Holonomic Quantum Computation *

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Abstract

Quantum computers promise to solve certain problems such as factoring large integers and quantum physics simulation far faster than any conceivable classical computers. The physical implementation of quantum computation requires a series of accurately controllable quantum gates. These gates can be of dynamical origin or of geometric origin. The geometric gates depend on the global feature of the path executed and are resilient to certain kinds of errors. In this paper, we will review the theory underlying the holonomic quantum computation and describe one of its physical implementations through explicit construction of the universal gates.

1 Introduction

The computational complexity theory has its foundations in a strengthening of Church-Turing thesis, which says that any reasonable model of computation can be efficiently simulated by a probabilistic Turing Machine. Here, "reasonable" means physically realizable. However, the Turing Machine is based on classical physics. It fails to capture all physically realizable computing devices for a simple reason that the Universe is governed by quantum theory. This poses two fundamental questions. The first one is whether we can have a new kind of computing devices based on quantum theory. Deutsch answered this question affirmatively by constructing a universal quantum computer that can simulate any given quantum machines [1, 2]. The second question is whether computing devices based on quantum theory can perform computations faster than traditional Turing Machine. This question was affirmatively answered by Shor's quantum factoring algorithm[3], which can factor an integer N in polynomial time in $log(N)$ as there is no known efficient classical algorithm for solving this problem.

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In quantum computation, the data are encoded by quantum states, and computation consists of unitary transformations on these quantum states. Quantum computer can accept as input states the coherent superposition of many different inputs. Computation simultaneously affects each component of the superposition, constituting a massive parallel data processing. The power of quantum computation lies in this "quantum parallel", by which quantum computers can effeciently solve some problems believed to be intractable with any classical computers. In the quantum circuit model for quantum computation[4], the data are encoded with qubits which are basically two-dimensional quantum systems (we denotes its basis with $|0\rangle$ and $|1\rangle$), and the unitary transformations are specified or approximated by networks of quantum logic gates. These quantum gates are unitary transformations on finite number of qubits. Just as in the classical Boolean circuits where all logic gates can be constructed from a small set of gates, there exist a set of universal quantum gates from which we can build all quantum gates to an arbitrary accuracy. For example, single qubit gates and CNOT gate are such a universal set. Thus, the implementation of quantum computation is reduced to the implementation of these simple, universal quantum gates.

There exist different kinds of proposals for the physical implementation of the quantum computation. These implementations can be of dynamical origin or of geometric origin. The all-geometric approach, under the name of holonomic quantum computation [5, 6], implements the universal gates solely based on geometric phases. The geometric phases or holonomies are acquired when the system evolves under the adiabatic cycling of parameters in the governing Hamiltonian. In mathematical language, adiabatic change realizes parallel transport of quantum states in parameter space and induces a Gauge structure, and the phase is the flux of the related two-form. These phases are non-integrable and depend only on the geometry of the path trasversed in parameter space. Due to this property, the holonomic quantum computation has some built-in fault-tolerant feature and is expected to be resilient to certain types of errors.

The paper is organized as follows. In section 2, we describe the mathematical foundation of holonomy. In section 3, we give a detailed account of the theory of holonomic quantum computation. We then descirbe, in section 4, the physical implementation of holonomic gates based on laser manipulation of the trapped ions. Finally, in section 5, we briefly discuss the adiabatic condition.

2 Mathematical foundation of holonomy

Since their discovery the geometric phases or holonomies in quantum theory have had great impact in physics as well as mathematics[8]. This is due on

the one hand to their universality in physcical systems, and on the other hand to their beauty in the sense that they admit elegant formulation in terms of concepts from differential geometry and topology. Here, we will give a brief introduction to the mathematical concepts behind holonomy. For a physical description, please see reference [8].

Our setting is a n-dimensional Hilbert space $\mathcal{H}$ over C, with family of Hamiltonians H_λ parametrized by a parameter manifold $\mathcal{M}$. Choose a point $\lambda_0 \in \mathcal{M}$ and set $H_0 = H_{\lambda_0}$, and assume it has a n-dimensional degenerate eigenspace with normalized eigenvalue 0

$$H_0 v_i = 0, \quad i = 1, \cdots, n, \tag{1}$$

where $\{v_i\}$ are chosen to be orthonormal and span a state space $\mathcal{S}_0$. We further assume a family of unitary operators parametrized by $\mathcal{M}$

