Excellent report. See paper of Rausel, Marsden & Ratiu wrt connections on Lie groups!

ON CONTROLLABILITY AND SYMMETRIES IN SIMPLE MECHANICAL SYSTEMS

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ABSTRACT. Mechanical systems with symmetries are an extremely interesting class of nonlinear control systems, which include rigid bodies with external forces as well as locomotion systems. This report presents a unifying theoretical framework for the analysis of the controllability property of these systems. The analysis is based on the geometric properties of the configuration space of these systems: a principal fiber bundle with an invariant metric. On this Riemannian manifold the Levi Civita connection is characterized in terms of two "integrability" tensors: the mechanical connection of the bundle and the second fundamental form of the fibers.

The reduced Euler-Lagrange equations are recovered in both intrinsic and local coordinates versions. Regarding the controllability problem, various cases are analyzed: Lie group systems, locomotion systems and chains of rigid bodies with full fiber actuation. Numerous examples illustrate the theory.

1. INTRODUCTION

Control of mechanical systems with symmetries is motivated by lots of fun examples [1], see pictures at page 18. A common feature shared by these examples is the Lagrangian being sum of kinetic and potential energy. Consequently our analysis will assume this additional structure and thus relies on Riemannian geometry tools. They indeed turn out to be well suited, as both the equations of motions and the controllability properties can be computed within the same framework in an intrinsic way.

When symmetries are added, the configuration manifold splits, at least locally, into the product of two spaces: the shape space and the structure group. Both spaces are naturally endowed with a Riemannian structure. This paper investigates this in geometric, mechanics and control theoretical terms.

[1] Being fun is ultimately related to our interest to "motion control problems".

Geometric Mechanics picture: π being a Riemannian submersion implies the following exact sequence of maps

$$G\cdot q \stackrel{\text{group action}}{\longrightarrow} Q \stackrel{\pi}{\longrightarrow} Q/G.$$

A natural Riemannian metric is induced in both the base B and the fiber $G\cdot q$: the fiber is a submanifold on Q (with metric specified by the locked inertia tensor $\mathbb{I}$) and, regarding the base, horizontally lifting vector fields on the base and taking their inner product on TQ, is a well-defined procedure. Two important tensors also have geometric meaning: the mechanical connection (and its curvature), and the second fundamental form of the immersion $G\cdot q \to Q$. The key idea in this work is that the covariant derivative on the total space is characterized by these two tensors, together with the Riemannian structures on base and fiber.
Aside: The mechanical connection is related to holonomy ideas, while the second fundamemntal form relates to mechanical systems with constraints (Marsden & Ratiu 1994), and to the notion of totally geodesic submanifolds (Kobayashi & Nomizu 1963*a*). Also, see Marsden, Montgomery & Ratiu (1990) and O'Neill (1966), for an elegant description of these concepts.

Problem statement: The goal of this paper is to split the controllability computations on the total bundle Q into computations on the single spaces. This is a *reduction problem.* Consistent with this view, we would also like to perform Lagrangian reduction all the way to the geodesic equations on the two reduced spaces. The purpose is to decompose the Euler-Lagrange's equation into geodesic equations on two reduced spaces. This is a way to study controllability, given the relativa ease in computing the accessibility distributions of the geodesic equations.

Literature description: Previous work by various authors focuses on the two important subclass of pure Lie group systems and locomotion systems. We refer to Bullo & Lewis (1996*a*) for the first case and talk about the second here. Kelly & Murray (1995) focuse on the case of horizontal forces (that is forces that preserve the total spatial angular momentum): the mechanical connection and its curvature determine the controllability of the vertical part of the system in a purely kinematic way. Here we face the fully dynamic case allowing for both horizontal and vertical way: the reduced Euler-Lagrange equation of Marsden & Scheurle (1993) are recovered as well as Kelly's results. Additionally we are able to deal with a class of systems where it's the group directions affect the motion of the internal variables.

Apologies on notation: We face a difficul choice of symbols in describing the Riemannian structure of all the spaces involved. The final setup s described as follows:

manifold	metric	covariant derivative
Q	$\langle\langle\cdot,\cdot\rangle\rangle$	∇
$B=Q/G$	$\langle\langle\cdot,\cdot\rangle\rangle_B$	${}_B\nabla$
$G\cdot q$	$\mathbb{I}(q)$	${}_{G\cdot q}\nabla$

Additionally, the connection on the fiber ${}_{G\cdot q}\nabla$ induces two bilinear maps on the Lie algebra $\mathfrak{g}$, which we denote with ${}_l\nabla$ and ${}_r\nabla$ (intuitively covariant derivatives of left invariant and right invariant vector fields can be reduced to operations on the Lie algebra, see Appendix A for a precise description).

