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Math 189 project report

1993 (?)

-: Stabilization of rigid body dynamics :-

This project is about the survey and understanding of the recent literature about stabilization of rigid body by using external torques (gas jets) and internal torques (internal rotors). It is mainly based on referances [1] & [2] given at the back. This also gives asymptotic stabilization laws for the rigid body.

Free Rigid Body & the nature of its equilibrium points

As the free rigid body equations are discussed in detail in the class notes, they will be just stated here.

If we write down the rigid body equations in body co-ordinates,

$$\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\,\omega_2\omega_3 \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_3}\,\omega_3\omega_1 \qquad\qquad (1) \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\,\omega_1\omega_2
\end{aligned}$$

where I_i are the moments of inertia about principle axes and ω_i are body angular velocities.

It will also be assumed that $I_1 > I_2 > I_3$

Actually the lagrangian for this case is $L(A,\dot{A}) = \omega_B I \omega_B$

Thus by applying Legendre transform we get the classical momentum variables

$$m_i = I_i \omega_i$$

So the above equations become

$$\begin{aligned} \dot{m}_1 &= a_1 m_2 m_3 \\ \dot{m}_2 &= a_2 m_3 m_1 \qquad (2) \\ \dot{m}_3 &= a_3 m_1 m_2 \end{aligned}$$

where $a_1 = \dfrac{I_2 - I_3}{I_2 I_3}$; $a_2 = \dfrac{I_3 - I_2}{I_1 I_3}$; $a_3 = \dfrac{I_1 - I_2}{I_1 I_2}$

& because of assumption about I_i's,

$$a_1 > 0, \quad a_2 < 0, \quad |a_2| > |a_1|, \quad a_3 > 0.$$

And by direct check $a_1 + a_2 + a_3 = 0$.

we can check that

$$H = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_2}\right)$$

is a constant of motion.

<u>Direct check</u> $\rightarrow$ $\underline{\dfrac{\partial H}{\partial m}} \dot{m} = \dfrac{\dot{m}_1 m_1}{I_1} + \dfrac{\dot{m}_2 m_2}{I_2} + \dfrac{\dot{m}_3 m_3}{I_3}$

$$= m_1 m_2 m_3 \left(\frac{a_1}{I_1} + \frac{a_2}{I_2} + \frac{a_3}{I_3}\right)$$

$$= 0$$

Actually this H is the KE expressed in momentum variables and it is actually the hamiltonian.

Similarly we can show that the total angular momentum M is conserved.

$$\because \quad M^2 = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2\right)$$

and by using the fact that $(a_1 + a_2 + a_3) = 0$, we can show that the derivative of M^2 along the trajectories of (2) is zero.

Actually equations (2) are euler equations given by

$$\frac{d}{dt} m = m \times \omega$$

Now we can analyse the stability of the equilibrium points of this system.

The eqns (2) have 4 equilibrium points given by

$$(m_1, m_2, m_3) = (0,0,0),\ (\bar{M},0,0),\ (0,\bar{M},0),\ (0,0,\bar{M})$$

where $\bar{M}$ is a constant.

[equilibrium pts. are the pts. where $\frac{d}{dt}\underline{m} = 0$]

Now, if we initially start with $M^2 = 0$ i.e. $m_i = 0$ $i = 1, 2, 3$ then we will stay there and then we will never go anywhere in the phase space (without any torque).

Now if we initially start with nonzero momentum, then because M^2 is conserved, we can never reach the eq. pt. $(0,0,0)$ so let us concentrate on remaining three equilibrium points.

The jacobian linearisation of (2) has the jacobian given by

$$Jac = \begin{bmatrix} 0 & a_1 m_3 & a_1 m_2 \\ a_2 m_3 & 0 & a_2 m_1 \\ a_3 m_2 & a_3 m_1 & 0 \end{bmatrix}$$

At the equilibrium $(0, \bar{M}, 0)$

$$Jac\Big|_{(0,\bar{M},0)} = \begin{bmatrix} 0 & 0 & a_1 m_2 \\ 0 & 0 & 0 \\ a_3 m_2 & 0 & 0 \end{bmatrix}$$

whose eigenvalues are given by

$$\lambda^3 + a_1 \bar{M} \, (-a_3 \bar{M} \lambda) = 0 \quad \Rightarrow \lambda = 0 \,, \; \pm \sqrt{a_1 a_3} \bar{M}$$

Thus this has one real positive eigenvalue & its unstable.

At the other two relative equilibria $(M,0,0)$ & $(0,0,M)$ the linearisation has one zero and two imaginary eigenvalues. It can be shown by energy-casimir method that these are stable.

<u>verification</u> → Consider energy-casimir function

$H + C$ where $C = \varphi(M^2)$ where φ is a smooth function

[Actually if φ is C^3, its enough]

Let us consider the relative equilibrium $(0, 0, \bar{M})$

Now if $(H+C)$ has a critical point at equilibrium & its second variation is definite, then we can prove stability.

Now

$$\delta(H+C) = \frac{m_1 \delta m_1}{I_1} + \frac{m_2 \delta m_2}{I_2} + \frac{m_3 \delta m_3}{I_3}$$
$$+ \varphi'(M^2)(m_1 \delta m_1 + m_2 \delta m_2 + m_3 \delta m_3)$$

Now for $(H+C)$ to have a critical point at $(0, 0, \bar{M})$, its first variation must be zero.

i.e.

$$\frac{m_1}{I_1} + \varphi'(M^2) \cdot m_1 = 0$$

$$\frac{m_2}{I_2} + \varphi'(M^2) \cdot m_2 = 0$$

$$\frac{m_3}{I_3} + \varphi'(M^2) \cdot m_3 = 0.$$

These equations are satisfied at $(0,0,\bar{M})$ if $\varphi'(M^2) = -\frac{1}{I_3}$

Now

$$\delta^2(H+C) = \frac{(\delta m_1)^2}{I_1} + \frac{(\delta m_2)^2}{I_2} + \frac{(\delta m_3)^2}{I_3} + \varphi'(M^2)\left[(\delta m_1)^2 + (\delta m_2)^2 + (\delta m_3)^2\right]$$
$$+ \varphi''(M^2)\left[m_1 \delta m_1 + m_2 \delta m_2 + m_3 \delta m_3\right]^2$$

so at $(0,0,\bar{M})$

$$\delta^2(H+C) = (\delta m_1)^2\left(\frac{1}{I_1} - \frac{1}{I_3}\right) + (\delta m_2)^2\left(\frac{1}{I_2} - \frac{1}{I_3}\right) + \varphi''(M^2)\bar{M}^2(\delta m_3)^2$$

Now because $I_1 > I_2 > I_3$, first two terms are negative so if we take $\varphi''(M^2) < 0$, then second variation of $(H+C)$ is negative definite at $(0,0,\bar{M})$

$\therefore$ The relative equilibrium $(0,0,\bar{M})$ is stable.

similarly we can show that $(\bar{M}, 0, 0)$ is stable.

