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(?)

19S FINAL PROJECT

Excellent Paper!

The Calogero hamiltonian

$$H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} 1/(x_i - x_j)^2$$

is shown to be integrable by showing that it is a reduction of the motion on $T^* u(n)^*$ due to the hamiltonian $H = \frac{1}{2} \mathrm{Tr}\, B^2$ (where B is momentum on $T^* u(n)^*$). The group action in question is the lift of the coadjoint action to $T^* u(n)^*$.

The work is based on the paper

Hamiltonian Group Actions and Dynamical Systems of Calogero Type

by Kazhdan Kostant and Sternberg

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– 195 PROJECT

Consider $\mathcal{U}(n)$. $\forall\, U \in \mathcal{U}(n)$ we can write

$$U = e^{iH} \qquad H \text{ hermitian}$$

In particular we can represent curves near I as

$$U(t) = e^{iHt}$$

which give the tangent vectors

$$U'(0) = iH$$

$\therefore$ $\mathfrak{g} = u(n)$ consists of skew hermitian matrices iH which can be identified with the space of hermitian matrices H. $u(n)^*$ can similarly be identified with the hermitian matrices using the scalar product

$$\underset{u(n)^*}{\langle A}, \underset{u(n)}{B\rangle} = \mathrm{Tr}\, A^+ B$$

Then $\mathcal{U}(n)$ acts on $u(n)$ according to

$$Ad_U H = U H U^{-1} = U H U^+$$

To find the action on $u(n)^*$ consider

$$\langle A, Ad_{U^{-1}} B \rangle = \mathrm{Tr}\, A^+ (U^+ B U)$$

$$= \mathrm{Tr}\; u A^{+} u^{+} B$$

$$= \mathrm{Tr}\left((u A u^{+})^{+} B \right)$$

$$= \langle u A u^{+} , B \rangle$$

$$= \langle Ad^{*}_{u^{-1}} A , B \rangle$$

hence $Ad^{*}_{u^{-1}} A = u A u^{+}$ $\quad A \in u(n)^{*}$
which is the same action as on $u(n)$

We can form $T^{*} u(n)^{*} \simeq u(n)^{*} \oplus u(n)$ identifying $u(n)^{**} \simeq u(n)$ ie regard $u(n)^{*}$ as the configuration space. This can then be given the standard symplectic form. The action of $U(n)$ on $u(n)^{*}$ can then be lifted to $T^{*} u(n)^{*}$ by the cotangent lift

since $\Phi_{u} A = Ad_{u^{-1}} A = u A u^{+}$ is linear in A, $T Ad_{u^{-1}}$ is given by the same expression

This $\langle \xi , T^{*} \Phi_{u^{-1}} \eta \rangle = \langle T \Phi_{u^{-1}} \xi , \eta \rangle$

$$= \langle u^{+} \xi u , \eta \rangle = \langle \xi , u \eta u^{+} \rangle$$

hence the lifted action is

$$(A, B) \xrightarrow{T^*\Phi_g} (uAu^{-1}, uBu^{-1}) \qquad (1)$$

$u(n)^* \oplus u(n)$

$\boxed{\text{SYMPLECTIC FORM}}$

The canonical 1-form θ is defined by 1-4-2 of the notes

$$\theta(A,B)(\xi,\eta) = \langle T\tau^*_{g^*} \cdot (\xi,\eta), B \rangle$$

$(A,B) \in g^* \oplus g$

$\xi, \eta \in T_{(A,B)} g^* \oplus g \simeq g^* \oplus g$

(hence forth I write g^* for $u(n)^*$ etc)

(note: because of the identification $g^{**} \simeq g$ the brackets $\langle \, , \rangle$ have backwards entries)

$$= \langle \xi, B \rangle$$

$$\therefore \quad \boxed{\theta(A,B)(\xi_{(A,B)}, \eta_{(A,B)}) = \langle \xi, B \rangle} \qquad (2)$$

differentiating gives

$$\Omega(A,B)\left((\xi^1_{(A,B)}, \eta^1_{(A,B)}), (\xi^2_{A,B}, \eta^2_{A,B})\right)$$

