Approximation of Constrained Dynamics via Augmented Potentials

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Abstract

In this paper, we consider the dynamics of mechanical systems where a holonomic constraint has been replaced with an additional potential energy term whose minimum lies along the constraint. Specifically, we consider the convergence of a sequence of integral curves corresponding to solutions to a sequence of dynamic systems where the additional potential is scaled by increasingly large constants. We present the results, due to Rubin and Ungar [7], which show that if the initial velocity lies along the constraint, then the integral curves converge to the constrained motion. If, however, the initial velocity has a component normal to the constraint, then the sequence exhibits oscillatory motion in that velocity component, and the motion along the constraint is affected as well. The theorem is illustrated by a simple example.

1 Introduction

Many mechanical systems exhibit holonomic constraints; that is, the allowable positions of the body within its configuration space are restricted to a submanifold of the configuration space. One example is the motion of a simple pendulum, where the fixed length of the pendulum restrict the motion to lie in S^1 rather than $\mathbb{R}^2$. The dynamics of such a system exhibit additional restoring forces which cause the body to remain on the submanifold [4].

Another way of introducing constraining forces of this type is by adding a potential term to the system Lagrangian whose basin lies along the submanifold. Such a term would produce forces in the direction of the gradient of the potential, and which would therefore cause the body to return to the submanifold. The slope of the potential well would determine the size of the constraining forces. It is possible that this would provide a good approximation for the constrained dynamics, or, conversely, that systems with large potentials could be approximated by constraints. This could lead to simplified models for diverse systems such as rigid body motion [7], [8], flexible body motion, fluid dynamics [2], statistical mechanics and quantum mechanics [8].

In this paper, we consider the validity of this hypothesis by asking whether the dynamics of a system with augmented potentials converges to the constrained dynamics as the slope of the potential well is increased; that is, as the potential is multiplied by a sequence of increasing constants whose limit is infinite. The body of the paper presents a result due to Rubin and Ungar [7] in this area. The motion of a simple pendulum is simulated in Section 5 to demonstrate the predictions of the theorem.

2 Constraints and Potentials: a little Geometry

Before presenting the results of [7], we consider how one generates potentials that are consistent with constraints in a geometric setting. Let M be a Riemannian manifold with dimension m, and let Let $L : TM \to \mathbb{R}$ be a Lagrangian on M of the usual form $L = T(x, \dot{x}) - V(x)$. Let $G : M \to \mathbb{R}^s$, $s < m$ be a submersion on M. By the Submersion Theorem [1], $N = G^{-1}(0)$ defines a submanifold of M with dimension $m - s$. We define this to be the submanifold to which we wish to constrain the motion. We can also generate an augmented potential by modifying the Lagrangian to be

$$L_k(x, \dot{x}) = L(x, \dot{x}) - \mu_k \|G(x)\|^2 \tag{1}$$

where $\{\mu_k\}_{k=1}^{\infty}$ is an increasing sequence of positive real numbers such that $\mu_k \to \infty$. Clearly, the additional term has N as its set of (local) minimum points, and hence generates restoring forces which point toward N. As k increases, the size of the restoring forces increases. Note that for a given submanifold N, there are infinitely many submersions G which generate N. We will have to address the effects of the choice of G on the resulting dynamics.

3 Constrained Motion with a Tangential Initial Velocity

We now present the results of [7]. This paper takes $M = \mathbb{R}^m$. For this reason, we now leave the geometric framework, and consider the theorem as a result in analysis, namely, on the convergence of a sequence of real functions, and take $T = \frac{1}{2}\dot{x} \cdot \dot{x}$. Insofar as the theorem is local, it is possible that is could be extended to any Riemannian manifold, but the technicalities of that are not considered here. It is clear, however, that this theorem is not valid for infinite- dimensional manifolds. In [2], a proof of this theorem is presented which applies to that case as well, but exposition of that proof is beyond the scope of this paper.

