GENERATING FUNCTIONS OF LAGRANGIAN SUBMANIFOLDS

Tony Faradjian
Math 189
Prof. J. E. Marsden
5/20/95

ABSTRACT

M.J. Sewell and I. Roulstone discuss generating functions and Legendre transformations in "Anatomy of the canonical transformation," Phil. Trans. R. Soc. Lond. A (1993), 345, 577-598. I will compare their paper to W. M. Tulczyjew's, entitled "The Legendre Transformation," in Annales de l'Institut Henri Poincaré - Section A - Vol. XXVII, n° 1 - 1977, which is a more abstract and sophisticated treatment.
A short note on thermodynamics is included.

Smell and Ralston (S & R)

S & R have conveyed very well both the meaning and the importance of canonicalness, doubtless because of their commitment to concrete examples in the lowest possible dimension, i.e. mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. It is just the feature of their exposition which does so much to dispel the mystery surrounding generating functions.

Let $X(x,y)$, $Y(x,y)$ be a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, and suppose x and y satisfy Hamilton's equations for some Hamiltonian $h(x,y,t)$:

$$\dot{x} = -\frac{\partial h}{\partial y}, \qquad \dot{y} = \frac{\partial h}{\partial x}. \tag{1}$$

Let H be the Hamiltonian h expressed in terms of X and Y, i.e. $H(X,Y,t) = h(x,y,t)$, with X and Y each related to x and y as above. We compute the time derivatives of X and Y using the chain rule:

$$\dot{X} = \frac{\partial X}{\partial x}\dot{x} + \frac{\partial X}{\partial y}\dot{y} = -\frac{\partial X}{\partial x}\frac{\partial h}{\partial y} + \frac{\partial X}{\partial y}\frac{\partial h}{\partial x}$$

$$\dot{Y} = \frac{\partial Y}{\partial x}\dot{x} + \frac{\partial Y}{\partial y}\dot{y} = -\frac{\partial Y}{\partial x}\frac{\partial h}{\partial y} + \frac{\partial Y}{\partial y}\frac{\partial h}{\partial x}$$

Denoting by j the Jacobian of $X(x,y), Y(x,y)$, we have:

$$j\frac{\partial \dot{x}}{\partial Y} = \left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right)\left(\frac{\partial h}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial Y}\right)$$
$$= \frac{\partial X}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial h}{\partial x}.$$

(Here I have used $\frac{\partial Y}{\partial y}\frac{\partial y}{\partial Y} = 1$ and $\frac{\partial X}{\partial x}\frac{\partial x}{\partial Y} = \frac{\partial X}{\partial Y} = 0$, though I am not totally confident of the second one) – it is necessary to obtain S & K's results). Comparing this with the above expression for $\dot{X}$ and proceeding similarly with $j\frac{\partial H}{\partial X}$, we obtain:

$$\dot{X} = -j\frac{\partial H}{\partial Y} \quad ; \quad \dot{Y} = j\frac{\partial H}{\partial X} \qquad (2).$$

If $j=1$ on a region U of (x,y) space, (2) becomes Hamilton equations for X, Y. It is just the condition which S & K use to derive canonicalness. Evidently the mapping $(x,y) \longrightarrow (X,Y)$ must be a diffeomorphism ($j \neq 0$) for it to be invertible $(x = X(x,y), Y = Y(x,y))$; but it must have a <u>constant Jacobian</u> over some region to be a canonical diffeomorphism (if $j \neq 1$, simply rescale the variables).

Examples

1) [illegible]; $X = \frac{y+x}{\sqrt{2}}$ $Y = \frac{y-x}{\sqrt{2}}$. $h = x^2 + y^2$.

Then $x = \frac{X-Y}{\sqrt{2}}$, $y = \frac{X+Y}{\sqrt{2}}$, and $H = X^2 + Y^2$.

$\frac{\partial H}{\partial X} = 2X$, $\frac{\partial H}{\partial Y} = 2Y$,

$\dot{X} = \frac{1}{\sqrt{2}}(\dot{x} + \dot{y}) = \frac{1}{\sqrt{2}}\left(-\frac{\partial h}{\partial y} + \frac{\partial h}{\partial x}\right) = \sqrt{2}(-y+x) = -2Y = -\frac{\partial H}{\partial Y}$; and

$\dot{Y} = \frac{1}{\sqrt{2}}(\dot{y} - \dot{x}) = \frac{1}{\sqrt{2}}\left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\right) = \sqrt{2}(x+y) = 2X = \frac{\partial H}{\partial X}$.