$$R : \mathcal{M} \to U(n), \quad R(\lambda_0) = I_n, \tag{2}$$

and H_λ is given by the isospectral family

$$H_\lambda = R(\lambda) H_0 R^{-1}(\lambda). \tag{3}$$

Note that equation (3) corresponds to adiabatic condition, and each H_λ has a n-dimensional degenerate eigenspace. Now define a Stiefel manifold

$$St = \{u = (u_1, \cdots, u_n) | u_i \in \mathcal{H}, i = 1, \cdots, n, and\ u^+ u = I_n\}, \tag{4}$$

we see the map R induces another map

$$P : \mathcal{M} \to St. \tag{5}$$

Define $Stm = P(\mathcal{M})$, which is also a Stiefel manifold, together with unitary operator $U(n)$ on $\mathcal{H}$, we have a principal $U(n)$ bundle[7]

$$\{U(n), Stm, \mathcal{M}\}. \tag{6}$$

The connection (one-form) or gauge potential $\mathcal{A}$ of this bundle is given by

$$\mathcal{A} = \mathcal{A}_\mu d\lambda^\mu = \langle v | R^{-1}(\lambda) dR(\lambda) | v \rangle, \tag{7}$$

where $v = (v_1, \cdots, v_n)$, and its curvature (two-form) is given by

$$\mathcal{F} = \mathcal{F}_{\mu\nu} d\lambda^\mu \wedge d\lambda^\nu = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}. \tag{8}$$

Let γ be loop in $\mathcal{M}$ at λ_0

$$\gamma : [0, 1] \to \mathcal{M}, \gamma(0) = \gamma(1) = \lambda_0, \tag{9}$$

associated with this loop, a holonomy is defined by

$$\Gamma(\gamma) = \mathcal{P}exp\{\oint_\gamma \mathcal{A}\} \quad \in U(n), \tag{10}$$

where $\mathcal{P}$ denotes path ordering. For $n = 1$ the holonomy is the Berry phase, and for $n > 1$ the holonomy is sometimes refered to as non-abelian geometric phase[8].

We further define a loop space at λ_0

$$\mathcal{L} = \{\gamma : [0,1] \to \mathcal{M} | \gamma(0) = \gamma(1) = \lambda_0\}, \tag{11}$$

where exists a composition law

$$(\gamma_2 \cdot \gamma_1)(t) = \theta(1/2 - t)\gamma_1(2t) + \theta(t - 1/2)\gamma_2(2t - 1) \tag{12}$$

with unity element $\gamma_0(t) = \lambda_0$ and inverse $\gamma^{-1}(t) = \gamma(1 - t)$. Since integral is linear in the intergal domains, the definition (10) combined with the composition law induces a group $Hol(\mathcal{A}) = \Gamma(\mathcal{L})$ called holonomy group. This group is a subgroup of $U(n)$, its action on $\mathcal{S}_0$ is: $x \to \Gamma(\gamma)x$, where $x \in \mathcal{S}_0$. When $Hol(\mathcal{A}) = U(n)$, the connection $\mathcal{A}$ is called irreducible. We will see the properties of this group is important to the holonomic quantum computation.

3 Basic ideas of holonomic quantum computation

As mentioned in the introduction, for quantum computation, the data are encoded by quantum states and the computation consists of unitary transformations on these states. A quantum algorithm for solving a computational problem is to design networks of quantum gates to realize these unitary transformations and to specify a measurement prescription to extract relevant information from which we infer the computational result. To be specific, in the proposed holonomic quantum computation the data are encoded by the states in degenerate eigenspace $\mathcal{S}_0$, and the unitary transformations are given by holonomies $\Gamma(\gamma)$ with different choices of the loop γ.

Along this road to quantum computation, there're two important questions that need to be addressed. The first one is related to the capacity with which the holonomic quantum computer can do computations: how many unitary transformations, i.e., computations, can we obtain by different choices of the loop? From last section, we know that the holonomy group $Hol(\mathcal{A})$ is a subgroup of $U(n)$, which means that the computations a holonomic quantum computer can perform may be of very limit kinds. We would like the connection $\mathcal{A}$ to be irreducible, thus $Hol(\mathcal{A}) = U(n)$ and the universal computation over $\mathcal{S}_0$ can be realized. Fortunately, in the