In their paper, they consider the convergence of a sequence of integral curves $q_k(t)$ generated from an initial condition $x_0, \dot{x}_0$. The theorem states the following:

Theorem: *Consider initial conditions $(x_0, \dot{x}_0) \in TN$, and let $x_k(t)$ be the integral curve corresponding to those initial conditions and the Lagrangian L_k. Then the following are true:*

1. *There exists $\delta > 0$ such that $x_k(t)$ is defined for $t \in [0, \delta]$ for all k.*
2. *The sequence $\{x_k(t)\}_{k=1}^{\infty}$ converges uniformly to a continuous function $x(t)$ for which $x(0) = x_0$.*
3. *$x(t) \in N \;\; \forall t \in [0, \delta]$.*
4. *The sequence $\{\dot{x}_k(t)\}_{k=1}^{\infty}$ converges uniformly to a continuous function $\dot{x}(t)$ for which $\dot{x}(0) = \dot{x}_0$.*
5. *There exist continuous functions $\lambda^i(t)$, $i = 1..s$ such that $\ddot{x} + \nabla V(x) + \sum_{i=1}^{s} \lambda_i(t) \nabla G^i(x) = 0$ identically in t.*

Proof: We begin by constructing the Hamiltonian H_k associated with each L_k:

$$H_k = \frac{1}{2}\dot{x} \cdot \dot{x} + V(x) + \mu_k \|G(x)\|^2 \tag{2}$$

Of course, H_k is constant along trajectories, and its initial value is given by

$$H = \frac{1}{2}\dot{x}_0 \cdot \dot{x}_0 + V(x_0) \tag{3}$$

since $G(x) = 0$ by assumption. Note that H is independent of μ_k, and hence the subscript has been dropped.

Equation (2) implies that the kinetic energy is bounded by

$$\frac{1}{2}\dot{x}_k \cdot \dot{x}_k \leq H - V(x_k) \tag{4}$$

Choose D to be a closed region of M which contains x_0. Since $V(q)$ is assumed to be C^1, it is bounded on D, and hence the family of functions $\{\dot{x}_k\}$ is uniformly bounded on D. This implies that $\dot{x}_k$ is complete on D for all k [9], meaning solutions $x_k(t)$ exist up to the boundary of D. It also guarantees the existence of each solution on a common interval $t \in [0, \delta]$, since the boundary of D can only be reached in some minimum finite time corresponding to the maximum possible velocity. Thus, Conclusion 1 is proven. Since each $x_k(t)$ is confined to D, $\{x_k(t)\}$ is uniformly bounded as well. Furthermore, the uniform boundedness of $\{\dot{x}_k\}$ on D bounds the growth of $x_k(t)$, implying that $\{x_k(t)\}$ is equicontinuous. Together, the equicontinuity and uniform boundedness of $\{x_k(t)\}$ imply by Arzela's theorem [3] that $\{x_k(t)\}$ is relatively compact in $C_{[0,\delta]}$. This implies [5] that $\{x_k(t)\}$ is sequentially compact, *i.e.*, that there exists a subsequence of $\{x_k(t)\}$ which converges to some function $x(t)$. This proves Conclusion 2 for a subsequence. That $x(t)$ is the limit of the entire sequence will be addressed later.

Next, note that equation (2) also implies that

$$\mu_k \|G(x_k)\|^2 \leq H - V(x_k) \tag{5}$$

Since the right hand side is bounded, and $\mu_k \to \infty$, we conclude that $\|G(q_k)\|^2 \to 0$, meaning that $q(t) \in N$, proving Conclusion 3 (for the convergent subsequence).