As in 1.1.2 $(2')$

$$= \langle \xi^1, \eta^2 \rangle - \langle \xi^2, \eta^1 \rangle$$

$\boxed{\text{INFINITESEMAL GENERATOR}}$

Given $H \in \mathfrak{g}$ then this corresponds to the curve $u(t)$ $\quad u'(0) = iH$

From this we get $\exp Ht = e^{iHt}$. Then the infintesmal generator on $\mathfrak{g}^* \oplus \mathfrak{g}$ is given by corresponding to $g \in \mathfrak{g}$ is

$$g_p\Big|_{A,B} = \frac{d}{dt} \exp gt \cdot (A,B)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(e^{igt} A e^{-igt},\ e^{igt} B e^{-igt}\right)\Big|_{t=0}$$

Now $\quad e^{igt} A e^{-igt} = A + it[g,A] + O(t^2)$

$\forall g, A$ — Baker Hausdorff formula

Sledge-hammer for something simple.

$$\Rightarrow \frac{d}{dt}\left(e^{igt} A e^{-igt}\right)\Big|_{t=0} = i[g,A]$$

thus $$\boxed{g_p\Big|_{(A,B)} = \left(i[g,A],\ i[g,B]\right)} \qquad (3)$$

MOMENTUM MAP

The momentum map is now determined by

$$J(\xi)\,(A,B) = \theta(A,B)\,.\,\xi_P\big|_{A,B}$$

$$= \theta(A,B)\,.\,\left(i[\xi,A]\ ,\ i[\xi,B]\right)$$

$$= \langle i[\xi,A]\ ,\ B\rangle \qquad \text{by } (2)$$

$$= \mathrm{Tr}\ i(\xi A - A\xi)B$$

$$= \mathrm{Tr}\ i\xi AB - \mathrm{Tr}\ iA\xi B$$

$$= \mathrm{Tr}\ i\xi AB - \mathrm{Tr}\ i\xi BA$$

$$= \mathrm{Tr}\ i\xi[A,B]$$

$$= \langle i[A,B]\ ,\ \xi\rangle$$

$$\Rightarrow \quad \boxed{\underset{\sim}{J}\big((A,B)\big) = i[A,B]} \qquad (4)$$

Theory has it that the momentum map for a lifted action is equivariant and we can immediately see that this is the case since

$$
\begin{aligned}
u \cdot \underset{\sim}{J}(A,B) &= u\left(i\,[A,B]\right)u^{-1} \\
&= i\left[uAu^{-1},\, uBu^{-1}\right] \\
&= \underset{\sim}{J}\left(uAu^{-1},\, uBu^{-1}\right) \\
&= \underset{\sim}{J}\left(u\cdot(A,B)\right)
\end{aligned}
$$

Now we consider reduction to $\tilde{\mathcal{J}}^{-1}(\mu)/G_\mu$ for a particular μ of the form

$$\mu = \lambda I + |v\rangle\langle v| \in \mathfrak{g}^* \quad \text{where } v \in \mathbb{C}^n$$

In this case we get

$$\tilde{\mathcal{J}}^{-1}(\mu) = \{ (A,B) \in \mathfrak{g}^* \oplus \mathfrak{g} \mid i[A,B] = \lambda I + |v\rangle\langle v| \}$$

The isotropy group of μ is then given by

$$G_\mu = \{ C \in U(n) \mid |Cv\rangle\langle Cv| = |v\rangle\langle v| \}$$

(where I use $\Phi_C \circ \mu = C(\lambda I + |v\rangle\langle v|)C^\dagger = \lambda I + |Cv\rangle\langle Cv|$)

$$= \{ C \in U(n) \mid Cv = e^{i\theta} v \}$$

Thus, to form $\tilde{\mathcal{J}}^{-1}(\mu)/G_\mu$ we identify pairs (A,B)
* which are unitary equivalent by unitary matrices satisfying

$$Cv = e^{i\theta} v$$

SPECIALISATION

Now take v to be of the form

$$v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$ ie all 1's

and $\lambda = -1$

so $\mu = |v\rangle\langle v| - I$

$$= \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \\ 1 & & & 0 \end{pmatrix}$$ ie zero's on the diagonal, 1's everywhere else