Thus this means that rotations of rigid body about its minor or major axis are stable [locally, nonlinearly stable] whereas rotations about intermediate axis are unstable.

Thus the problem is that of stabilizing the rigid body about its intermediate axis.

Stabilization →

This problem is discussed in ref[1] & there the claim is that if you use the feedback law

$$u = -\varepsilon \frac{I_1 - I_2}{I_3} \omega_1 \omega_2$$

for the external torque along the minor (major) axis, then

(i) The system is stabilized about $(0, \bar{M}, 0)$

(ii) The resulting system is still hamiltonian with a different H & a different lie-poissoin structure.

Now, the equations of rigid body with external torque along the minor axis are

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2} \omega_3 \omega_1$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3} \omega_1 \omega_2 + u$$

The same equations can be written in terms of momenta as

$$\dot{m}_1 = a_1 m_2 m_3$$
$$\dot{m}_2 = a_2 m_3 m_1$$
$$\dot{m}_3 = a_3 m_1 m_2 + u.$$

& the quadratic feedback will be

$$u = -\varepsilon a_3 m_1 m_2$$

So the last equation becomes

$$\dot{m}_3 = (1-\varepsilon) a_3 m_1 m_2.$$

Now again by direct calculation, we can show that

$$H_F = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\frac{1}{1-\varepsilon}\right) \qquad \varepsilon \neq 1$$

& $$M_F^2 = \frac{1}{2}\left(m_1^2(1-\varepsilon) + m_2^2(1-\varepsilon) + m_3^2\right)$$

are constants of motion.

Actually H_F is the hamiltonian

Now to prove that the above feedback law stabilizes system, we can check by energy casimir method.

Let $H_F + \psi(M_F^2)$ be the Energy casimir function

Then for this to have critical point at $(0, \bar{M}, 0)$, we need first variation of energy casimir function to be zero.

$$\Rightarrow \quad \delta\left[H_F + \psi(M_F^2)\right] = 0$$

$$\Rightarrow \quad \frac{m_1\,\delta m_1}{I_1} + \frac{m_2\,\delta m_2}{I_2} + \frac{m_3\,\delta m_3}{I_3(1-\varepsilon)} + \psi'(M_F^2)\left[m_1\,\delta m_1(1-\varepsilon) + m_2\,\delta m_2(1-\varepsilon) + m_3\,\delta m_3\right] = 0$$

At $(0, \bar{M}, 0)$

$$\Rightarrow \quad \frac{m_2}{I_2}\,\delta m_2 + \psi'(M_F^2)\,m_2\,\delta m_2\,(1-\varepsilon) = 0$$

$$\Rightarrow \quad \psi'(M_F^2) = \frac{-1}{I_2(1-\varepsilon)} \qquad \varepsilon \neq 1$$

Now,

$$\delta^2\left[H_F + \psi(M_F^2)\right]\Big|_{(0,\bar{M},0)} = \left(\frac{1}{I_1} - \frac{1}{I_2}\right)(\delta m_1)^2 + \left[\frac{1}{(1-\varepsilon)}\left(\frac{1}{I_3} - \frac{1}{I_2}\right)\right](\delta m_3)^2 + \psi''(M_F^2)\,\bar{M}^2(1-\varepsilon)^2(\delta m_2)^2$$

Now if $\varepsilon > 1 \Rightarrow$ Both first two terms are negative & then choosing $\psi''(M_F^2) < 0$ will make the second variation negative definite thus proving the stability of relative equilibrium $(0, M, 0)$ for $\varepsilon > 1$.

Now for $\varepsilon < 1 \rightarrow$ in the linearisation, we still have a saddle point and so the system is still unstable about the relative equilibrium $(0, \bar{M}, 0)$

For $\varepsilon = 1 \Rightarrow$ the equations (2) become

$$\dot{m}_1 = a_1\, m_2\, m_3(0)$$
$$\dot{m}_2 = a_2\, m_1\, m_3(0)$$
$$\dot{m}_3 = 0.$$

$m_3(0) =$ initial condition on m_3, const.

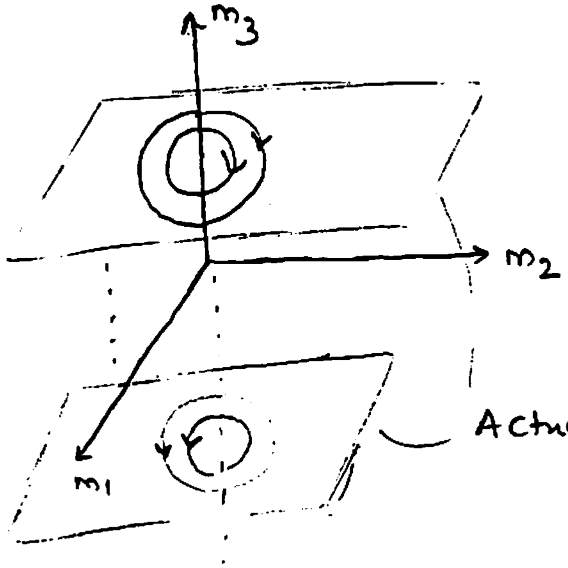
Here m_3 is always const & in the remaining 2 dimensional system,

$$\dot{m}_1 = \alpha_1 m_2 \qquad \text{where } \alpha_1 = a_1 m_3(0)$$
$$\dot{m}_2 = \alpha_2 m_1 \qquad \alpha_2 = a_2 m_3(0)$$

$$a_1 > 0$$
$$a_2 < 0.$$

Thus this system is linear. Has eigenvalues on $j\omega$ axis. & M^2 is still conserved.

$\therefore$ we will get continuum of limit cycles or in other words, the eq. pt. of this linear system is a center.



So the system is Gyroscopically stable

Actually these planes will be tangential to the surface

$$\beta m_1^2 + m_2^2 + m_3^2 = \text{const.}$$

where $\beta = \dfrac{a_2}{a_1}$

So Basically the idea in stabilizing control is that with $\epsilon < 1$, we don't have definiteness in the second variation of the Energy-Casimir function which we get, when we choose $\epsilon > 1$.

Also by making $\epsilon > 1$, we have shifted a real +ve & a real −ve e.value (saddle) to $j\omega$ axis (center).