(This is a 45° rotation of the axes with a paraboloid hamiltonian, so both Hamilton's equations and the form of H itself are invariant).

2) (S&E's) $X = \sqrt{x}\cos 2y$, $Y = \sqrt{x}\sin 2y$

$j = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x} = \left(\frac{1}{2}x^{-1/2}\cos 2y\right)\left(2\sqrt{x}\cos 2y\right) - \left(-2\sqrt{x}\sin 2y\right)\left(\frac{1}{2}x^{-1/2}\sin 2y\right)$

$= \cos^2 2y + \sin^2 2y = \boxed{1. = j}$ No matter what hamiltonian h we choose, the fact that $j = 1$ tells us X and Y will satisfy Hamilton's equations for H if x and y do for h.

This more sophisticated example of S&E's leads to the next level of detail in the study of canonical transformations.

Generating Functions

While the Jacobian in the previous example is indeed 1, nevertheless each of the partial derivatives vanishes periodically. S & R call this kind of a degeneracy an [illegible] singularity. In the absence of such a singularity we can perform a "partial" inversion of the canonical transformation [illegible] obtain a set of interesting relations. Consider the mapping $X = x$, $Y = x^2 + y$; the Jacobian is equal to 1, but also $\frac{\partial X}{\partial x} \equiv 1$ and $\frac{\partial Y}{\partial y} \equiv 1$, which means we can write $y = y(x, Y)$ as $y = Y - x^2$. Since $X = X(x, Y)$ trivially we have swapped independent variables: now we have $(x, Y) \longmapsto (X, y)$. Furthermore we can find a function $A(x, Y)$ such that $X = \frac{\partial A}{\partial Y}$ and $y = \frac{\partial A}{\partial x}$: $A(x, Y) = xY - \frac{1}{3}x^3$. The function A is a generating function for the canonical transformation ($(x, y) \to (X, Y)$ is canonical because $j = 1$). S & R generalize this observation in two theorems, which I merge into one.

Theorem 1 (S & R)

For a mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$, $X = X(x,y)$, $Y = Y(x,y)$ with $j = 1$:

i) if $\frac{\partial Y}{\partial y} \neq 0, \pm\infty$, then $X = X(x,Y)$, $y = y(x,Y)$, $\frac{\partial X}{\partial x} = \frac{\partial y}{\partial Y}$.

and $\exists\, A(x,Y)$ s.t. $X = \frac{\partial A}{\partial Y}$ and $y = \frac{\partial A}{\partial x}$;

ii) if $\frac{\partial X}{\partial y} \neq 0, \pm\infty$, then $y = y(x,X)$, $Y = Y(x,X)$, $\frac{\partial Y}{\partial x} = -\frac{\partial y}{\partial X}$,

and $\exists\, B(x,X)$ s.t. $y = -\frac{\partial B}{\partial x}$ and $Y = \frac{\partial B}{\partial X}$;

iii) if $\frac{\partial X}{\partial x} \neq 0, \pm\infty$, then $x = x(X,y)$, $Y = Y(X,y)$, $\frac{\partial Y}{\partial y} = \frac{\partial x}{\partial X}$,

and $\exists\, C(X,y)$ s.t. $x = -\frac{\partial C}{\partial y}$, and $Y = -\frac{\partial C}{\partial X}$;

iv) if $\frac{\partial Y}{\partial x} \neq 0, \pm\infty$, then $X = X(Y,y)$ $x = x(Y,y)$, $\frac{\partial X}{\partial y} = -\frac{\partial x}{\partial Y}$,

and $\exists\, D(Y,y)$ s.t $X = -\frac{\partial D}{\partial Y}$, and $x = \frac{\partial D}{\partial y}$.