Now take $(A, B) \in \underset{\sim}{J}^{-1}(\mu)$ and diagonalise A with the unitary matrix C

ie $CAC^{-1} = D$ diagonal

If we write $CBC^{-1} = E$ then,

$$i[D, E] = -I + |w\rangle\langle w|$$

where $w = Cv$

The diagonal entries of $[D, E]$ are zero since

$$([D,E])_{ii} = \sum_j D_{ij} E_{ji} - E_{ij} D_{ji} \qquad \left(\begin{matrix} D \text{ diagonal} \\ \Rightarrow i=j \text{ term} \\ \text{contributes} \end{matrix} \right)$$

$$= D_{ii} E_{ii} - E_{ii} D_{ii}$$

$$= 0$$

This implies that $|w\rangle\langle w| - I$ has zero diagonal entries, but these are just $w_i^* w_i - 1$, so,

$$|w_i| = 1$$

$$w_i = e^{i\theta_i}$$

If we conjugate A, B by $C' = \text{diag}(e^{-i\theta_i})\, C$ rather than C we get

$$C' A C'^{-1} = D \qquad \left(\begin{matrix} \text{unchanged since 2 diagonal} \\ \text{matrices commute} \end{matrix} \right)$$

$$C' B C'^{-1} = E'$$

But now $C' v = (\text{diag } e^{-i\theta_i})(\underset{\sim}{e^{i\theta_i}}) = v$!

By comparing with $*$ we now see that (A,B) and (D, E') are in the same equivalence class of $\bar{\Phi}^{-1}(\mu)/G_\mu$ ie they both ~~contract~~ project to the same point. It turns that (D, E') can be used to represent the class of (A, B) in which case we can regard $\mathcal{J}^{-1}(\mu)/G_\mu$ as consisting of elements of the form (D, E') where D is diagonal (whose entries consist of the eigenvalues of the A's in the class of $\mathcal{J}^{-1}(\mu)/G_\mu$). To find out what E' is like consider the following.

$$([D, E'])_{ij} = \sum_k D_{ik} E'_{kj} - E'_{ik} D_{kj}$$

$$= x_i E'_{ij} - E'_{ij} x_j$$

(where $x_i = D_{ii}$ are the eigenvalues of A)

Using $i[D, E'] = \mu$ gives $i E'_{ij}(x_i - x_j) = 1 - \delta_{ij}$

$\Rightarrow$ ① $i \neq j \Rightarrow x_i \neq x_j$

② $E'_{ij} = \dfrac{-i}{x_i - x_j}$

Thus the off diagonal elements depend on x_i. The diagonal element however are in-dependent and we can denote these by y_i. Thus we have

$$D = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \qquad E' = \begin{pmatrix} y_1 & & \frac{-i}{x_i - x_j} \\ & \ddots & \\ \frac{-i}{x_i - x_j} & & y_n \end{pmatrix}$$

From ① over we can order the x_i's in a unique way

$$x_1 < x_2 < \cdots < x_n$$

Hence for a given class we can choose a representative in a unique way namely (D, E') with the ordering above. Hence we can identify $J^{-1}(\mu)/G_\mu$ as the space of

$$\left\{ (D, E') \;\middle|\; D = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}_{x_1 > x_2 > \cdots > x_n}, \quad E' = \begin{pmatrix} y_1 & & \frac{-i}{x_i - x_j} \\ & \ddots & \\ \frac{-i}{x_i - x_j} & & y_n \end{pmatrix} \right\}$$

(Its clear that we can reverse the argument earlier to show any pair of this form is in $\Phi^{-1}(\mu)/G_\mu$ and so the x's and y's are all independent)

hence a coordinate system is given on $\Phi^{-1}(\mu)/G_\mu$ by

$$(x_1, \cdots x_n, y_1 \cdots y_n)$$

Now in the paper it is shown that the coordinates $(x_1, \cdots x_n, y_1 \cdots y_n)$ are actually canonical. I propose to show this by a different route, namely by using

$$\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$$

good!