The most interesting thing about this paper is that with above feedback law, the system is still hamiltonian with H_F as new hamiltonian & the Lie-poissoin structure

$$\{F, G\}_F = -\nabla M_F^2 \cdot (\nabla F \times \nabla G)$$

We can check this by verifying that

$$\dot{m}_i = \{m_i, H_F\}_F$$

To check for m_1,

$$\{m_1, H_F\}_F = -\nabla M_F^2 \cdot (\nabla m_1 \times \nabla H_F)$$

$$= -\nabla m_1 \cdot (\nabla H_F \times \nabla M_F^2)$$

$$= -[1 \;\; 0 \;\; 0] \cdot \left(\left[\frac{m_1}{I_1} \;\; \frac{m_2}{I_2} \;\; \frac{m_3}{I_3(1-\varepsilon)}\right]^T \times \left[m_1(1-\varepsilon) \;\; m_2(1-\varepsilon) \;\; m_3\right]^T \right)$$

$$= [1 \;\; 0 \;\; 0] \cdot \left[\frac{I_2 - I_3}{I_2 I_3} m_2 m_3 \;\; \frac{I_3 - I_1}{I_3 I_1} m_1 m_3 \;\; \frac{I_1 - I_2}{I_1 I_2}(1-\varepsilon) m_1 m_2\right]$$

$$= a_1 m_2 m_3 = \dot{m}_1$$

& similarly for m_2 & m_3.

Thus the invariance group for lie poissoin bracket changes from $SO(3)$ to $SO(2,1)$ when ε changes from $\varepsilon < 1 \longrightarrow \varepsilon > 1$.

Now because this system still remains hamiltonian under the feedback, we can realise the external torque feedback by internal torque (internal rotor) feedback.

Stabilization by internal torque feedback

If we consider a rigid body carrying three symmetric rotors, then configuration space is $SO(3) \times S^1 \times S^1 \times S^1$.

After going through necessary analysis

(i.e. $L \longrightarrow$ Legendre transform $\longrightarrow$ phase space)

we get the eq^{ns} of motion as

$$\mathring{m} = m \times \Omega = m \times (I_{lock} - I_{rotor})^{-1}(m - l)$$

$$\dot{l} = u$$

where l = momenta of rotor @ principle axes relative to body frame

I_{lock} = inertia tensor of full system (with rotors locked)

I_{rotor} = diagonal matrix of rotational inertias about principal axes.

Now to extract the single external feedback torque from three rotors,

let $\pi = m - l$ & $I = I_{lock} - I_{rotor}$

If we let I to be diagonal

$$\Rightarrow \mathring{\pi} = (\pi + l) \times I^{-1}\pi - u$$

$$\mathring{l} = u$$

where u is the torque applied to the rotors.

Now first of all, let us set

$$u(\Pi, l) = l \times I^{-1} \cdot \Pi - \bar{u}(\Pi)$$

$$\Rightarrow \quad \dot{m} = \Pi \times I^{-1}\Pi + u'(\Pi)$$

$$\dot{l} = l \times I^{-1}\Pi - u'(\Pi)$$

& now choose $u'(\Pi) = \begin{bmatrix} 0 \\ 0 \\ -a_3 \varepsilon \Pi_1 \Pi_2 \end{bmatrix}$

& then if $I = (I_1\ I_2\ I_3)^T$ is st. $I_1 > I_2 > I_3$

Then the original system (2) is recovered. as the eq^{ns} of motion become,

$$\left.\begin{aligned} \dot{\Pi}_1 &= a_1 \Pi_2 \Pi_3 \\ \dot{\Pi}_2 &= a_2 \Pi_3 \Pi_1 \\ \dot{\Pi}_3 &= a_3(1-\varepsilon)\, \Pi_1 \Pi_2 \end{aligned}\right\} \text{ same as (2)}$$

$$\dot{l}_1 = \frac{l_2 \Pi_3}{I_3} - \frac{l_3 \Pi_2}{I_2}$$

$$\dot{l}_2 = \frac{l_3 \Pi_1}{I_1} - \frac{l_1 \Pi_3}{I_3}$$

$$\dot{l}_3 = \frac{\Pi_2}{I_2}(l_1 + \varepsilon \Pi_1) - \frac{\Pi_1}{I_1}(l_2 + \varepsilon \Pi_2)$$

Here actually, we have applied feedback of the type $u = k\,(m \times (I_{lock} - I_{rotor})^{-1}(m - l))$

Now if k is st. $J = (I_d - k)^{-1}(I_{lock} - I_{rotor})$ is symmetric then the system with this feedback is still hamiltonian.

& now it is hamiltonian with the standard lie-poissoin structure

$$\{F, G\}(m) = -m\cdot(\nabla F \times \nabla G)$$

This is a very important result.

Also it is shown in the paper that the dual spin case as in ref[2] is a special case of this when $k\, I_{lock} = I_{rotor}$.

We can also consider rigid body with a single rotor. Suppose the rotor is aligned with third principal axis & has moments of inertia $J_1 = J_2$ & J_3.

Then let $\mathrm{diag}(\lambda_1\ \lambda_2\ \lambda_3) = \mathrm{diag}(J_1+I_1,\ J_2+I_2,\ J_3+I_3)$ be locked rotor inertia.

Then the momenta will be $m_i = \lambda_i w_i \quad i=1,2$

$$m_3 = I_3 w_3 + l_3$$

$$l_3 = J_3(w_3 + \dot{\alpha}) \qquad \text{where } \dot{\alpha} = \text{ang. velocity of rotor.}$$

Now the equations of motion become

$$\dot{m}_1 = m_2 m_3\left(\frac{1}{I_3} - \frac{1}{\lambda_2}\right) - \frac{l_3 m_2}{I_3}$$

$$\dot{m}_2 = m_1 m_2\left(\frac{1}{\lambda_1} - \frac{1}{I_3}\right) + \frac{l_3 m_1}{I_3} \qquad — (4)$$

$$\dot{m}_3 = m_1 m_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$$

$$\dot{l}_3 = u$$

Now if we choose $u = k\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) m_1 m_2$

$\Rightarrow$

$$\dot{m}_1 = m_2\left[\frac{(1-k)\, m_3 - p}{I_3}\right] - \frac{m_3 m_2}{\lambda_2}$$

$$\dot{m}_2 = -m_1\left[\frac{(1-k)\, m_3 - p}{I_3}\right] + \frac{m_3 m_1}{\lambda_1}$$

$$\dot{m}_3 = a_3 m_1 m_2$$

where $p = \text{const.}$

Now we can show that these equations are hamiltonian on $so(3)^*$ w.r.t. usual Lie-Poissoin structure with

$$H = \frac{1}{2}\left[\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda_2} + \frac{((1-k)\, m_3 - p)^2}{(1-k)\; I_3}\right] + \frac{1}{2}\,\frac{p^2}{J_3(1-k)}$$

Now, we can take $m^2 = m_1^2 + m_2^2 + m_3^2$

& using $(H+C)$ where $C = \varphi(m^2)$ as a Energy-Casimi function, we can prove by exactly same method as that given for external torque that if

$$k > 1 - \frac{I_3}{\lambda_2}$$

Then system is stable about $(0, \bar{M}, 0)$

← Asymptotic Stabilization →

For asymptotic stabilization, Lyapunov stability theory along with the Lasalle's principle will be used.