Until now, the proof has not required the imposition of special coordinates on the problem. To prove the remainder, however, we require two changes of coordinates. The first change of coordinates uses the constraint equation as the first s coordinates, *i.e.* $q^i = G^i(x), i = 1, .., s$. For the remaining $m - s$ equations, we introduce coordinates on each submanifold representing an inverse image of some point in $Rng(G)$ in such a fashion that curves of constant q^i across submanifolds are C^1. In this fashion, we generate a change of coordinates whose Jacobian J is nonsingular. In these coordinates, Equation (2) becomes

$$\frac{1}{2}\dot{q}^T(J^TJ)^{-1}\dot{q} + U(q) + \mu_k \sum_{i=1}^{s}(q^i)^2 = H \tag{6}$$

where U is V in q-coordinates. Next, we introduce coordinates

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = s+1, \ldots, m \tag{7}$$

which replace the corresponding $\dot{q}^i$. We then transform the Lagrangian to the Routhian R via

$$R = U(q) - \frac{1}{2}\dot{q}^T(J^TJ)^{-1}\dot{q} + \sum_{i=s+1}^{m} p_i\dot{q}^i \tag{8}$$

where $\dot{q}^i$ is considered a function of p_i for the final $m-s$ terms. This is reminiscent of the Legendre transform, except that we do not apply it to all the coordinates. We represent R using the following generic polynomial expansion:

$$R = U(q) - \sum_{i,j-1}^{s} a_{ij}(q)\dot{q}^i\dot{q}^j + 2\sum_{i=s+1}^{m}\sum_{j=1}^{s} a^i_j(q)p_i\dot{q}^j + \sum_{i,j=s+1}^{m} a_{ij}(q)p_ip_j \tag{9}$$

The Routhian coordinates have the effect of rendering the equations of motion Lagrangian for the first s coordinates and Hamiltonian for the remaining coordinates:

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}^i}\right) = \frac{\partial R}{\partial q^i} + 2\mu_k q^i, \quad i = 1, \ldots, s \tag{10}$$

and

$$\dot{p}_i = -\frac{\partial R}{\partial q^i}, \quad \dot{q}^i = \frac{\partial R}{\partial p_i}, \quad i = s+1, \ldots, m \tag{11}$$

In these coordinates, equation (3) can be written:

$$H = \sum_{i=s+1}^{n} p_i\dot{q}^i - R + 2U + \mu_k \sum_{i=1}^{s}(q^i)^2 \tag{12}$$

or

$$H = \sum_{i=s+1}^{n} p_i \frac{\partial R}{\partial p_i} - R + 2U + \mu_k \sum_{i=1}^{s} (q^i)^2 \tag{13}$$

Evaluating the first two terms using Equation (9) reveals that the $p_i \dot{q}^j$ terms will cancel and H becomes

$$H = \sum_{i,j=1}^{s} a_{ij}(q)\dot{q}^i\dot{q}^j + \sum_{i,j=s+1}^{n} a^{ij}(q)p_i p_j + U(q) + \mu_k \sum_{i=1}^{s} (q^i)^2 \tag{14}$$

We now see the significance of these coordinates, namely that the split the kinetic energy into two separate quadratic functions of the $\dot{q}^i$ and p_i terms with no coupling between them. Note that the $\dot{q}^i$ terms represent motion normal to the constraint submanifold, and the p_i terms represent motion along the constraint submanifold.

We now consider the convergence of each set of terms separately. Both sets of terms are uniformly bounded as a consequence of $\dot{x}(t)$ being uniformly bounded on D, as discussed earlier. Furthermore, Equation (11) implies that $\dot{p}$ is uniformly bounded on D, and hence we conclude that a subsequence of the $\dot{p}$ terms converge uniformly, by the same argument use earlier for the convergence of $x_k(t)$ sequence. For the $\dot{q}$ terms, however, Equation (10) reveals that $\ddot{q}_k(t)$ is not uniformly bounded in k, so another argument must be presented.