space of connections on $\mathcal{M}$, the irreducible ones are an open dense set, thus irreducibility is the generic situation. Here, we will show the connections associated with non-abelian geometric phase are irreducible. Note that the condition of irreducibility can be stated in terms of the curvature defined in (8): if the $\mathcal{F}_{\mu\nu}$'s span the whole Lie algebra $u(n)$, then $\mathcal{A}$ is irreducible. For simplicity, we consider the generic example of a system with $n+1$ levels, of which n levels are degenerate with normalized eigenvalue 0[12]. Extend the Hamiltonian H_0 defined in section 2 to act on $(n+1)$-dimensional space, and accordingly make change to unitary operator $R(\lambda)$. To be specific, H_0 can be denoted by a $(n+1)$-dimensional matrx with entries $(H_0)_{ij} = 0$ unless $i = j = n+1$, and $R(\lambda)$ can be written as

$$R(\lambda) = exp(\lambda_n T_{n,n+1} - h.c.) \cdots \exp(\lambda_2 T_{2,n+1} - h.c.) \exp(\lambda_1 T_{1,n+1} - h.c.), \tag{13}$$

where $(T_{i,h})_{jk} = \delta_{i,j}\delta_{k,h}$, and the parameter manifold $\mathcal{M}$ is the coset space $SU(n+1)/U(n) \cong CP^n$. Using formula (7)-(8) and applying the projection onto the first n degenerate eigenstates, we obtain the curvature at $\lambda = 0$

$$(\mathcal{F}_{\mu\nu}(0))_{jk} = i^{j+k}((-1)^j T_{\nu\mu} - (-1)^k T_{\mu\nu}). \tag{14}$$

It's easy to check that the components of $\mathcal{F}$ do span the Lie algebra $u(n)$, thus the associated connection is irreducible and $Hol(\mathcal{A}) = U(n)$.

The second question is related to the realization of unitary transformations in practice. We shouldn't expect that we will choose arbitrary loop corresponding to arbitrary given unitary transformation. Could we obtain the whole holonomy group with arbitrary accuracy with the compositions of just several loops? This is equivalent to finding a set of universal quantum gates. Recall that, for a Lie algebra, there might exist several elements $\{g_i\}$ from which we can obtain all other elements through scalor product and Lie-brocket operations. If this is the case, choose a group element U_i generated by each g_i, we obtain a set of group elements $\{U_i\}$ from which we can get almost all the other group elments by group multiplication. Specific to our non-abelian geometric phases, it turns out that two generic loops generate a universal set of gates over $\mathcal{S}_0$[5], i.e., the group generated by composing the holonomies from these two loops and their inverses is dense in $Hol(\mathcal{A}) = U(n)$.

Note that, if we concern with a specific physical implemention of holonomic computation, there exists a simple way to prove the universality. Recall that, for example, a nontrivial two-qubit gate with single qubit gates are universal. If we can directly construct these gates, we've proved that holonomic computer can realize universal computation. We will see how to construct these gates in the next section.

4 Physical implementation of holonomic quantum computation

In the physical implementation of holonomic quantum computation, $\lambda \in \mathcal{M}$ represents the control parameters such as external fields and the interaction coefficients between subsystems. The computation effects through adiabatically changing the control parameters to induce the desired evolution of the computational space $\mathcal{S}_0$. Recently, several schemes for holonomic computation were proposed by using nuclear magnetic resonance[9], trapped ions[10] or superconducting nanocircuits[11]. Here, we focus on the scheme based on laser manipulation of trapped ions[10].

As mentioned before, the implementation of quantum computation is reduced to the physical implemetation of several universal gates. Each gate is specified by a gate Hamiltonian, the implementation of the computation is to engineer these Hamiltonians. The system we have in mind is a set of ions confined in a linear Pauli trap. Each ion has three relevant internal states $|0\rangle$, $|1\rangle$ and $|a\rangle$, of which $|0\rangle$ and $|1\rangle$ are used to encode data (one qubit) and $|a\rangle$ is used as an ancillary level for gate operations. These three states are coupled seperately with an excited state $|e\rangle$ by resonant classical lasers. The Hamiltionian in the rotating frame for each ion is

$$H_j = \hbar\{|e\rangle(\Omega_0\langle 0| + \Omega_1\langle 1| + \Omega_a\langle a|) + h.c.\}, \tag{15}$$

where Ω_0, Ω_1 and Ω_a are control parameters representing the Rabi frequencies. This Hamiltonian will serve to construct single qubit gates. We need to exploit the Coulomb interactions between ions to construct two-qubit gates. Using two-color laser manipulation and under the Lamb-Dicke approximation, we can obtain the interaction Hamiltonian[13]

$$H_{ij} = \kappa\{-|\Omega_1|^2 e^{i2\phi_1}|e\rangle_j\langle 1||e\rangle_k\langle 1| + |\Omega_a|^2 e^{i2\phi_a}|e\rangle_j\langle a||e\rangle_k\langle a|\}, \tag{16}$$

where κ is some constant, and ϕ_1 and ϕ_a are phases of Ω_1 and Ω_a, respectively.