Here is the statement of the Lasalle's theorm.

Lasalle's principle → (for autonomous (time independent) systems)

Let $V(x)$ be a positive definite function with contineous partial derivatives. Let Ω_c be the region where $V(x) \leq c$. Assume Ω_c is bounded.

Now inside Ω_c,

If $V(x) > 0 \quad \forall x \neq 0$ (gurenteed by positive definiteness)

& $\dot{V}(x) \leq 0$ — (but not negative definite)

along the trajectories of eqn $\dot{x} = X(x)$.

Now find out the largest invariant set in $\dot{V}(x) = 0$.

Let us call it N.

Then every solution $x(t)$ in Ω_c tends to N as $t \to \infty$.

Now consider equations of rigid body with one external torque along minor axis.

$$\dot{m}_1 = a_1 m_2 m_3$$

$$\dot{m}_2 = a_2 m_3 m_1$$

$$\dot{m}_3 = a_3 m_1 m_2 + u.$$

Select $V = \dfrac{m_1^2}{2} + \dfrac{m_2^2}{2} + \dfrac{m_3^2}{2}$ as the Lyapunov function.

Then

$$\dot{V} = \frac{\partial V}{\partial M}\dot{m} = m_1\dot{m}_1 + m_2\dot{m}_2 + m_3\dot{m}_3$$

$$= \cancel{(a_1+a_2+a_3)}^{\,0.}\, m_1 m_2 m_3 + m_3 u$$

Now if we choose $u = -k m_3$ where $k > 0$ const.

$\Rightarrow \dot{V} = -k m_3^2 \leq 0.$

Now let us apply Lasalle's principle,

Here $\Omega_c = \{m : \|m\| \leq M^2_{(0)}\}$ where $M^2_{(0)}$ is the initial momentum (squared)

Now $\dot{V} = 0 \Rightarrow m_3 \equiv 0$

$\Rightarrow \dot{m}_3 = 0 = a_3 m_1 m_2 \rightarrow m_1 = 0,\ m_2 = 0.$

$m_1 = 0$, $m_2 = k_1$ $m_2 = 0$, $m_1 = k_2$

where k_1 & k_2 are const.

Thus the largest invariant set is

$$\{m_1 = 0,\ m_2 = k_1,\ m_3 = 0\} \cup \{m_1 = k_2,\ m_2 = 0,\ m_3 = 0\}.$$

Thus according to lasalle's principle at $t \to \infty$

$m_3 = 0$ & $m_1 = 0,\ m_2 = k_1$

or

$m_1 = k_2,\ m_2 = 0.$

Thus we are gurenteed that asymptotically $m_3 \to 0$ & either m_1 or $m_2 \to 0$ & other has a const. value.

But we don't know which one of these m_1 or m_2 goes to zero. But it actually depends on initial conditions.

A similar analysis holds for external force along major axis where one chooses $u = -m_1$ & now we are gurenteed that $m_1 \to 0$ & one of m_2 and $m_3 \to 0$ and other is a constant.

Actually for getting asymptotic stability we are introducing a dissipative term into the system so the system no longer remains hamiltonian after the feedback.

If we obsewe the hamiltonian systems jacobian linearisation, then about an equilibrium, the jacobian eigenvalues are always symmetric with respect to the imaginary axis, so basically it will always have a paie of imaginary e-values or a saddle point. So a hamiltonian system by itself cannot be asymptotically stable.

Intuitively this is clear because if we add some friction like terms to a hamiltonian mechanical system, then we will get asymptotic stability.

Thus in above case, the eigenvalues of linearisation are changed to one zero & two complex conjugate eigenvalues in the left half plane.

Now, a similar analysis can be done in the case of rigid body with internal rotors. The only change is that we will not be realising the same control law as in case of external torque.

So with the three rotor case

$$\dot{\Pi} = \Pi \times I^{-1}\Pi + u'(\Pi)$$
$$\dot{l} = l \times I^{-1}\Pi - u'(\Pi)$$

supposing I is diagonal, then by choosing

$$u'(\Pi) = \begin{bmatrix} -a_1 \Pi_2 \Pi_3 - k_1 \Pi_1 \\ -a_2 \Pi_1 \Pi_3 - k_2 \Pi_2 \\ -a_3 \Pi_2 \Pi_1 - k_3 \Pi_3 \end{bmatrix}$$

we get three independent linear equations for body momenta

$$\dot{\Pi}_1 = -k_1 \Pi_1$$
$$\dot{\Pi}_2 = -k_2 \Pi_2$$
$$\dot{\Pi}_3 = -k_3 \Pi_3$$

which are exponentially stable.
Again after feedback, the system is not hamiltonian.

Similarly in case of rigid body with single rotor, if we first consider l_3 to be the control for first three equations, then

$$\dot{m}_1 = m_2 m_3 \alpha_1 - \frac{l_3 m_2}{I_3}$$

$$\dot{m}_2 = m_1 m_3 \alpha_2 + \frac{l_3 m_1}{I_3}$$

$$\dot{m}_3 = m_1 m_2 \alpha_3$$

where $\alpha_1 = \frac{1}{I_3} - \frac{1}{\lambda_2}$; $\alpha_2 = \frac{1}{\lambda_1} - \frac{1}{I_3}$; $\alpha_3 = \frac{1}{\lambda_2} - \frac{1}{\lambda_1}$

Then take

$$V = \frac{b_1 m_1^2}{2} + \frac{b_2 m_2^2}{2} + \frac{b_3 m_3^2}{2} \qquad b_i > 0$$

as Lyapunov function, then

$$\dot{V} = (b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3) m_1 m_2 m_3 + \left(\frac{b_2 - b_1}{I_3}\right) m_1 m_2 l_3$$

Now choose

$$l_3 = I_3 m_3 + I_3 m_1 m_2 k$$

& solve for b_1, b_2, b_3 st. $(b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + b_2 - b_1) = 0$

$\Rightarrow \dot{V} < 0$

& $\Rightarrow m_1 m_2 = 0$

so we get $m_1 = 0$ & $m_2 m_3 = 0$

or $m_2 = 0$ & $m_1 m_2 = 0$

by lasalle's theorm.

Thus again we are not sure which two momentas go to zero. But two of them go to zero & last one is a constant.

Now just by differentiating l_3 above, we can get u.

Energy Casimir method vs. Lyapunov + Lasalle's thm.

If we consider the hamiltonian systems where

$H =$ some form of energy of the system (KE + PE is the simplest)

then in that case the first choice for Lyapunov function will be $V = H$ itself.

But as H is conserved along the trajectories of the system, $\dot{V} \equiv 0$. & then we cannot conclude anything about stability from Lasalle's thm as the largest invariant set is the entire trajectory.