To address these terms, we begin by differentiating Equation (14) with respect to time:

$$\frac{d}{dt}\left(\sum_{i,j=1}^{s} a_{ij}(q)\dot{q}^i\dot{q}^j + \mu_k \sum_{i=1}^{s} (q^i)^2\right) = -\frac{d}{dt}\left(\sum_{i,j=s+1}^{n} a^{ij}(q)p_i p_j + U(q)\right) \tag{15}$$

and we evaluate the right hand side to yield the following equation:

$$\frac{d}{dt}\left(\sum_{i,j=1}^{s} a_{ij}(q)\dot{q}^i\dot{q}^j + \mu_k \sum_{i=1}^{s} (q^i)^2\right) = \sum_{i,j=1}^{s} A_{ij}\dot{q}^i\dot{q}^j + \sum_{i=1}^{s} B_i \dot{q}^i \tag{16}$$

where A_{ij}, B_i are continuous functions of q, p_i, and hence are also bounded on D. Integrating yields

$$\sum_{i,j=1}^{s} a_{ij}(q)\dot{q}^i\dot{q}^j + \mu_k \sum_{i=1}^{s} (q^i)^2 = \int_0^t \sum_{i,j=1}^{s} A_{ij}\dot{q}^i\dot{q}^j d\tau + \int_0^t \sum_{i=1}^{s} B_i \dot{q}^i d\tau \tag{17}$$

Examination of the left hand side reveals that it is zero at $t = 0$, since the initial velocity assumed to have no component normal to the constraint manifold. Thus, the constant of integration on the right hand side is zero. This is significant, as will be seen shortly. We now deal with each term separately, starting with the final term. We present the following technical lemma from [7] without proof:

Lemma 1: *If $\{g_k(t)\}$ is a sequence of continuous functions which converges uniformly to zero for $t \in [0, \lambda]$ and if the $g_k(t)$ possess uniformly bounded continuous derivatives $\dot{g}_k(t)$, and $\{h_k(t)\}$ is a sequence of continuous functions which converges uniformly as $k \to \infty$, then $\int_0^t h_k(\tau)\dot{g}_k(\tau)$ approaches zero uniformly in t for $t \in [0, \lambda]$ as $k \to \infty$.*

Applying this lemma to the final term in Equation (17), we observe that since B_i is a continuous function of q, p_i, it converges uniformly in k. Additionally, $q_k(t)$ is already known to have a subsequence converging to zero and to having uniformly bounded derivatives. Thus, the conditions of the lemma are satisfied (for the subsequence), and we may state that

$$\int_0^t \sum_{i=1}^{s} B_i \dot{q}_k^i d\tau \leq \epsilon_k \tag{18}$$

where ϵ_k converges to zero as $k \to \infty$. We may also remove the second term of the left hand side of (17) and state

$$\sum_{i,j=1}^{s} a_{ij}(q)\dot{q}^i\dot{q}^j \leq \int_0^t \sum_{i,j=1}^{s} A_{ij}\dot{q}^i\dot{q}^j d\tau + \epsilon_k \tag{19}$$

Next, we note that since A_{ij} is bounded, $\sum_{i,j=1}^{s} A_{ij}\dot{q}^i\dot{q}^j \leq C\sum_{i=1}^{s}(\dot{q}^i)^2$ for some C. Finally, we note that the remaining term on the left hand side is derived from the kinetic energy, and hence is positive definite at all q. Its eigenvalues are therefore bounded away from zero on D, and we may state that $\sum_{i,j=1}^{s} a_{ij}\dot{q}^i\dot{q}^j \geq C^* \sum_{i=1}^{s}(\dot{q}^i)^2$. Substituting into (19) yields

$$C^*\phi_k(t) \leq C\int_0^t \phi_k(\tau)d\tau + \epsilon_k \tag{20}$$

where $\phi_k(t) = \sum_{i,j=1}^{s} a_{ij}\dot{q}^i\dot{q}^j$.