To use this, first consider the nature of the tangent space to $\Phi^{-1}(\mu)/G_\mu$

Given curves $D(t) = \begin{pmatrix} x_1(t) & & 0 \\ & \ddots & \\ 0 & & x_n(t) \end{pmatrix}$

$$E(t) = \begin{pmatrix} y_1(t) & & \frac{-i}{x_i(t)-x_j(t)} \\ & \ddots & \\ \frac{-i}{x_i(t)-x_j(t)} & & y_n(t) \end{pmatrix}$$

we get

$$D'(0) = \begin{pmatrix} x_1'(0) & & 0 \\ & \ddots & \\ 0 & & x_n'(0) \end{pmatrix}$$

$$E'(0) = \begin{pmatrix} y_1'(0) & & +i\,\frac{x_i'(0)-x_j'(0)}{(x_i-x_j)^2} \\ & \ddots & \\ i\,\frac{x_i'(0)-x_j'(0)}{(x_i-x_j)^2} & & y_n'(0) \end{pmatrix}$$

Thus we can identify tangent vectors $(\xi_1 \cdots \xi_n, \eta_1 \cdots \eta_n)$ in $T_{\underline{x},\underline{y}}\,\Phi^{-1}(\mu)/G_\mu$ with the tangent vectors to $\mathfrak{g}^* \oplus \mathfrak{g}$ of the form

$$\left(\begin{pmatrix} \xi_1 & & 0 \\ & \ddots & \\ 0 & & \xi_n \end{pmatrix}, \begin{pmatrix} \eta_1 & & i\,\frac{\xi_i-\xi_j}{(x_i-x_j)^2} \\ & \ddots & \\ i\,\frac{\xi_i-\xi_j}{(x_i-x_j)^2} & & \eta_n \end{pmatrix} \right)$$

Writing out $\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$ explicitly, we find

$$\Omega_\mu \left(T_{(A,B)}\pi_\mu \, (\xi^1, \eta^1)_{(A,B)} \, , \, T_{(A,B)}\pi_\mu \, (\xi^2, \eta^2)_{(A,B)} \right)$$

$$= \Omega\left((\xi^1, \eta^1)_{(A,B)} \, , \, (\xi^2, \eta^2)_{(A,B)} \right)$$

$$= \langle \xi^1, \eta^2 \rangle - \langle \xi^2, \eta^2 \rangle$$

In particular we can take $(A,B) = (D,E)$ as constructed earlier in which case

$$\xi^\alpha = \begin{pmatrix} \xi_1^\alpha & & 0 \\ & \ddots & \\ 0 & & \xi_n^\alpha \end{pmatrix}$$

$$\eta^\alpha = \begin{pmatrix} \eta_1^\alpha & & i\dfrac{\xi_i^\alpha - \xi_j^\alpha}{(x_i - x_j)^2} \\ & \ddots & \\ \cdots & & \eta_n^\alpha \end{pmatrix} \qquad \alpha = 1, 2$$

in which case $T_{(A,B)}\pi_\mu (\xi^\alpha, \eta^\alpha)_{(D,E)} \sim \underbrace{(\xi_1^\alpha \cdots \xi_n^\alpha, \eta_1^\alpha \cdots \eta_n^\alpha)}_{(\underline{\xi}^\alpha, \underline{\eta}^\alpha)}$

so $\Omega_\mu \left((\underline{\xi}^1, \underline{\eta}^1), (\underline{\xi}^2, \underline{\eta}^1) \right)$

$$= \mathrm{Tr}(\xi^1 \eta^2) - \mathrm{Tr}(\xi^2 \eta^1)$$

Since ξ^α is diagonal only the diagonal elements of η^α contribute (ie the η_i^α) giving

$$\Omega_\mu\Big((\underline{\xi}^1,\underline{\eta}^1),(\underline{\xi}^2,\underline{\eta}^2)\Big) = \sum_i \left(\xi^1_i\,\eta^2_i - \xi^2_i\,\eta^1_i\right)$$

$$= \langle \underline{\xi}^1 | \underline{\eta}^2 \rangle - \langle \underline{\xi}^2 | \underline{\eta}^1 \rangle \qquad \left(\because\ x, y \text{ canonical coordinates!} \right)$$