Then we are stuck because finding a Lyapunov function is hard & there are no constructive methods for that.

Energy Casimir method really helps you a lot in these cases as this gives you a constructive method of proving stability (assuming that you already know the hamiltonian for the system & the equations of motion) & we have to just check definiteness of 2nd variation and critical point at the equilibrium point.

so basically Energy-Casimir method is excellent for proving stability of hamiltonian systems.

Exact linearisation [input-state] of rigid body with one external torque

with the equations given as in (2) we can write down

$$\dot{m} = f(m) + g(m)\,u$$

here $f(m) = \begin{pmatrix} a_1 m_2 m_3 \\ a_2 m_1 m_3 \\ a_3 m_1 m_2 \end{pmatrix}$ & $g(m) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

One of the necessary & sufficient conditions for exact input-state linearisation of this system are

$$\text{rank}\,[g \;\; ad_f g \;\; ad_f^2 g] = 3$$

here $[g \;\; ad_f g \;\; ad_f^2 g] = \begin{bmatrix} 0 & -a_1 m_2 & 0 \\ 0 & -a_2 m_1 & 0 \\ 1 & 0 & a_3 a_1 m_2^2 + a_3 a_2 m_1^2 \end{bmatrix}$

So this is not full rank

Thus this system is not fully linearisable & actually if you try to linearise it then the zero dynamics will not be minimum phase.

Thus this paper was a nice application of energy casimir method & the particular feedback law retained the hamiltonian form of the system. This energy casimir method seems to be very useful in future research.

Attached are a few simulations of the rigid body with internal & external torques.

The control laws in ref [1] only stabilize system & the system is not asymptotically stable is clear.

The asymptotic stabilization control laws reveal the fact that you cannot gurentee which of the momentum's go to zero.

References

[ ] Bloch, Marsden, Krishnaprasad, Sanchez De Alvarez –

Stabilization of rigid body dynamics by Internal & External torque

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[2] P. S. Krishnaprasad
Lie-Poisson structure, Dual spin Spacecraft, & asymtotic stability.

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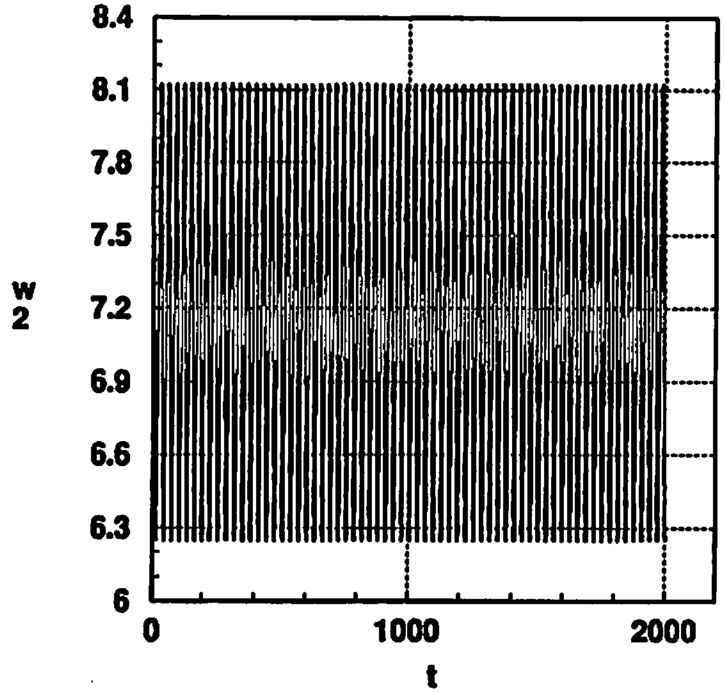
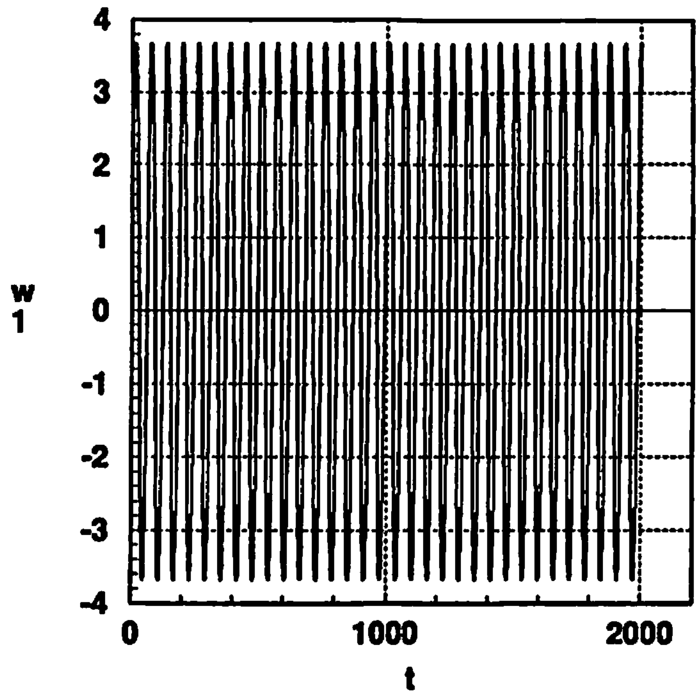
[3] Bloch, Marsden
Stabilization of rigid body dynamics by Energy Casimir method

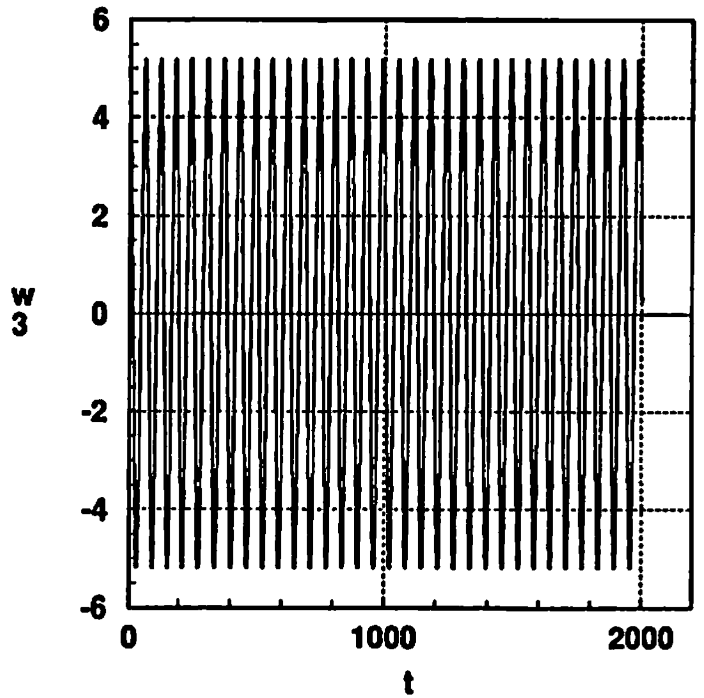
'Systems & Control Letters' Vol. 14. 1990 pp 341–346