Applying Gronwall's Inequality [Perko, p. 79] to this yields:

$$\phi_k(t) \leq \frac{\epsilon_k}{C^*} e^{\frac{C}{C^*} t} \tag{21}$$

from which we conclude that $\phi_k(t)$, and hence $\dot{q}_k(t)$ converge uniformly to zero. This, combined with the earlier result that $p_k(t)$ converges uniformly, leads to the conclusion that $\dot{x}_k(t)$ possesses a uniformly convergent subsequence, thus proving Conclusion 4 (for the subsequence). Again, we note that were the initial velocities normal to the manifold not zero, then the right-hand side would not converge uniformly to zero, and the proof would collapse.

Finally, we consider Hamilton's principle with respect to L_k and restrict the variations to satisfy $\delta x \cdot \nabla G^i(x_k) = 0$, meaning the variations are constrained to lie along the submanifold. Hamilton's principle states that

$$\delta \int_0^t \frac{1}{2} \dot{x}_k \cdot \dot{x}_k - U(x) - \mu_k G(x_k) \cdot G(x_k) d\tau = 0 \tag{22}$$

or

$$\int_0^t \delta x \cdot \dot{x}_k - \delta x \cdot \nabla U(x) - 2\mu_k \sum_{i=1}^{s} G^i(x_k) \delta x \cdot \nabla G^i(x_k) d\tau = 0 \tag{23}$$

and the final term is eliminated by the previous restriction on δx, yielding

$$\int_0^t \delta x \cdot \dot{x}_k - \delta x \cdot \nabla U(x_k) d\tau = 0 \tag{24}$$

Passing to the limit yields

$$\int_0^t \delta x \cdot \dot{x} - \delta x \cdot \nabla U(x) d\tau = 0 \tag{25}$$

from which we infer, using the theorem of Lagrange multipliers for constrained optimization problems, that $x(t)$ satisfies

$$\ddot{x} + \nabla U(x) + \sum_{i=1}^{s} \lambda_i(t) \nabla G^i(x) = 0 \tag{26}$$

for some continuous $\delta^i(t)$. Thus, Conclusion 5 is proven. Though not stated explicitly in [7], it follows from Conclusion 5 that $x(t)$ is in fact the solution for the constrained dynamics, since they, too, obey this equation. Equations (26) and the given initial conditions uniquely define $x(t)$. Since the μ_k terms were chosen arbitrarily, we conclude that any possible sequence of μ_k terms possesses a subsequence of solutions which converge to the same $x(t), \dot{x}(t)$, and conclude that the entire sequence must converge. [1]

4 Constrained Motion with Initial Velocity Normal to the Constraint

In the previous section, the assumption that the $\dot{q}^i$ terms normal to the manifold were zero was crucial to the proof. We now consider the case where those terms are nonzero. For simplicity, we consider a case where there is only one constraint, *i.e.* $s = 1$. Note first that Conclusions 1-3 did not depend on the tangency of the initial velocity, and hence we can already conclude that $q_k(t)$ converges, as does $p_k(t)$ from the previous section. To begin, we partition the energy equation (14) and define

$$e_k(t) = a_{11}(q_k)\left(\dot{q}_k^1\right)^2 + \mu_k\left(q_k^1\right)^2 \tag{27}$$

which is the energy corresponding to the motion normal to the constraint. We can thus rewrite equation (14) as

$$e_k(t) = H - \sum_{i,j=2}^{n} a^{ij}(q_k)p_{ik}p_{jk} - U(q_k) \tag{28}$$

The variables on the right hand side are already known to converge, so we can pass to the limit:

$$e(t) = H - \sum_{i,j=2}^{n} a^{ij}(q)p_i p_j - U(q) \tag{29}$$

[1]The final step in Rubin and Ungar's proof appears somewhat mysterious. They introduce the constraint on the variation for each $x_k(t)$, even though those solutions are not constrained to lie on the submanifold. Perhaps the justification is that they already know that the limiting case $x(t)$ will lie on the submanifold, and therefore the inclusion of the constraint on δx does not corrupt the derived equations of motion for x(t).