Proof

S & R prove (i); I'll prove (iv). The existence of $X(Y,y)$ and $x(Y,y)$ is clear. To show $\frac{\partial X}{\partial y} = -\frac{\partial x}{\partial Y}$, use the chain rule: $X = X(x(Y,y),y)$ means $\frac{\partial X}{\partial y} = \frac{\partial X}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial X}{\partial y}$ (slight abuse of notation!), and $\frac{\partial x}{\partial y} = 0 = \frac{\partial x}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial x}{\partial y}$.

So $\frac{\partial X}{\partial y} = \frac{\partial X}{\partial y} - \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\frac{\partial x}{\partial Y} = -j\frac{\partial x}{\partial Y}$. Since $j = 1$, we've shown $\frac{\partial X}{\partial y} = -\frac{\partial x}{\partial Y}$. Now obtain the function D by integrating $-X$ with respect to Y, so that $\frac{\partial D}{\partial Y} = -X$. Then $\frac{\partial^2 D}{\partial y \partial Y} = -\frac{\partial X}{\partial y} = \frac{\partial x}{\partial Y} = \frac{\partial^2 D}{\partial Y \partial y} \Rightarrow \frac{\partial D}{\partial y} = x$. $\boxed{\text{QED}}$

The Legendre Quartet

The mapping $X = \frac{y+x}{\sqrt{2}}$, $Y = \frac{y-x}{\sqrt{2}}$ of Example 1 has $j=1$ but also has constant partial derivatives, so all four generating functions are available. We construct the generating functions $C(y,X)$ and $D(y,Y)$ by rewriting the transformation as a function of (y,X) and as a function of (y,Y): $x = \sqrt{2}X - y$ & $Y = \sqrt{2}y - X$ gives $C(y,X) = \frac{1}{2}y^2 - \sqrt{2}Xy + \frac{1}{2}X^2$; and $x = y - \sqrt{2}Y$ & $X = \sqrt{2}y - Y$ gives $D(y,Y) = \frac{1}{2}y^2 - \sqrt{2}Yy + \frac{1}{2}Y^2$. Computing $C+D$ we obtain:

$$
\begin{aligned}
C+D &= y^2 + \tfrac{1}{2}(X^2+Y^2) - \sqrt{2}\,y\,(X+Y) \\
&= \left(\frac{X+Y}{\sqrt{2}}\right)^2 + \tfrac{1}{2}(X^2+Y^2) - \sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right)(X+Y) \\
&= \tfrac{1}{2}(X^2+2XY+Y^2) + \tfrac{1}{2}(X^2+Y^2) - (X^2+2XY+Y^2) \\
&= -XY.
\end{aligned}
$$

We see, then, that C and D related by a <u>Legendre transformation</u>. Indeed we recognize $Y = -\frac{\partial C}{\partial X}$ as the analog of $p = \frac{\partial L}{\partial \dot{q}}$, where L is a Lagrangian, $\dot{q}$ a velocity and p the corresponding momentum. (The same can be said for $X = -\frac{\partial D}{\partial Y}$, reflecting the

<u>duality</u> of the Legendre transformation, a matter treated in Lanczos, <u>The Variational Principles of Mechanics</u>, Chapter VI, Section 1). It is straightforward to generalize this discussion:

<u>Theorem 2 (S & R)</u>

If all four generating functions of the canonical transformation $X(x,y)$, $Y(x,y)$ exist, they are related by a <u>quartet of Legendre transformations</u>:

$$A + B = XY \qquad C + D = -XY$$

$$B + C = -xy \qquad D + A = xy\,.$$

The utility of this construction, this "quartet," is not clear to me. As long as two generating functions with a common independent variable ("passive variable") exist, they will be related by the Legendre dual transformation. The only noteworthy feature to me is that in the sums $A + B$, ..., the products on the right side of the equation involve variables from the same space—the domain or the range

FORMALISM

In 1977, W.M. Tulczyjew discussed the relationship between Legendre transformations and generating functions in "The Legendre Transformation." He took as his point of departure A. Weinstein's notion of Lagrangian submanifold and articulated the theory in the language of symplectic geometry. Providentially, he supplemented his abstract discussion with sections that spell things out in coordinates.