[4] P. S. Krishnaprasad, C. A. Berenstein
On equilibria of rigid spacecrafts with rotors

'Systems & Controls letters' 4 (1984) pp 157–163

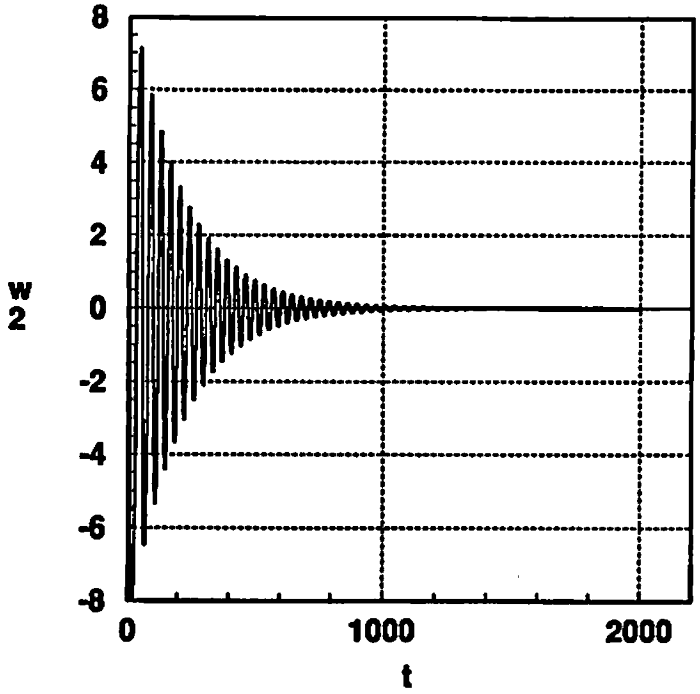


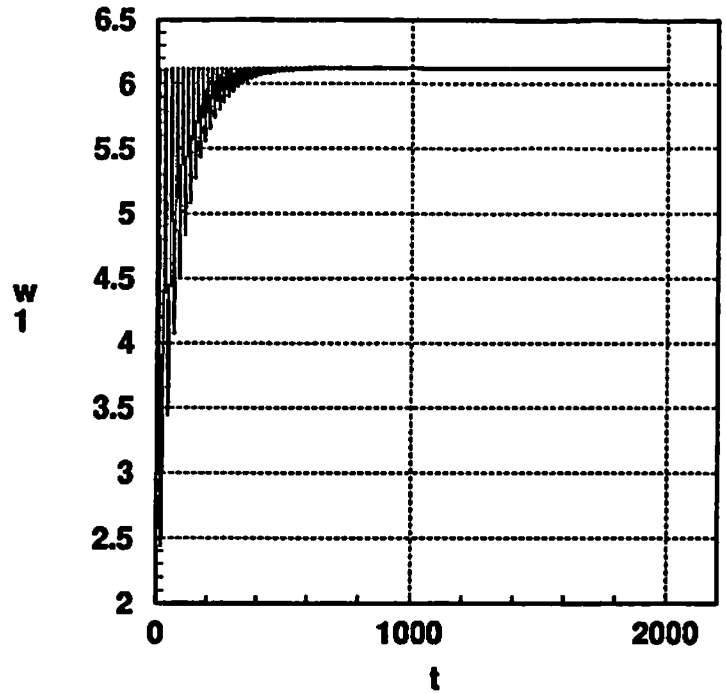


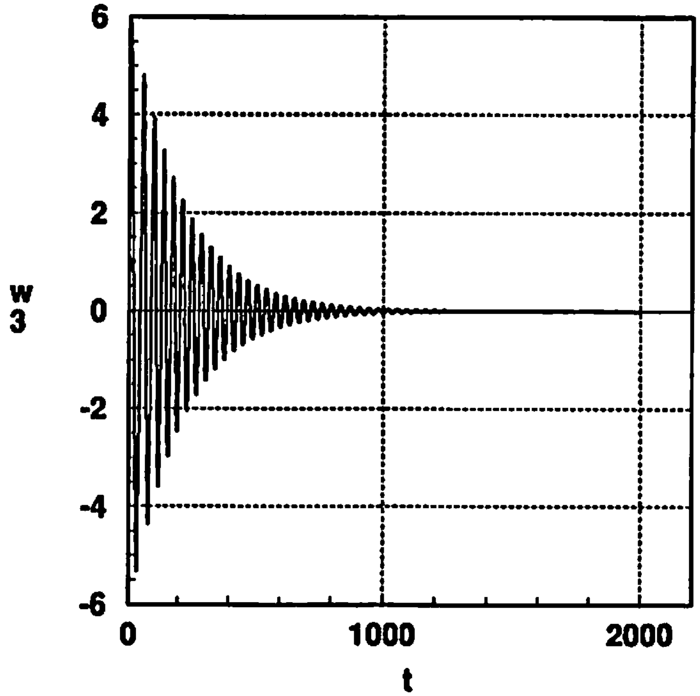


rigid body with external torque
along minor axis

stabilization law - Bloch, Marsden ...