From here, we first seek equations of motion for $q^i, p_i,\ i = 2..n$. Returning to Equation (11), we evaluate the right hand side using Equation (9) to get

$$\dot{q_k}^i = 2a_1^i(q_k)\dot{q}_k^1 + 2\sum_{i,j=2}^{n} a^{ij}(q_k)p_{k_j} \tag{30}$$

which integrates to

$$q_k^i(t) - q^i(0) = \int_0^t 2a_1^i(q_k)\dot{q}_k^1 + 2\sum_{j=2}^{n} a^{ij}(q_k)p_{k_j}d\tau. \tag{31}$$

Using Lemma 1, we conclude that the first term on the right hand side goes to zero uniformly as $k \to \infty$, and hence in the limit we have

$$q^i(t) - q^i(0) = \int_0^t 2\sum_{j=2}^{n} a^{ij}(q_k)p_{k_j}d\tau \tag{32}$$

or

$$\dot{q}^i = 2\sum_{j=2}^{n} a^{ij}(q)p_j \tag{33}$$

Similarly, for $\dot{p_k}$, we have

$$\begin{aligned} p_{ik}(t) - p_i(0) \quad = \quad & -\int_0^t \left[\frac{\partial V(q_k)}{\partial q^i} - \frac{\partial a_{11}(q_k)}{\partial q^i}\left(\dot{q}_k^1\right)^2 + \right. \\ & \left. 2\sum_{j=2}^{n} \frac{\partial a_1^j(q_k)}{\partial q^i} p_{ik}\dot{q}_k^i + \sum_{j,l=2}^{n} \frac{\partial a^{jl}(q_k)}{\partial q^i} p_{kj}p_{kl}\right] \ d\tau \end{aligned} \tag{34}$$

When we pass to the limit, the first and last terms simply lose the k subscript as previously, and the third term goes to zero by Lemma 1. To deal with the second term, we cite another technical lemma:

Lemma 2: *If $\{h_k(t)\}$ is a sequence of continuous functions which converges uniformly for $t \in [0, \lambda]$ as $k \to \infty$ to a function $h(t)$, then some subsequence of $\int_0^t h_k(\tau)a_{11}(q_j)\left(\dot{q}_k^1(\tau)\right)^2 d\tau$ and $\int_0^t h_k(\tau)\mu_k\left(q_k^1(\tau)\right)^2 d\tau$ converge uniformly to the same limit, namely $\frac{1}{2}\int_0^t h(\tau)e(\tau)d\tau$.*

We can apply this lemma to the second term where

$$h_k(t) = \frac{\partial a_{11}(q_k)}{\partial q^i} \frac{1}{a_{11}(q_k)} \tag{35}$$

and conclude that in the limit,

$$\dot{p}_i = -\frac{\partial V(q)}{\partial q^i} + \frac{\partial a_{11}(q)}{\partial q^i} \frac{e(t)}{2a_{11}(q)} - \sum_{j,l=2}^{n} \frac{\partial a^{jl}(q)}{\partial q^i} p_j p_l \tag{36}$$

These equations, combined with Equation (29), represent the equations of motion when the initial velocity is not tangent to the constraint submanifold.

Noting that equations for $q(t), p(t)$ effectively involved removing from R the terms corresponding to $q^1, \dot{q}^1$, we can consider the new equations Hamiltonian be defining

$$R_0 = R|_{q^1=\dot{q}^1=0} \tag{37}$$

which yields

$$\dot{q}^i = \frac{\partial R_0}{\partial p_i}, \;\; \dot{p}_i = -\frac{\partial R_0}{\partial q^i} + \frac{\partial a_{11}(q)}{\partial q^i} \frac{e(t)}{2a_{11}(q)} \tag{38}$$

we then have

$$e(t) = H - R_0(q) \tag{39}$$

which, if we differentiate and substitute in Equation (38), yields

$$\dot{e}(t) = -e(t) \sum_{i=2}^{n} \frac{\partial a_{11}(q)}{\partial q^i} \frac{e(t)}{2a_{11}(q)} \dot{q}^i \tag{40}$$