$\varphi(x,y) = (Y(x,y), X(x,y))$ is a mapping from a symplectic manifold P_1 to another, P_2, where $P_1 = \mathbb{R}^2$ and $P_2 = \mathbb{R}^2$. The graph of φ, which I denote by $gr(\varphi)$, is a submanifold of $(P_2 \times P_1, \omega_2 \ominus \omega_1)$, where ω_2 and ω_1 are the symplectic forms of P_2 and P_1, respectively, and $\omega_2 \ominus \omega_1 = dY \wedge dX - dy \wedge dx$. Let us evaluate $\omega_2 \ominus \omega_1$ on $gr(\varphi)$.

$$\omega_2 \ominus \omega_1 \big|_{gr(\varphi)} = dX \wedge dY - dy \wedge dx$$
$$= \left(\frac{\partial X}{\partial x}\,dx + \frac{\partial X}{\partial y}\,dy\right) \wedge \left(\frac{\partial Y}{\partial x}\,dx + \frac{\partial Y}{\partial y}\,dy\right) - dy \wedge dx$$

$$= \frac{\partial \chi}{\partial x}\frac{\partial \eta}{\partial y}\, dx\, dy + \frac{\partial \chi}{\partial y}\frac{\partial \eta}{\partial x}\, dy\, dx - dy\, dx$$

$$= (j-1)\, dy\, dx ,$$

where j is the Jacobian of φ. We see, then, that <u>$\omega_2 \ominus \omega_1 | gr(\varphi)$ vanishes if and only if φ is a symplectic (canonical) transformation from P_1 to P_2.</u> Tulczyjew proves this in the most general case as follows.

<u>Definition</u> Let (P, ω) be a symplectic manifold, and let N be a submanifold of P with $\dim P = 2 \dim N$. If $\omega | N = 0$ then N is called a <u>lagrangian submanifold.</u>

<u>Theorem 1</u> The graph of a symplectic diffeomorphism φ of (P_1, ω_1) onto (P_2, ω_2) is a lagrangian submanifold of $(P_2 \times P_1, \omega_2 \ominus \omega_1)$. (W.M. Tulczyjew)

By definition, $\omega_2 \ominus \omega_1 = pr_2^* \omega_2 - pr_1^* \omega_1$, where pr_1, pr_2 are the canonical projections of $P_2 \times P_1$ onto P_1 and P_2, and, of course, $\omega_1 = \varphi^* \omega_2$. This theorem is also proved in Marsden and Kato ("Intro to Mechanics and Symmetry", p. 157. Tulczyjew next gives this more coordinate expression for $\omega_2 \ominus \omega_1|_{gr}(f)$, from which we wrote $\omega_2 \ominus \omega_1 | gr(f)$ for the case $f(x,y) = (f(x,y), g(x,y))$ above.

<u>Generating Functions, or: "How to make a Lagrangian submanifold.</u>

Let Q be a manifold and F a function on a submanifold K of Q. Let N denote the image of K under $\underline{d}F$; N is a section of T^*Q. Let $\rho = \pi_Q|N$, where π_Q is the cotangent bundle projection; let Θ_Q denote the canonical 1-form on T^*Q; let $\alpha_k \in N$, and let $u_k \in T_k K$. Then:

$$\Theta_{Q,\alpha_k}(u_k) = \langle \alpha_k , T\pi_Q \cdot u_k \rangle$$

$$= \langle \underline{d}F(k), T\pi_Q \cdot u_k \rangle$$

$$= (\rho^* \underline{d}F)_{\alpha_k} (u_k).$$

This means $\Theta_Q|N = \rho^* \underline{d}F$. Differentiating, we get $\omega_Q|N = \underline{d}(\Theta_Q|N) = \underline{d}\rho^*\underline{d}F = \rho^*\underline{d}(\underline{d}F) = 0$. This is the proof of Tulczyjew's

<u>Proposition</u> Let K be a submanifold of Q, and $F: K \longrightarrow \mathbb{R}$. Then $N = \{\alpha_k \in T^*Q \mid k \in K$ and $\langle \alpha_k, u_k \rangle = \underline{d}F(k) \cdot u_k \;\; \forall\, u_k \in T_k K\}$ is a Lagrangian submanifold of (T^*Q, ω_Q).