For our purpose, we choose the universal set of gate operations to be $U_1^j = e^{i\varphi_1|1\rangle_j\langle 1|}$, $U_2^j = e^{\varphi_2(|0\rangle_j\langle 1| - |1\rangle_j\langle 0|)}$ and $U_3^{jk} = e^{i\varphi_3|11\rangle_{jk}\langle 11|}$. The universality of this set of gates can be proved by checking the Lie algebra generated by $|1\rangle_j\langle 1|$, $|0\rangle_j\langle 1| - |1\rangle_j\langle 0|$ and $|11\rangle_{jk}\langle 11|$. To get the gate U_1^j, we set $\Omega_0 = 0$ and choose $\Omega_1 = -\Omega e^{i\phi} sin(\theta/2)$ and $\Omega_a = \Omega cos(\theta/2)$ in Hamiltonian (15). The corresponding eigenstate with zero eigenvalue is

$$|\psi\rangle = cos(\theta/2)|1\rangle_j + e^{i\phi} sin(\theta/2)|a\rangle_j. \tag{17}$$

We see that θ and ϕ are effective control parameters and Ω is irrelevant. When the parameters make an adiabatic cyclic evolution with the starting

and ending points at $\theta = 0$, we obtain the gate operation U_1^j with Berry phase[8]

$$\varphi_1 = \oint \langle\psi|d\psi\rangle = \frac{1}{2}\int\int sin\theta d\theta d\phi. \tag{18}$$

Note that the acquired Berry phase is one-half of the enclosed solid angle $\int d\Omega$ swept by the vector pointing to the (θ, ϕ) direction.

To get the gate U_2^j, we choose $\Omega_0 = \Omega sin\theta cos\phi$, $\Omega_1 = \Omega sin\theta sin\phi$ and $\Omega_a = \Omega cos\theta$ in the Hamiltonian (15). The corresponding degenerate eigenstates with zero eigenvalue are

$$|\psi_1\rangle = cos\theta cos\phi|0\rangle_j + cos\theta sin\phi|1\rangle_j - sin\theta|a\rangle_j, \tag{19}$$

$$|\psi_2\rangle = -sin\phi|0\rangle_j + cos\phi|1\rangle_j. \tag{20}$$

When the parameters θ and ϕ make the similar adiabatic cyclic evolution with staring and ending point at $\theta = 0$, we obtain the gate U_2^j with the phase

$$\varphi_2 = \oint \langle\psi_1|d\psi_2\rangle = \int\int sin\theta d\theta d\phi. \tag{21}$$

Note that U_1^j and U_2^j obtained are noncommutable, i.e. are non-abelian holonomies.

To get the gate U_3^{jk}, we choose $|\Omega_1|^2/|\Omega_a|^2 = tan(\theta/2)$ and $\phi_1 - \phi_a = \phi/2$ in Hamiltonian (16). The corresponding adiabatic state is

$$|\psi_3\rangle = cos(\theta/2)|11\rangle_{jk} + e^{i\phi}sin(\theta/2)|aa\rangle_{jk}. \tag{22}$$

When the parameters make an adiabatic cyclic evolution with the starting and ending points to be $\theta = 0$, we obtain the gate operation U_3^{jk} with the phase

$$\varphi_3 = \oint \langle\psi_3|d\psi_3\rangle = \frac{1}{2}\int\int sin\theta d\theta d\phi. \tag{23}$$

We see that the gate operations are determined only by the global property (the swept solid angle) but does not depend on the details of the adiabatic path in the parameter space. This is an advantage of holonomic quantum computation, and makes it resilient to certain kinds of errors such as the local random errors along the adiabatic path caused by some unwanted interaction.

5 Discussion

The above construction proves the feasibility of holonomic quantum computation. Of course, besides the adiabatic condition, there exist other conditions and constraints in its implementation. We will not discuss these issues since they are not mathematical questions.

We've seen that the adiabatic condition is the prerequistite of the holonomic quantum computation. However, the quantum system is very fragile and the operation time for quantum computer needs to be shorter than the decoherence time. Thus, in practice, the adiabatic condition is not easy to meet. We need to study the non-adiabatic corrections[14], and quantify the errors induced to the computation and find ways to counteract them.

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