which in turn integrates to

$$\ln e(t) = \ln(a_{11}(q)^{-1/2}) + K \tag{41}$$

or, more simply

$$\sqrt{a_{11}(q)}\, e(t) = K \tag{42}$$

where K is a constant determine by the initial conditions. If we redefine the Hamiltonian to be

$$\tilde{R}_0 = R_0 + \frac{K}{\sqrt{a_{11}}} \tag{43}$$

then we recover Hamilton's equations for $q(t), p(t)$:

$$\dot{q}^i = \frac{\partial R_0}{\partial p_i}, \;\; \dot{p}_i = -\frac{\partial R_0}{\partial q^i} \;\; i = 2, \ldots, m \tag{44}$$

This result can be extended to the case where multiple constraints exist.

5 Example

In this section we use the simple pendulum as an example to demonstrate the theorem. The simple pendulum is given with the following Lagrangian in $\mathbb{R}^2$:

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + gy \tag{45}$$

where g is the gravitational constant, and subject to the constraint

$$G(x, y) = x^2 + y^2 - 1 = 0 \tag{46}$$

This yields the well known pendulum equation

$$\ddot{\theta} = -g \sin\theta \tag{47}$$

where $\theta = \tan^{-1}(y/x)$. We consider as approximations to that system the augmented Lagrangian given in (1), which yields the following equations:

$$\begin{aligned} \ddot{x} &= -2\mu_k G(x, y)x && (48) \\ \ddot{y} &= -g - 2\mu_k G(x, y)y && (49) \end{aligned}$$

We begin by simulating with zero initial velocity, which obviously lies in TN. Figure 1 shows the response of x and y for $\mu_k = 1, 10, 100, 1000$ in the dashed lines, and the response of Equation (47) in the solid line. The convergence of the augmented potentials to the true solution is apparent. In fact, the $\mu_k = 1000$ case is indistinguishable from the true solution.

Next, we simulate responses for with a large non-tangential initial velocity. We now consider two different constraints: $G(x, y)$ as given before, and $G_1(x, y) = e^x G(x, y)$. The extra e^x term is positive everywhere, so $G_1 = 0$ still generates the same constraint submanifold. This new constraint generates the equations

$$\begin{aligned} \ddot{x} &= -\mu_k e^x (G^2(x, y) + 2G(x, y)x) && (50) \\ \ddot{y} &= -g - 2\mu_k e^x G(x, y)y && (51) \end{aligned}$$

Figure 2 shows the converged responses for the x and y variables ($\mu_k = 10000$). The response due to $G(x, y)$ is shown using the dashed line, and $G_1(x, y)$ using the dot-dashed line. As is seen there, both solutions converge, but to different curves, and neither of them corresponds to the correct solution. Though it is not obvious from the plots, both solutions lie on the unit