Since Θ_Q is the canonical 1-form on T^*Q, $\omega_Q = d\Theta_Q$ (or minus) is canonical 2-form on T^*Q. If you have a manifold Q and a function F on Q, then you have Lagrangian submanifold of (T^*Q, ω_Q). F is called the <u>generating function of the Lagrangian submanifold</u> and N is said to be <u>generated by F.</u>

If Sewell and Roulstone are to be trusted, F should also be a generating function of <u>canonical transformations</u> – indeed, the graphs of the transformations must be the Lagrangian submanifold. I've come up with a few examples which we can use as tests. For all of them $Q = \mathbb{R}^2$, $T^*Q = \mathbb{R}^4$.

<u>Example 3</u> $F(x,y) = x^2+y^2$. $dF = 2x\,dx + 2y\,dy$.

So $N = \{(x,y,\Psi,\chi) \in \mathbb{R}^4 \mid \Psi = 2x, \chi = 2y\}$.

$\varphi_1(x,\chi) = (\Psi, y) = (2x, \frac{\chi}{2})$ has $j = \begin{vmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = 1$.

Yes! φ_1 is a canonical transformation $(x,\chi) \mapsto (\Psi, y)$.

Since it is a diffeomorphism, it's inverse, $(\Psi, y) \mapsto (x, \chi)$ is also a canonical transformation.

<u>Example 4</u> $F(x,y) = e^{x+y}$; $dF = e^{x+y}\,dx + e^{x+y}\,dy$. The Lagrangian submanifold is $N = \{(x,y,\xi,\chi) \in \mathbb{R}^4 \mid \xi = \chi = e^{x+y}\}$.

$\varphi_1(y,\chi) = (\xi, x) = (\chi, \ln\chi - y)$ has $j = \begin{vmatrix} 0 & 1 \\ -1 & \frac{1}{\chi} \end{vmatrix} = 1$.

$\varphi_2(x,\chi) = (\xi, y) = (\chi, \ln\chi - x)$ has $j = \begin{vmatrix} 0 & 1 \\ -1 & \frac{1}{\chi} \end{vmatrix} = 1$.

So here we've found <u>two</u> canonical transformations.

<u>Example 5</u> $F(x,y) = xy^2$; so $dF = y^2\,dx + 2xy\,dy$. The Lagrangian submanifold is $N = \{(x,y,\xi,\chi) \in \mathbb{R}^4 \mid \xi = y^2, \chi = 2xy\}$.

$\varphi(y,\chi) = (x,\xi) = \left(\frac{\chi}{2y}, y^2\right)$ has $j = \begin{vmatrix} -\frac{\chi}{2y^2} & \frac{1}{2y} \\ 2y & 0 \end{vmatrix} = 1$.

So φ is symplectic

<u>Example 6</u> $F(x,y) = \frac{1}{2}x^2 - \sqrt{2}\,xy + \frac{1}{2}y^2$.

$dF = (x - \sqrt{2}\,y)\,dx + (-\sqrt{2}\,x + y)\,dy$. The Lagrangian submanifold is $N = \{(x,y,\xi,\chi) \in \mathbb{R}^4 \mid \xi = x - \sqrt{2}\,y,\ \chi = -\sqrt{2}\,x + y\}$.

$\varphi_1(x,\chi) = (\xi, y) = (-x - \sqrt{2}\,\chi, \chi + \sqrt{2}\,x)$ has $j = \begin{vmatrix} -1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{vmatrix} = 1$.

$\varphi_2(y,\chi) = (\xi, x) = \left(\frac{-\chi - y}{\sqrt{2}}, \frac{y - \chi}{\sqrt{2}}\right)$ has $j = \begin{vmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{vmatrix} = 1$.

φ_2 is Example 1 (above) in disguise.

In the above examples I have not employed any particular method for extracting the symplectic maps from the graph — it's been trial and error. But S & K tell us that we need only find one symplectic map, compute its partial derivatives and look for internal singularities (Theorem 1 (S & K) above). In Example 6, for instance, no such singularities exist, and a moment's study shows that I have listed only two of four possible canonical transformations.

Tulczyjew has abstracted these concepts by introducing the special symplectic manifold.