Outline: The report is organized as follows: in Section 2 we briefly review principal fiber bundles and basics of Riemannian geometry. Then we compute expressions for ∇ on Q as a function of ${}_B\nabla$, ${}_{G\cdot q}\nabla$, mechanical connection and second fundamental form of the fibers. In Section 5, we show how to use this formalism to rederive the reduced Euler-Lagrange equations. In Section 6 we study controllability properties of mechanical systems with symmetries and we then present some examples in the following section.

In Appendix A we characterize Riemannian connections on Lie groups and fibers of principal fiber bundles. In Appendix B we show that the equations we derived in the previous sections agree with the ones in Marsden & Scheurle (1993).

Contents

2. Some geometry

We refer to Kobayashi & Nomizu (1963*a*) and Abraham & Marsden (1987, Section 2.7) for an introduction to Riemannian geometry and to the theory of connections on principal fiber bundles.

2.1. Elements of Riemannian geometry. Let M be a Riemannian manifold, denote with $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ its metric tensor and with the symbols $^\flat : TM \to T^*M$ and $^\sharp : T^*M \to TM$ the musical isomorphisms. An *affine connection* on M is a map that assigns to each pair of smooth vector fields X, Y a smooth vector field $\nabla_X Y$ such that

i) $\nabla_{fX} Y = f\nabla_X Y$ and
ii) $\nabla_X fY = f\nabla_X Y + (\mathcal{L}_X f)\, Y$ for all $f \in C^\infty(M)$.

Given any three vector fields X, Y, Z on M, we say that the affine connection ∇ on M is *torsion-free* if

$$[X, Y] = \nabla_X Y - \nabla_Y X \tag{2.1}$$

and is *compatible* with the metric $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ if

$$\mathcal{L}_X \langle\!\langle Y, Z \rangle\!\rangle = \langle\!\langle \nabla_X Y, Z \rangle\!\rangle + \langle\!\langle Y, \nabla_X Z \rangle\!\rangle. \tag{2.2}$$

There exists a unique torsion-free affine connection ∇ on M compatible with the metric. We call this ∇ the Riemannian (or Levi-Civita) connection: it satisfies the Koszul formula:

$$2\langle\!\langle X, \nabla_Z Y \rangle\!\rangle = \mathcal{L}_Z \langle\!\langle X, Y \rangle\!\rangle + \langle\!\langle Z, [X, Y] \rangle\!\rangle + \mathcal{L}_Y \langle\!\langle X, Z \rangle\!\rangle + \langle\!\langle Y, [X, Z] \rangle\!\rangle \\ - \mathcal{L}_X \langle\!\langle Y, Z \rangle\!\rangle - \langle\!\langle X, [Y, Z] \rangle\!\rangle. \tag{2.3}$$

Any Riemannian connection ∇ can be decomposed into the sum of a symmetric and skew part

$$\nabla_X Y = \frac{1}{2} \langle X : Y \rangle + \frac{1}{2} [X, Y], \tag{2.4}$$

where $\langle X : Y \rangle \triangleq \nabla_X Y + \nabla_Y X$ is the symmetric product introduced in (Lewis & Murray 1996).

2.2. Principal fiber bundles and connections. Let B be a manifold and G a Lie group G. A *principal fiber bundle* with *base* B and *structure group* G consists of a manifold Q and a free and proper action of G on Q, such that $B = Q/G$ and the canonical projection $\pi : Q \to B$ is a smooth surjection. The group action is denoted by

$$\Phi : G \times Q \to Q : (g, q) \mapsto \Phi_g(q) = g \cdot q.$$

and the fiber over $q \in Q$ is denoted by $G \cdot q = \pi^{-1}(q)$. We define the *vertical subspace* of the principal fiber bundle Q at the point $q \in Q$ to be $\mathrm{ver}_q = T(G \cdot q)$. A vector which lies in ver_q is tangent to the orbit of q under the action of G. A *connection* on the bundle Q is specified by a *connection one-form* $\mathcal{A} : TQ \to \mathfrak{g}$ satisfying

i) $\mathcal{A}(\xi_Q) = \xi$ and
ii) $\mathcal{A}(T_q \Phi_g(v_q)) = \mathrm{Ad}_g\, \mathcal{A}(v_q)$,

where ξ_Q denotes the infinitesimal generator corresponding to the Lie algebra element ξ, Ad_g denotes the adjoint action of G on $\mathfrak{g}$, and $v_q \in T_qQ$. We can locally trivialize the bundle as $Q = B \times G$ and we write correspondingly $q = (r, g)$. If $v_q = \dot{q} = (\dot{r}, \dot{g}) \in T_qQ$, the *local connection form* $\mathcal{A}_{loc} : TB \to \mathfrak{g}$ satisfies

$$\mathcal{A}_{(r,g)} \cdot (\dot{r}, \dot{g}) = \mathrm{Ad}_g(g^{-1}\dot{g} + \mathcal{A}_{loc}(r) \cdot \dot{r}).$$

