{L}_{-[X_g, X_h]} f \quad (5)$$

$$= \mathcal{L}_{X_{\{g, h\}}} f \quad (6)$$

$$= -\{\{g, h\}, f\} \quad (7)$$

4 Arnold

Lemma 4 The Lie Bracket of two vector fields

$$[X, Y] = L_X L_Y - L_Y L_X$$

is a first order differential operator.

PROOF.

Let the local coordinates of P be (x_1, \dots, x_n) , and let X and Y have components (X_1, \dots, X_n) and (Y_1, \dots, Y_n) , respectively, then

$$\begin{aligned} L_X L_Y \varphi &= \sum_i X_i \frac{\partial}{\partial x_i} \left(\sum_j Y_j \frac{\partial}{\partial x_j} \varphi \right) \\ &= X_i \frac{\partial Y_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + X_i Y_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}. \end{aligned}$$

By equality of second partials, the second partials in φ vanish when $L_Y L_X \varphi$ is subtracted. Hence,

$$[X, Y] = \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial \varphi}{\partial x_j}.$$

Jacobi Proof 3

The sum $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}$ is solely a linear combination of the second-order partial derivatives of f , g , and h . The terms which contribute to the second derivatives of f are

$$\{\{f, g\}, h\} = L_{X_h} \{f, g\} = L_{X_h} L_{X_g} f,$$

and

$$\{\{h, f\}, g\} = L_{X_g} \{h, f\} = -L_{X_g} L_{X_h} f.$$

Hence,

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} = -[X_g, X_h] f.$$

By Lemma 4, $[X_g, X_h]$ is a first-order differential operator, so there are no second-order partial derivatives of f which contribute to the sum. The same statement can be made for g and h . Hence, the Jacobi identity holds.

The statement that the Jacobi sum is a "linear combination of second-order partial derivatives" is not obvious. With this assumption, all first-order partial derivatives in f must sum to zero, giving

$$-[X_g, X_h] f - X_{\{g, h\}} f = 0,$$

which is equivalent to Equation 1. Hence, this statement together with Equations 3–7 found in the Abraham and Marsden proof completes the Jacobi identity without using Lemma 4.

5 Libermann and Marle

The proof of Libermann and Marle first shows that $(d\{f, g\})^\sharp = [(df)^\sharp, (dg)^\sharp]$, or equivalently $X_{\{f, g\}} = -[X_f, X_g]$, by computations using the formulas for Lie derivatives and interior products and the fact that Ω is closed.

These formulas are:

$$i_{[X, Y]}\alpha = \mathcal{L}_X i_Y \alpha - i_Y \mathcal{L}_X \alpha, \quad (8)$$

and

$$\mathcal{L}_X \alpha = di_X \alpha + i_X d\alpha. \quad (9)$$

Jacobi Proof 4

PROOF.

$$\begin{aligned} i_{[X_f, X_g]}\Omega &= \mathcal{L}_{X_f} i_{X_g} \Omega - i_{X_g} \mathcal{L}_{X_f} \Omega \\ &= \left(di_{X_f} i_{X_g} \Omega + i_{X_f} di_{X_g} \Omega \right) - i_{X_g} \left(di_{X_f} \Omega + i_{X_f} d\Omega \right) \\ &= di_{X_f} i_{X_g} \Omega + i_{X_f} d^2 g - i_{X_g} d^2 f \\ &= di_{X_f} i_{X_g} \Omega \\ &= -d\{f, g\} \end{aligned}$$

Hence, Equation 1 holds.

The Jacobi identity now proceeds as in Equations 3– 7 found in the proof of Abraham and Marsden.

6 Souriau

Souriau proves the Jacobi identity in terms of derivations, so it is necessary to first define derivations and then write the Lie derivative, Lie Bracket, and other necessary formulas in terms of derivations.

Derivation

A derivation δ on a manifold P with vector field $X \in \mathcal{X}(P)$ produces a new variable

$$\delta y = Dg(x) X(x),$$

where $y = g(x)$.

Letting $y = x$ gives $\delta x = X(x)$, so any vector field can be written in the form $x \mapsto \delta x$.

Lie bracket

Given two vector fields, $x \mapsto \delta x$ and $x \mapsto \delta' x$, on P , there exists another derivation denoted $[\delta, \delta']$ called the Lie bracket of δ and δ' . This derivation is defined by

$$[\delta, \delta'] y = \delta [\delta' y] - \delta' [\delta y],$$

if $\delta \delta' y$ and $\delta' \delta y$ exist.