Definition (Tulczyjew)

Let (P, Q, π) be a differentiable fibration and Θ a 1-form on P. The quadruple (P, Q, π, Θ) is called a special symplectic manifold if there is a diffeomorphism $\alpha : P \to T^*Q$ such that $\pi = \pi_Q \circ \alpha$ and $\Theta = \alpha^* \Theta_Q$, where Θ_Q is the canonical 1-form on T^*Q. α will be unique.

If (P, Q, π, Θ) is a special symplectic manifold, then $(P, \omega) = (P, d\Theta)$ is a symplectic manifold called the underlying symplectic manifold of (P, Q, π, Θ).

Although I am not familiar with differentiable fibrations, the picture here is quite clear: Tulczyjew is using P as a substitute cotangent bundle (T^*Q), and π and Θ as substitutes for τ_Q and Θ_Q, all via the diffeomorphism α. The following facts strike me as remarkable, but Tulczyjew states them without proof; conceptually, they are easy to accept.

1) If (P, Q, π, Θ) is a special symplectic manifold, K a submanifold of Q, and $F : K \longrightarrow \mathbb{R}$, then the set $N = \{ p \in P \mid \pi(p) \in K$ and $\langle \Theta, v \rangle = \langle dF, T\pi(v) \rangle \; \forall\, v \in TP$ s.t. $\tau_P(v) = p$ and $T\pi(v) \in TK \}$ is a Lagrangian submanifold of $(P, d\Theta)$ said to be generated with respect to (P, Q, π, Θ) by F.

2) The diffeo $\alpha : P \longrightarrow T^*Q$ maps N onto the Lagrangian submanifold of (T^*Q, ω_Q) generated by F!

Next, Tulczyjew establishes $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \Theta_2 \ominus \Theta_1)$ as a special symplectic manifold, given $(P_2, Q_2, \pi_2, \Theta_2)$ and $(P_1, Q_1, \pi_1, \Theta_1)$ are themselves special symplectic manifolds and the definition of $\Theta_2 \ominus \Theta_1$ is directly analogous to that of $\omega_2 \ominus \omega_1$ in Theorem 1 (Tulczyjew). The proof uses $\alpha_{21} : P_2 \times P_1 \longrightarrow T^*(Q_2 \times Q_1) : (p_2, p_1) \longmapsto (\alpha_2(p_2), -\alpha_1(p_1))$ as the unique "special diffeomorphism" (I'm inventing the term — my meaning should be clear from the definition above); α_{21} has the property that $\tau_2 \times \tau_1 = (\pi_{Q_2} \times \pi_{Q_1}) \circ \alpha_{21}$ and $\Theta_2 \ominus \Theta_1 = \alpha_{21}^*(\Theta_{Q_2} \oplus \Theta_{Q_1})$, where $\Theta_{Q_2} \oplus \Theta_{Q_1} = pr_2^* \Theta_{Q_2} + pr_1^* \Theta_{Q_1}$ is identified with $\Theta_{Q_2 \times Q_1}$. I don't have a rock solid grasp of this proof, but these constructions seem reasonable to me.

Finally, Tulczyjew defines the generating function G of a symplectic diffeomorphism φ as the function G on a submanifold M of $Q_2 \times Q_1$ that generates its graph (a Lagrangian submanifold of $(P_2 \times P_1, \omega_2 \ominus \omega_1)$) with respect to the special symplectic structure $(P_2 \times P_1, Q_2 \times Q_1, \tau_2 \times \tau_1, \Theta_2 \ominus \Theta_1)$.

The reason Tulczyjew defines special and underlying symplectic manifolds will become clear shortly, when I discuss the generalized definition of canonical transformation.

Local Expressions

Tulczyjew provides local expressions of the formalism above. They are very helpful in making the connection to the linear examples I introduced above while discussing S & K's paper.

Let (x^i), $1 \le i \le n$ be coordinates on a manifold Q_1 and let (x^i, y_i) be canonical coordinates on $P_1 = T^*Q_1$. Let K_1 be a submanifold of Q_1 defined by

$$U^\kappa(x^i) = 0, \quad 1 \le \kappa \le k,$$

and let $F_1(x^i)$ be the continuation of F_1 to Q_1. Tulczyjew gives

$$y_i dx^i = \underline{d}\left(F_1(x^i) + \lambda_\kappa U^\kappa(x^i)\right) \qquad (1)$$

as the equation defining N_1, the Lagrangian submanifold of P_1 generated by F_1. If $K_1 = Q_1$, then the expression reduces to $y_i dx^i = \underline{d}F_1(x^i)$, which means $y_i = \frac{\partial F_1}{\partial x^i}$.