(There is a confusion here between the matrices ξ, η and the vectors $\underline{\xi}, \underline{\eta}$ which I try to eliminate by writing bars under the vectors. Also I distinguish the innerproducts in each case by

OK

$$\langle \xi, \eta \rangle = \mathrm{Tr}\, \xi \eta$$

$$\langle \underline{\xi} | \underline{\eta} \rangle = \sum_i \eta_i \xi_i$$

↑ central bar)

From this we see that the coordinates $(x_1 \cdots x_n, y_1 \cdots y_n)$ are in fact <u>canonical</u> as claimed! so

$$\boxed{\Omega_\mu = dx^i \wedge dy_i}$$

Now consider the function on $\mathfrak{g}^* \oplus \mathfrak{g}$ defined by

$$(A, B) \longmapsto \frac{1}{2} \mathrm{Tr}\, B^2 = H \quad \text{say}$$

This is a ? G invariant function since conjugation by a unitary matrix does not change the trace. While we didn't do this in class your 1972 paper on reduction (Corollary 3) shows that since H is $\mathfrak{g}$ invariant its flow on $\mathfrak{g} \oplus \mathfrak{g}^*$ projects to the flow on $J^{-1}(\mu)/G_\mu$ which corresponds to the hamiltonian $\tilde{H}$ on it defined by $\tilde{H} \circ \pi_\mu = H|_{J^{-1}(\mu)}$

Specifically

$$\tilde{H}(x_1 \cdots x_n, y_1 \cdots y_n) = H\left[\left(D(\underline{x}, \underline{y}), E(\underline{x}, \underline{y})\right)\right]$$

$$= \frac{1}{2} \mathrm{Tr}\, E^2$$

$$= \frac{1}{2} \sum y_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$$

which is the Calogero Hamiltonian!

The point is that the flow of $\frac{1}{2} \mathrm{Tr}\, B^2 = \frac{1}{2} \langle B, B \rangle$ on $\mathfrak{g}^* \oplus \mathfrak{g}$ is easier to integrate than the flow on the reduced system given by the Calogero hamiltonian! For this "free particle" hamiltonian we get the flow

$$(A, B) \longmapsto (A + tB, B)$$

To make the relationship between this flow and the Calogero flow more transparent consider putting coordinates on $J^{-1}(\mu)/G_\mu$ as before except now by diagonalising B not A. The procedure is exactly the same except that the roles of A, D and B, E are reversed. In particular this leads to a change of sign in the off diagonal elements of D since they are determined by the relationship

$$i[D,E] = \mu$$

with D, E reversed. Now we identify $J^{-1}(\mu)/G_\mu$ with the points

$$\left\{ (\mathcal{D}, \mathcal{E}) \;\middle|\; \mathcal{D} = \begin{pmatrix} \mu_1 & & \frac{i}{\lambda_i - \lambda_j} \\ & \ddots & \\ \frac{i}{\lambda_i - \lambda_j} & & \mu_n \end{pmatrix}, \; \mathcal{E} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \right\}$$

Exactly the same reasoning as before gives

$$d\Omega_\mu = d\mu^i \wedge d\lambda_i$$

But now in these coordinates we get

$$\tilde{H} = \frac{1}{2} \operatorname{Tr} \mathcal{E}^2 = \frac{1}{2} \sum \lambda_i^2$$

which gives the flow $\begin{cases} \mu_i \to \mu_i + t\lambda_i \\ \lambda_i \to \lambda_i \end{cases}$

Thus the relation-ship between the Calogero system and $\mathfrak{g} + \mathfrak{g}^*$ has led us to the canonical transformation

$$x_i, y_i \longrightarrow \mu_i, \lambda_i$$

for which $H = \frac{1}{2}\sum y_i^2 + \frac{1}{2}\sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \longrightarrow \frac{1}{2}\sum_i \lambda_i^2$

which demonstrates the integrability of the Calogero system!