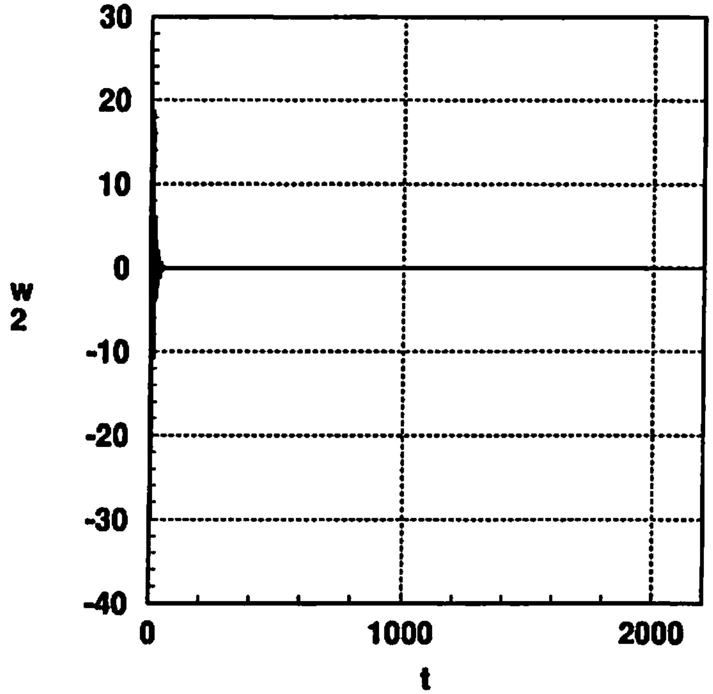


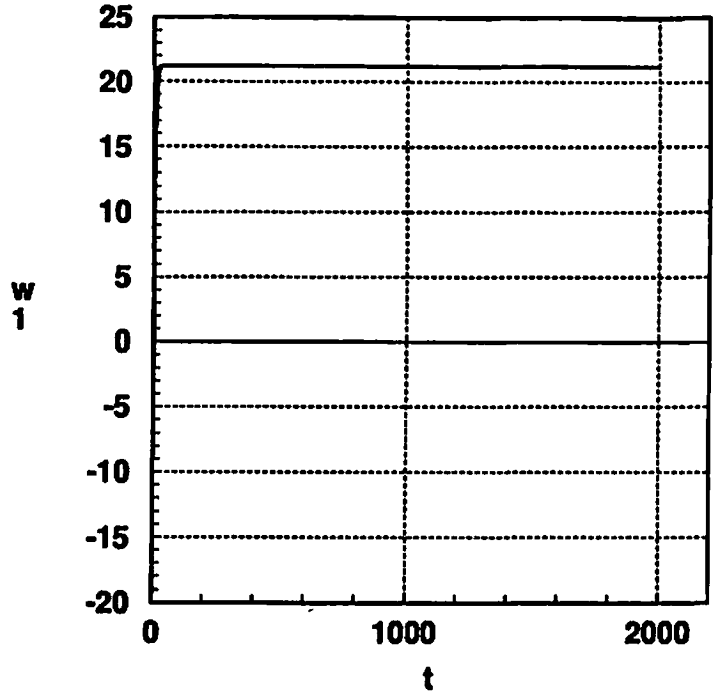


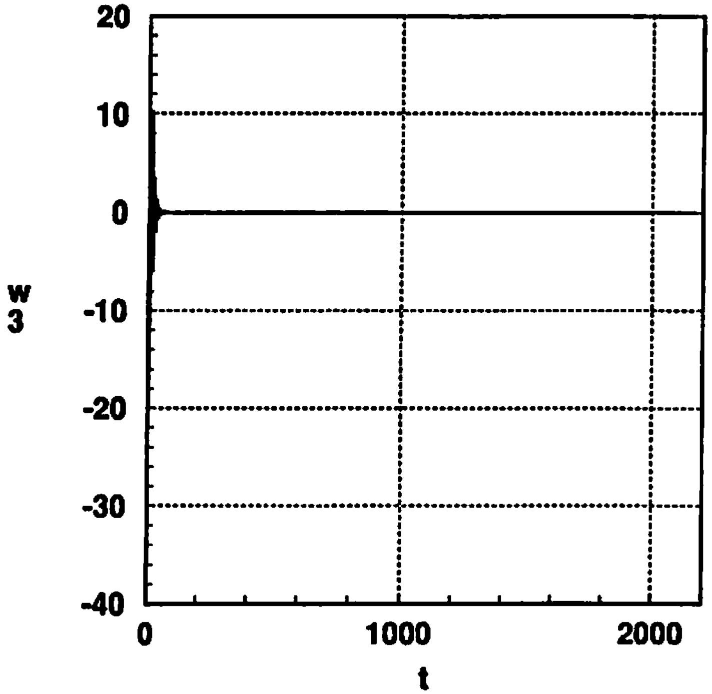
Asymptotic stabilization law

rigid body with external torque along minor axis

gain $k = 1.5$

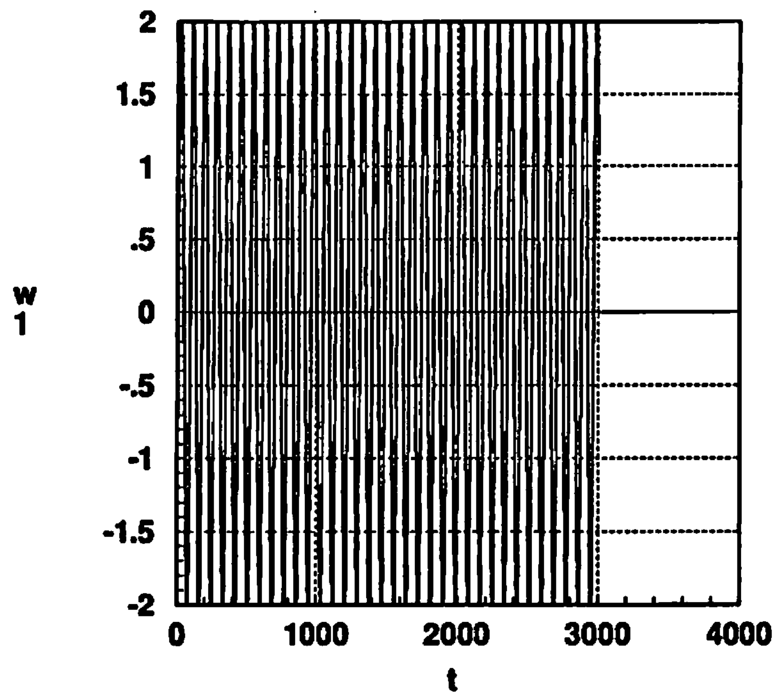


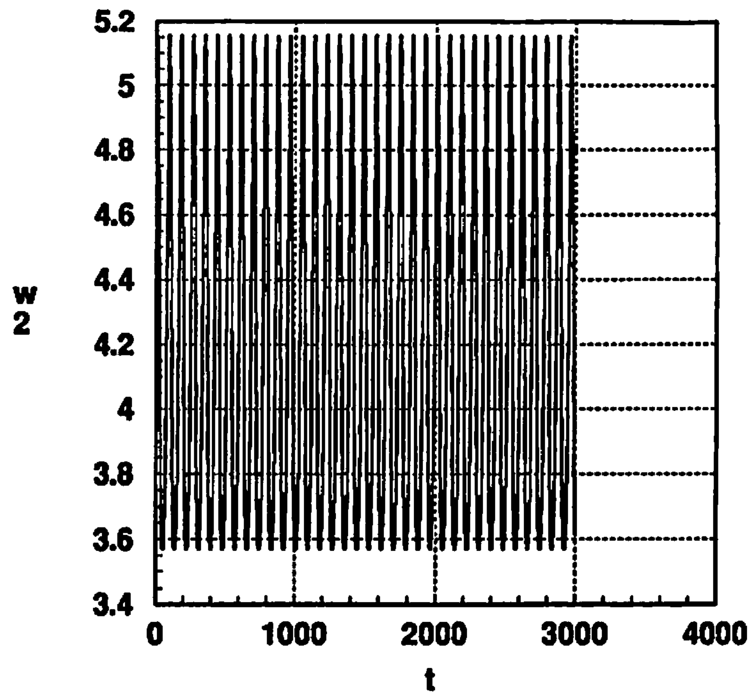


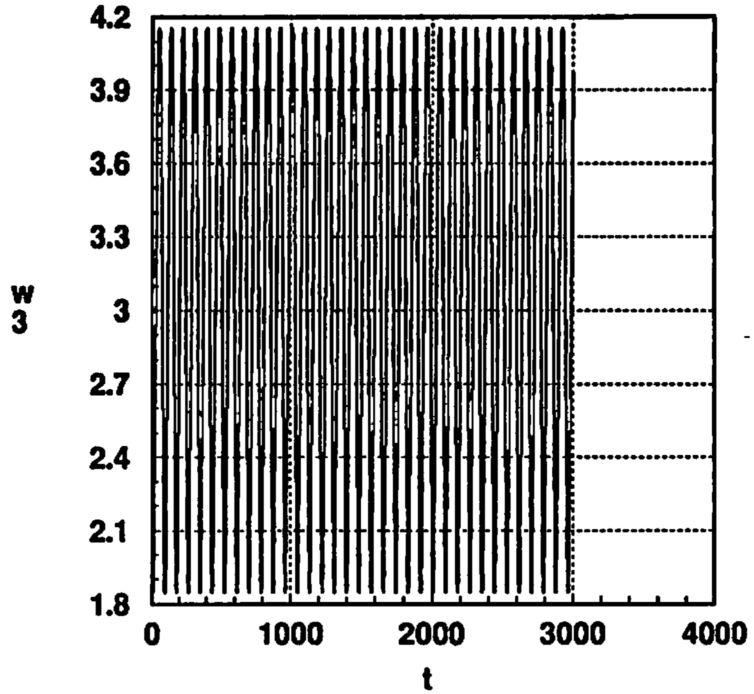