Lie derivative

Given a vector field, $f(x)$, and a field of covariant operators of degree m , $g(x)$, on P , the Lie derivative of the field g in the direction of the vector field f is a field of covariant operators on degree m on P denoted by $[f, g]$. If $X(x) = \delta x$ and $g(x) = \varphi$ then another notation for the Lie derivative is

$$\delta_{\mathcal{L}} \varphi = [f, g](x).$$

The general definition of the Lie derivative for a field of covariant operators of degree m is not given here, but two special cases of m are:

- If the field φ is a vector field $\delta' x$ on P , ie $\varphi = \delta' x$, then the Lie derivative becomes the Lie bracket:

$$\delta_{\mathcal{L}} \delta' x = [\delta, \delta'] x. \quad (10)$$

- If the field φ is a scalar field f on P , ie $f \in \mathcal{F}(P)$, the Lie derivative reduces to

$$\delta_{\mathcal{L}} f = \delta f, \quad (11)$$

when the δf exists.

A few formulas useful in proving the Jacobi Identity will now be stated. Given a tensor field $x \mapsto \varphi$ of degree $m \geq 1$ and a vector field $x \mapsto \delta x$,

a contraction $x \mapsto \varphi(\delta x)$ is a $m-1$ tensor field. The Lie derivative of this contraction in the direction of another vector field $x \mapsto \delta' x$ is given by

$$\delta'_L [\varphi(\delta x)] = [\delta'_L \varphi](\delta x) + \varphi(\delta'_L \delta x). \quad (12)$$

Next is Cartan's formula written in the language of derivations:

$$\delta_L \varphi = [d\varphi](\delta x) + d[\varphi(\delta x)]. \quad (13)$$

Noticing that $d\varphi(\delta x) = \delta_L \varphi$, it follows from Cartan's formula that

$$\delta_L [d\varphi] = [d^2\varphi](\delta x) + d[d\varphi(\delta x)] = d[\delta_L \varphi]. \quad (14)$$

Notation

In the notation of Souriau, the equivalent of a Hamiltonian vector field is denoted by $\text{grad } f$, where

$$df = -\Omega(\text{grad } f).$$

In more familiar notation, $\text{grad } f = -X_f$ and $-\Omega(\text{grad } f) = i_{X_f} \Omega$.

Definition 2 *The derivation δ^f associated with $f \in \mathcal{F}(P)$ is defined by*

$$\delta^f x = \text{grad } f.$$

Lemma 5 *The Lie derivative $\delta^f_L \Omega$ is zero for all $f \in \mathcal{F}(P)$.*

PROOF.

$$\begin{aligned} \delta^f_L \Omega &= [d\Omega](\delta^f x) + d[\Omega(\delta^f x)] \text{ by eqn 13} \\ &= 0 + d[-df] = 0. \end{aligned}$$

The Poisson bracket in the notation of Souriau becomes

$$\begin{aligned} \{f, g\} &= \Omega(\text{grad } f)(\text{grad } g) \\ &= -[df](\text{grad } g) = [dg](\text{grad } f), \end{aligned}$$

or in terms of derivations,

$$\{f, g\} = \delta^f g = -\delta^g f. \quad (15)$$

Jacobi Proof 5

PROOF.

Apply $\delta_{\mathcal{L}}^f$ to $\mathbf{d}g = -\Omega(\delta^g x)$. (The right hand side written in more familiar notation is just $\mathbf{i}_{X_g}\Omega$.)

LHS:

$$\begin{aligned}\delta_{\mathcal{L}}^f[\mathbf{d}g] &= \mathbf{d}[\delta_{\mathcal{L}}^f g] \text{ by eqn 14} \\ &= \mathbf{d}[\delta^f g] \text{ by eqn 11} \\ &= \mathbf{d}\{f, g\} \text{ by eqn 15} \\ &= -\Omega(\text{grad}\{f, g\})\end{aligned}$$

RHS:

$$\begin{aligned}\delta_{\mathcal{L}}^f[-\Omega(\delta^g x)] &= -[\delta_{\mathcal{L}}^f \Omega](\delta^g x) - \Omega(\delta_{\mathcal{L}}^f \delta^g x) \text{ by eqn 12} \\ &= -\Omega([\delta^f, \delta^g] x) \text{ by eqn 10 and Lemma 5}\end{aligned}$$

Hence,

$$[\delta^f, \delta^g] x = \text{grad}\{f, g\},$$

or using Definition 2,

$$[\delta^f, \delta^g] = \delta^{\{f, g\}}, \tag{16}$$

which is Equation 1 written in derivations.

Apply Equation 16 to $h \in \mathcal{F}(P)$ yields

$$\delta^f[\delta^g h] - \delta^g[\delta^f h] = \delta^{\{f, g\}} h,$$

and, by equation 15, this is

$$\{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\},$$

or

$$0 = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}.$$

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