Indeed the naive Examples 3, 4, 5, and 6 which I concocted above agree with this procedure.

For instance, the Lagrangian submanifold N of Example 5 is defined by $N = \{(x^1, x^2, y_1, y_2) \mid y_1 = (x^2)^2, y_2 = 2x^1x^2\}$, and y_1, y_2 were identified with $\frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^2}$, where $F(x^1, x^2) = x^1(x^2)^2$. If I had chosen Example 3 instead, I might choose to restrict $F(x^1, x^2) = (x^1)^2 + (x^2)^2$ to the x^1-axis (the condition $u^K(x^i) = 0$ becomes $x^2 = 0$), in which case we must use ($\underline{1}$) to obtain N.

The Legendre Transformation

Tulczyjew makes the same observation about generating functions and their relationship via Legendre transformations as did S & K (or, rather, the other way around, since S & K's paper is 16 years later). It goes as follows.

Let (P, ω) be the underlying symplectic manifold of two special symplectic manifolds $(P, Q_1, \pi_1, \Theta_1)$ and $(P, Q_2, \pi_2, \Theta_2)$. A Lagrangian submanifold of (P, ω) may be generated with respect to both special symplectic structures.

<u>Definition 5.1 (Tulczyjew)</u> The transition from the representation of Lagrangian submanifolds of (P, ω) by generating functions w.r.t. $(P, Q_1, \pi_1, \Theta_1)$ to the representation by generating functions w.r.t. $(P, Q_2, \pi_2, \Theta_2)$ is called the <u>Legendre transformation</u>

CASE STUDY: Thermodynamics.

The internal energy of an ideal gas is given by S & R as: $U(v,S) = c_v v^{1-\gamma} e^{S/c_v}$, where v is the specific volume (volume per mole), and S is the entropy. Taking gradients, we find:

$$\frac{\partial U}{\partial v} = (1-\gamma) c_v v^{-\gamma} e^{S/c_v} = p \quad ; \quad \text{and}$$

$$\frac{\partial U}{\partial S} = v^{1-\gamma} e^{S/c_v} = \frac{pv}{pv^\gamma} e^{S/c_v} = \frac{RTe^{S/c_v}}{Re^{S/c_v}} = T,$$

using $pv = RT$, $pv^\gamma = Re^{S/c_v}$, and $c_v = \frac{R}{\gamma-1}$. S & R tell us to solve the second of these for S and substitute into the first; we obtain a mapping $(p,v) \longmapsto (S,T)$:

$$S(p,v) = c_v \ln\left(\frac{pv^\gamma}{R}\right)$$

$$T(p,v) = \frac{pv}{R}.$$

The determinant of the Jacobian is

$$\begin{vmatrix} \frac{\partial S}{\partial v} & \frac{\partial S}{\partial p} \\ \frac{\partial T}{\partial v} & \frac{\partial T}{\partial p} \end{vmatrix} = \begin{vmatrix} \frac{\gamma c_v}{v} & \frac{c_v}{p} \\ \frac{p}{R} & \frac{v}{R} \end{vmatrix} = \frac{c_v}{R}(\gamma - 1) = \boxed{1}$$

Evidently, <u>$S(p,v)$, $T(p,v)$ is a canonical transformation with generating function U.</u>

Tulczyjew similarly devotes a section to thermostatics and remarks that U (internal energy), F (the Helmholtz function), G (the Gibbs potential), and H (enthalpy) are all generating functions of the canonical transformation(s (Lagrangian submanifolds). He lists examples of the twelve possible Legendre transformations.

CONCLUSION

While S&R's exposition does not match Tulczyjew's in sophistication and generality, their paper has been vital to a clear conceptual understanding of what it means for a map to be symplectic. They have not said anything new, however. Tulczyjew's paper is not cited in their references, and yet 16 years before them he had fully detailed the theory in the general language of symplectic geometry. The study of both papers is a fruitful undertaking, however, as they each shed light one or the other.