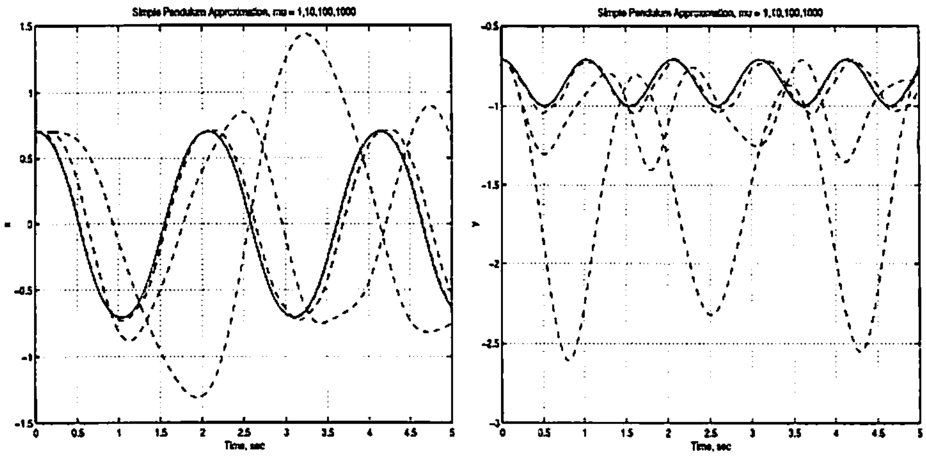

Figure 1: Convergence of Solutions with increasing μ_k

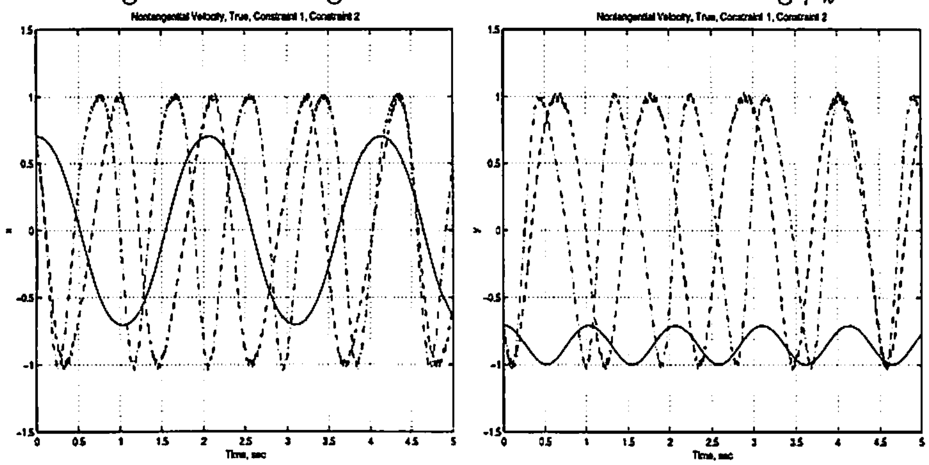

Figure 2: Solutions with Nontangential Velocity, Different Constraints, $\mu_k = 10000$

circle. This is consistent with the theory, which states that the solutions will converge to some curve on the submanifold, though not necessarily the correct curve, and that the choice of constraint affects the result. (Responses of $\dot{x}, \dot{y}$ were difficult to interpret due to noise from the integration, and hence are not shown).

6 Conclusions and a little more Geometry

The theorems presented show that care must be taken in approximating constrained dynamics by a constraining potential. While the theorem does confirm the intuitive notion that the integral curves ought converge to the constrained motion, the proof reveals that the convergence of the motion normal to the constraint to zero is not trivial. The second result, that initial conditions normal to the constraint submanifold fail to converge to the correct result, is indicative of the care with which the approximate equations must be used. (Note that it is possible to carefully construct a potential such that the dynamics along the constraint submanifold will still converge to the correct curve, as pointed in both [7] and [8]) Finally, the equations derived for the oscillatory dynamics normal to the constraint submanifold are also useful, as shown in [7], where these equations are used to derive a well-known equation for the motion of a charged particle in an axisymmetric magnetic field.

The theorem, as presented, is fundamentally a theorem in analysis. However, it is clear that several geometric concepts are lurking beneath the surface. As shown earlier, generation of both constraints and potentials by a single submersion can be done for arbitrary manifolds. A logical starting point for a geometric formulation of the proof would be the use of Routhian coordinates. In particular, the splitting of the kinetic energy into separate quadratic terms in $\dot{q}$ and p begs a geometric interpretation. The major hurdle to an intrinsic formulation of the proof appears carrying over concepts like uniform boundedness to an arbitrary manifold. It is possible that a Riemannian manifold has sufficient structure to use these concepts. In [2], a similiar theorem is proven for an arbitrary Riemannian manifold. A comparison of the two proofs would no doubt be instructive in this regard.

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