A connection $\mathcal{A}$ on the bundle Q assigns to each point $q \in Q$ a *horizontal subspace* hor_q of T_qQ:

$$\mathrm{hor}_q = \{v_q \in T_qQ : \mathcal{A}(v_q) = 0\}.$$

It follows from the properties of $\mathcal{A}$ that $T_qQ = \mathrm{hor}_q \oplus \mathrm{ver}_q$ and $\mathrm{hor}_{\Phi_g q} = T\Phi_g(\mathrm{hor}_q)$ for $g \in G$. Indeed, given a subspace hor_q, there is in general a well defined connection such that $\mathrm{hor}_q = \ker \mathcal{A}_q$. If $v_q \in T_qQ$, its decomposition with respect to a given connection is written

$$v_q = \mathrm{hor}\, v_q + \mathrm{ver}\, v_q.$$

Consider now the projection map $\pi : Q \to B$: for each $q \in Q$, the associated tangent map $T_q\pi : T_qQ \to T_{\pi(q)}B$ is a linear isomorphism from the horizontal subspace hor_q onto the tangent space to the base $T_{\pi(q)}B$. Hence its inverse is well-defined and is called the *horizontal lift*. Given a vector field X on B, we denote by X^h the unique horizontal vector field on Q which projects via $T\pi$ onto X. By construction, a horizontally lifted vector field is π-related to its projection and thus, for all $X, Y \in T_{\pi(q)}Q$, it follows

$$\mathrm{hor}_q \left[X^h, Y^h\right] = [X, Y]_B{}^h, \tag{2.5}$$

where $[\cdot,\cdot]_B$ denotes the Lie bracket on the base space.

Given a connection form $\mathcal{A} : TQ \to \mathfrak{g}$, the corresponding *curvature form* $\mathcal{B} = D\mathcal{A} : TQ \times TQ \to \mathfrak{g}$ is given by its covariant exterior derivative

$$\mathcal{B}(v_1, v_2) = d\mathcal{A}(\mathrm{hor}\, v_1, \mathrm{hor}\, v_2), \qquad v_1, v_2 \in T_qQ.$$

From this definition one has

$$\mathcal{A}([\mathrm{hor}\, v_1, \mathrm{hor}\, v_2]) = -\mathcal{B}(v_1, v_2). \tag{2.6}$$

In practice, the curvature of a connection is computed via the *structure equation*

$$\mathcal{B}(v_1, v_2) = d\mathcal{A}(v_1, v_2) - [\mathcal{A}(v_1), \mathcal{A}(v_2)]_{\mathfrak{g}}, \qquad v_1, v_2 \in T_qQ.$$

The *local curvature form* $\mathcal{B}_{loc} : TB \times TB \to \mathfrak{g}$ satisfies the analogous equation

$$\mathcal{B}_{loc}(r)(u_1, u_2) = d\mathcal{A}_{loc}(u_1, u_2) - [\mathcal{A}_{loc}(u_1), \mathcal{A}_{loc}(u_2)]_{\mathfrak{g}}, \qquad u_1, u_2 \in T_rB, \tag{2.7}$$

where $\mathcal{B}_{(r,g)}(X_q, Y_q) = \mathrm{Ad}_g\, \mathcal{B}_{loc}(r)(T_q\pi \cdot X_q, T_q\pi \cdot Y_q)$.

2.3. The mechanical connection. The geometric construction above applies directly to the study of mechanical systems with symmetry. Let $\pi : Q \to B$ be the configuration space, endowed with a G invariant metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Define the mechanical connection $\mathcal{A}$ by declaring horizontal subspace to be the perpendicular complement of the vertical subspace with respect to the metric. Denote with $\mathcal{B}$ its curvature.

Define the locked inertia tensor $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ as

$$\langle \mathbb{I}(q)\xi, \eta \rangle = \langle\!\langle \xi_Q(q), \eta_Q(q) \rangle\!\rangle, \qquad \xi, \eta \in \mathfrak{g}.$$

Its local form is denoted by

$$I(r) = \mathbb{I}(r, e).$$

In a local trivialization $Q = B \times G$, we have both a left and a right action of G on itself. The vertical subspace can now described in terms of both (left) invariant vector fields and infinitesimal generators (right invariant!). We denote

$$\xi^r \triangleq \xi_Q = (0,\ T_e R_g(\xi)) \qquad \text{and} \qquad \xi^\ell = (0,\ T_e L_g(\xi)),$$

where L_g and R_g are the canonical translations. The component of an horizontal vector can be written as

$$X^h = (X\ ,\ -g\mathcal{A}_{loc}(r) \cdot X),$$

where $X \in T_r B$.

