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CONSTRUCTION OF A POISSON BRACKET FOR MAXWELL'S EQ'S USING REDUCTION.

ABSTRACT: WE START WITH THE CANONICAL SYMPLECTIC STRUCTURE ON THE PHASE SPACE AND APPLY REDUCTION BY THE GAUGE INVARIANCE SYMMETRY TO GET A POISSON ~~MANIFOLD~~ BRACKET ON THE REDUCED SPACE OF E AND B.

THE CONFIGURATION SPACE FOR MAXWELL'S EQUATIONS IS TAKEN AS THE SPACE U OF VECTOR FIELDS A ON $\mathbb{R}^3$. THE COTANGENT BUNDLE T^*U HAS ELEMENTS (A, Y) WHERE AN ELEMENT Y OF THE DUAL OF U IS A VECTOR FIELD DENSITY ON $\mathbb{R}^3$ AND THE PAIRING BETWEEN A AND Y IS GIVEN BY INTEGRATION.

THE CANONICAL SYMPLECTIC STRUCTURE ω ON T^*U IS GIVEN BY

$$\omega((A_1, Y_1), (A_2, Y_2)) = \int (Y_2 \cdot A_1 - Y_1 \cdot A_2)\, d^3x$$

SO GIVEN A FUNCTION $F(A, Y)$ ON T^*U WE CALCULATE ITS ~~FLOW~~ Ham Vf BY

$$\omega(X_F, V) = dF \cdot V = \int \left(\frac{\delta F}{\delta A} \cdot A + \frac{\delta F}{\delta Y} \cdot Y\right) d^3x$$

WHERE $V \equiv (A, Y) \in T^*U$

AND SINCE $\omega(X_F, V) = \int (Y A_F - Y_F A)\, d^3x$

WE GET:

$$X_F = \left(\frac{\delta F}{\delta Y}, -\frac{\delta F}{\delta A}\right)$$

SO THE ASSOCIATED POISSON BRACKET OF FUNCTIONS $F, G : T^*U \longmapsto \mathbb{R}$ IS

$$\{F, G\} = \omega(X_F, X_G) = \int \left(\frac{\delta F}{\delta A} \cdot \frac{\delta G}{\delta Y} - \frac{\delta F}{\delta Y} \cdot \frac{\delta G}{\delta A} \right) d^3x$$

NOW WE WRITE THE HAMILTONIAN

$$H(A, Y) = \frac{1}{2}\int |Y|^2 d^3x + \frac{1}{2}\int |\mathrm{curl}\, A|^2 d^3x$$

AND CHECK THAT IT GENERATES MAXWELL'S EQUATIONS

$$\left(\frac{\partial A}{\partial t}, \frac{\partial Y}{\partial t} \right) = X_H$$

$$X_H = \left(\frac{\delta H}{\delta Y}, -\frac{\delta H}{\delta A} \right) = (Y, -\mathrm{curl}\, A)$$

$$\therefore \frac{\partial A}{\partial t} = Y, \qquad \frac{\partial Y}{\partial t} = -\mathrm{curl}\,\mathrm{curl}\, A$$

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$$* \; \Delta H(A, Y)\Big|_{\delta A} = \int (\mathrm{curl}\, A \cdot \mathrm{curl}\, \delta A)\, d^3x$$

$$= \int (\mathrm{curl}\,\mathrm{curl}\, A) \cdot \delta A \, d^3x + \int \mathrm{div} \cdot [(\mathrm{curl}\, A) \cdot \delta A]\, d^3x$$

SURFACE INTEGRAL $= 0$ IF curl A AND δA GO TO ZERO AT INFINITY.

IF WE CALL $Y = -E$ AND $\mathrm{curl}\, A = B$ WE GET THE HAMILTONIAN

$H = \frac{1}{2}\int (|E|^2 + |B|^2)\, d^3x$ AND MAXWELL'S EQS :

$$\frac{\partial E}{\partial t} = \mathrm{curl}\, B, \qquad \frac{\partial B}{\partial t} = -\mathrm{curl}\, E$$

NOW WE NOTICE THAT THIS SYSTEM HAS A GAUGE INVARIANCE SYMMETRY : SINCE H DEPENDS ONLY ON CURL A, ADDING A GRADIENT TO $\underline{A}$ LEAVES H INVARIANT. SO THE GAUGE GROUP G CONSISTS OF REAL FUNCTIONS ϕ ON $\mathbb{R}^3$ AND THE GROUP ACTION CONSISTS OF ADDITION OF $\nabla\phi$ TO A:

$$A \longrightarrow A + \nabla\phi \quad \text{OR} \quad (A, Y) \longrightarrow (A + \nabla\phi, Y)$$

THE LIE ALGEBRA OF G CONSISTS OF THE REAL VALUED FUNCTIONS ON $\mathbb{R}^3$.

NOW WE CALCULATE THE MOMENTUM MAP $J : T^*U \longrightarrow \mathfrak{g}^*$ OF THE ACTION.