Asymptotic stabilization law
rigid body with external torque
along minor axis.

gain k = 20.

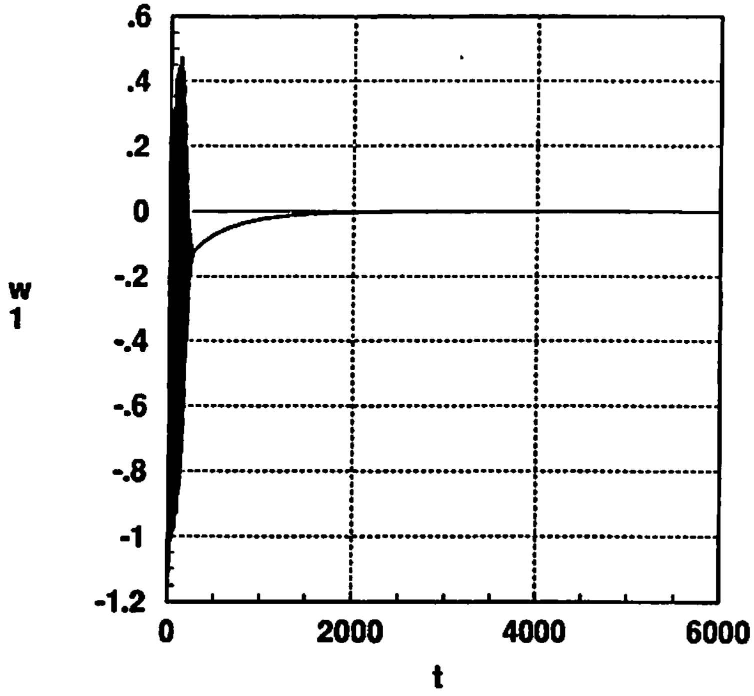


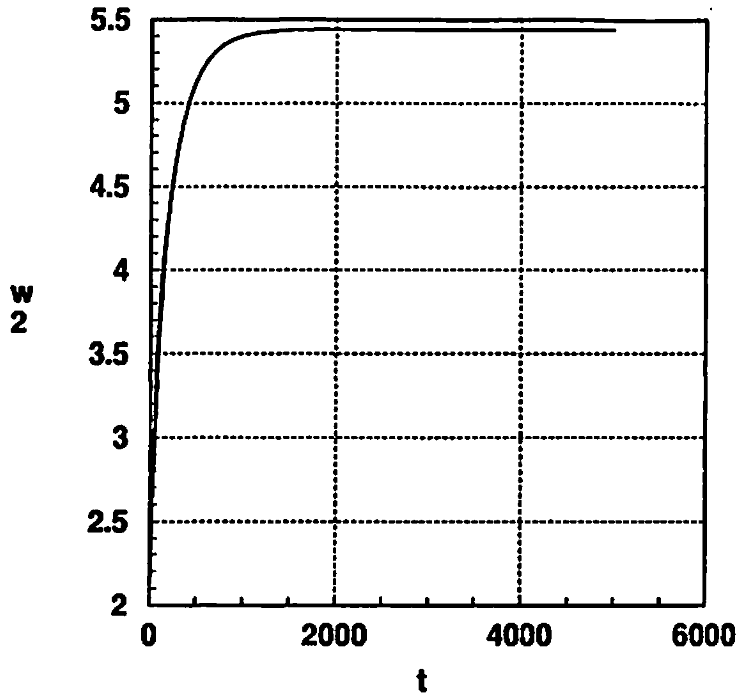


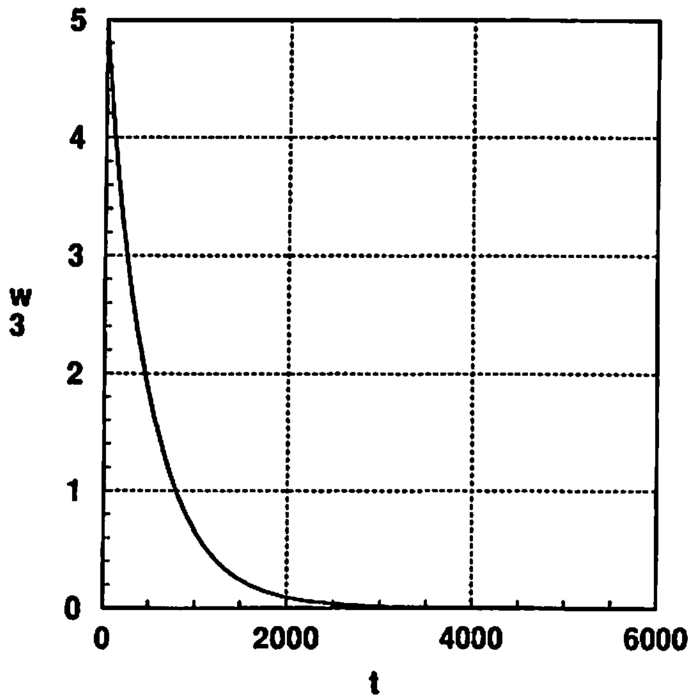


Rigid body with 1 internal rotor.

Stabilization law - Bloch, Marsden ...







Asymptotic Stabilization law.

rigid body with one internal rotor