To relate the μ_i's to the λ_i's a little further we note that the potential $\sum 1/(x_i - x_j)^2$ is repulsive and so the particles fly apart. We thus expect $1/x_i - x_j \longrightarrow 0$ as $t \to \infty$ and hence

$$E \to \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \text{ as } t \to \infty$$

Now from the construction of (D, E) and $(\mathcal{D}, \mathcal{E})$ it is clear that they are simultaneously unitary equivalent. Ie $\forall t$ there is a $\mathcal{U}$ such that

$$\mathcal{D} = \mathcal{U} D \mathcal{U}^{-1}, \quad \mathcal{E} = \mathcal{U} E \mathcal{U}^{-1}$$

From this we see that we must have

$$E \to \mathcal{E}, \quad \mathcal{U} \to \operatorname{diag} e^{i\psi_i} \text{ as } t \to \infty$$

From $E \to \mathcal{E}$ we get $\lambda_i = y_i(\infty)$.

Now if $U = (u_{ij})_{ij}$ then

$$(\mathcal{D})_{ii} = \mu_i = \sum u_{ij} D_{jk} u^*_{ik}$$
$$= \sum u_{ij} x_j u^*_{ij}$$
$$= \sum |u_{ij}|^2 x_j$$

hence since $|u_{ij}| \to \delta_{ij}$ $t \to \infty$
we see that

$$\mu_i \sim x_i \quad t \to \infty$$

$\therefore$ we have the interpretations

$$\lambda_i = y_i(\infty)$$

$$x_j \sim \mu_j \sim \mu_i + t\lambda_j \sim t\, y_j(\infty)$$

Then μ_i, λ_i can be taught of as the asymptotic forms of x_i, y_i as $t \to \infty$

A note on the Sutherland system

In a similar way it is shown in the paper that the hamiltonian

$$H = \sum y_i^2 + \sum_{j \neq k} 1 / \sin^2 \frac{\theta_j - \theta_k}{2}$$

is completely integrable. This time the pre-reduced dynamics takes place on the configuration space $U(n)$ with conjugation as the action. This time a momentum map

$$\underset{\sim}{J}(g, \alpha) = g \alpha g^{-1} - \alpha$$

where we identify $T^* U(n) \simeq U(n) \times u(n)^*$ but do <u>not</u> identify $u(n)^*$ with hermitian matrices this time but leave them as skew-hermitian.

As before we represent the classes of $\underset{\sim}{J}(\mu)/G_\mu$ (new μ = - old μ) by particular matrices, (e, β) $e \in U(n)$ $\beta \in u(n)^*$ this time of the form

$$e = \begin{pmatrix} e^{i\theta_1} & & 0 \\ & e^{i\theta_2} & \\ 0 & & \ddots \; e^{i\theta_n} \end{pmatrix}, \quad \beta = \begin{pmatrix} y_1 & & 1/e^{i(\theta_i - \theta_j)} - 1 \\ & \ddots & \\ \swarrow & & y_n \end{pmatrix}$$

and as before we find that $(\theta_1 \cdots \theta_n, y_1 \cdots y_n)$ are canonical variables. Then the Sutherland hamiltonian arises from the hamiltonian $H = \operatorname{Tr} \beta^+ \beta$

The corresponding flow on $T^* U(n)$ as determined by the hamiltonian

$$H(g,\alpha) = \mathrm{Tr}\, \alpha^{+}\alpha$$

is then trivially integrable being just momentum dependent and so we find that the Sutherland system is integrable