LET ξ BE AN ELEMENT OF $\mathfrak{g}$, $\xi = \partial\phi/\partial t$, SO THE INFINITESIMAL GENERATOR OF THE ACTION RELATED TO ξ IS

$$\frac{\partial}{\partial t}(A + \nabla\phi, Y) = (\nabla\xi, 0) \equiv \xi_U$$

NOW WE CALCULATE THE MOMENTUM MAP BY

$$dJ(\xi).(A, Y) = \omega(\xi_U, (A, Y)) = \omega((\nabla\xi, 0), (A, Y))$$

$$= \int Y . \nabla\xi \, dx$$

WHERE (A, Y) IS AN ARBITRARY VECTOR OF T^*U

SO $J(\xi)$ IS THE LINEAR FUNCTION

$$J(\xi)(A, Y) = \int Y . \nabla\xi \, dx = -\int (\operatorname{div} Y)\, \xi \, dx$$

SO $J(A, Y) = -\operatorname{div} Y$

SO WE KNOW THAT $\operatorname{div} Y = -\rho \in \mathfrak{g}^*$ IS A CONSTANT OF THE MOTION, OR IF WE WRITE IT IN TERMS OF E WE GET MAXWELL'S EQ. $\operatorname{div} E = \rho$.
(ACTUALLY, THE REASON MAXWELL FIRST INVENTED THE TERM $\partial E/\partial t$ OR "DISPLACEMENT CURRENT" IN MAXWELL'S EQS WAS TO CONSERVE CHARGE IN THE OLD ELECTRODYNAMICS)
NOW WE FORM $J^{-1}(\rho) = \{(A, Y) \in T^*U \mid \operatorname{div} Y = -\rho\}$ AND USE A GENERAL THEOREM ON REDUCTION (MARSDEN & WEINSTEIN, [2]) TO REDUCE THE CANONICAL FORM ω ON T^*U TO A SYMPLECTIC FORM ON $J^{-1}(\rho)/G$.

FIRST WE NOTICE THAT WE CAN IDENTIFY $J^{-1}(\rho)/G$ WITH $M_{ax} = \{(E, B) \mid \operatorname{div} E = \rho, \operatorname{div} B = 0\}$ BECAUSE IF WE ASSOCIATE TO EACH (A, Y) IN $J^{-1}(\rho)/G$ THE PAIR

$$(B, E) \doteq (\operatorname{curl} A, -Y)$$

AND BECAUSE EVERY $B \mid \nabla \cdot B = 0$ CAN BE WRITTEN AS A curl $B = \operatorname{curl} A$ UP TO THE INDEFINITE $A + \nabla \phi$ $(\operatorname{curl} \nabla \phi = 0)$ JUST THE EQUIVALENCE CLASS OF G, SO THE CORRESPONDENCE IS 1-1.
NOW TO GET THE BRACKET FOR FUNCTIONS F, G OF $M_{ax} \to \mathbb{R}$ WE PULL THEM BACK TO $J^{-1}(\rho)$ BY

$$\hat{F}(A, Y) = F(\operatorname{curl} A, -Y) = F(B, E)$$

AND NOW WE THINK OF $\hat{F}(A, Y)$ AS A FUNCTION ON T^*U AND CALCULATE THE CANONICAL BRACKET IN T^*U:

$$\{\{F, G\}\} = \{\hat{F}, \hat{G}\} = \int \left(\frac{\delta \hat{F}}{\delta A} \cdot \frac{\delta \hat{G}}{\delta Y} - \frac{\delta \hat{F}}{\delta Y} \frac{\delta \hat{G}}{\delta A} \right) d^3x$$

$$= -\int \left(\frac{\delta \hat{F}}{\delta A} \cdot \frac{\delta \hat{G}}{\delta E} - \frac{\delta \hat{F}}{\delta E} \cdot \frac{\delta \hat{G}}{\delta A} \right) d^3x$$

NOW WE USE THE DEFINITION OF FUNCTIONAL DERIVATIVE

$$\Delta F\Big|_{A'} = \int \frac{\delta \hat{F}}{\delta A} . A' d^3x = \int \frac{\delta F}{\delta B} . \text{curl}\, A' . d^3x$$

WHICH CAN BE INTEGRATED BY PARTS TO GIVE

$$\int \frac{\delta \hat{F}}{\delta A} . A' d^3x = \int A' . \text{curl}\, \frac{\delta F}{\delta B}\, d^3x$$

SO WE CAN WRITE THE BRACKET ON THE E, B SPACE:

$$\boxed{\{\{F, G\}\} = \int \left(\frac{\delta F}{\delta E} . \text{curl}\, \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} . \text{curl}\, \frac{\delta F}{\delta B} \right) d^3x}$$

THIS BRACKET WAS USED BY BORN AND INFELD [3] IN 1934 TO QUANTIZE THE ELECTROMAGNETIC FIELD, THEY DIDN'T DERIVE IT FROM A CANONICAL AS WE DID HERE, ON THE REFERED PAPER (3) THE BRACKET WAS ASSUMED AND THEY DIDN'T KNOW

? IT WAS related to A LIE ALGEBRA.

REFERENCES

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2. J. MARSDEN, A. WEINTEIN, REDUCTION OF SYMPLECTIC MANIFOLDS WITH SYMMETRY, REPORTS ON MATH. PHYS. 5 (1974) 121-130

3. M. BORN, L. INFELD, ON THE QUANTIZATION OF THE NEW FIELD THEORY, PROC. ROY. SOC. A 150 (1935) 141-