2.4. Isometries on Riemannian manifolds. The content of this subsection is taken from (Kobayashi & Nomizu 1963*a*, Chapter VI).

An infinitesimal isometries on $(M, \langle\!\langle \cdot, \cdot \rangle\!\rangle)$ is a vector field K that satisfies the Killing equation

$$\langle\!\langle X, \nabla_Y K \rangle\!\rangle + \langle\!\langle Y, \nabla_X K \rangle\!\rangle = 0 \tag{2.8}$$

for all $X, Y \in \mathfrak{X}(M)$. In other words, the $(1, 1)$ tensor field ∇K on M is skew symmetric with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Consider now the princiapl fiber bundle $\pi : M \to B$ and let ∇ be the Riemannian connection of an metric invariant under the group action. Then ∇ is said to be *invariant*, since

$$(\nabla_X Y)_{\Phi_g(q)} = T_q \Phi_g (\nabla_X Y)_q \tag{2.9}$$

for each pair of invariant vector fields $X, Y \in \mathfrak{X}(M)$. For all $\xi \in \mathfrak{g}$, the infinitesimal generator ξ_Q is a infinitesimal isometry and therefore is satisfies the Killing equation (2.8). Particularly interesting is the case of $M = G$ (hence M is a Lie group itself), since it is possible to characterize the connection ∇ in an algebraic way. This is described in detail in Appendix A.

3. Distributions on Riemannian manifolds

In this section we describe some properties of distributions on manifolds and in particular on Riemannian manifolds. A precise statement of the some of the following ideas is contained in Lewis (1996) and Kobayashi & Nomizu (1963*b*, Chapter VII (on submanifolds)).

3.1. Integrability and geodesic invariance. Let D be a distribution on the manifold Q, let X be a section in D and call $\phi_X^t(q)$ its flow. Frobenious theorem states the equivalence between involutivity and integrability, which we described as:

infinitesimal condition	*integral condition*	*description of flow* $\phi_X^t(q)$
involutivity	integrability	$\exists$ submfld Λ_0 of Q, s.t. $\phi_X^t(q_0) \in \Lambda_0\ \forall t$,

where Λ_0 is the leaf of the foliation induced by D, such that $q_0 \in \Lambda_0$.

Assume now the manifold Q has a Riemannian structure and denote now with $q(t)$ the geodesic flow on Q starting from q_0 with initial velocity $\dot{q} \in D$. The picture above becomes slightly more complicated, due to the second order nature of the geodesic equation. In particular, we have the two natural definitions:

Definition 1 (Integral conditions). We say that D is ***geodesically invariant*** if, for every geodesic $q : [a,b] \to Q$ such that $\dot{q}(a) \in D_{q(a)}$, $\dot{q}(t) \in D_{q(t)}$ for each $t \in (a,b]$.

If D is integrable and geodesically invariant, we say that it is ***totally geodesic.***

Definition 2 (Infinitesimal condition). We say that a distribution D is ***symmetric*** if $\langle X : Y \rangle \in D$, for every $X, Y \in D$. □

The result proven by Lewis (1996) is that, just like involutivity is equivalent to integrability, symmetry of D is equivalent to geodesic invariance. The global picture that hold in this generalized case (second order equations) is described in the following table:

infinitesimal condition	*integral condition*	*description of geodesic*
symmetry	geodesic invariance	$\exists$ subbundle Γ of TQ, s.t. $\dot{q}(t) \in \Gamma\ \forall t$,
involutivity & symmetry	totally geodesic	$\exists$ submfld Λ_0 of Q, s.t. $q(t) \in \Lambda_0\ \forall t$.

3.2. Involutivity and symmetry tensors. Given distribution D on Q, define the ***involutivity tensor*** $\mathcal{B}_q : D_q \times D_q \to T_qQ/D_q$ as

$$\mathcal{B}_q(X_q, Y_q) = [X_q, Y_q] \, mod \, D_q.$$

$\mathcal{B}$ measures the "lack of integrability" of D and is sometimes called the curvature of the distribution. Notice that $\mathcal{B}$ is well defined, since independent on the extension of $X_q, Y_q \in D \subset T_qQ$.

For distributions on Riemannian manifolds, we have a second notion of derivation, given by the covariant derivative and in particular by the symmetric product. We can therefore define the ***symmetry tensor*** $\mathcal{S}_q : D_q \times D_q \to T_qQ/D_q$ as

$$\mathcal{S}_q(X_q, Y_q) = \langle X_q : Y_q \rangle \, mod \, D_q.$$

$\mathcal{S}$ measures the "lack of geodesic invariance" of D and, if D is involutive, is called the second fundamental form of the integral manifold of D through the point q.

4. The Riemannian geometry of a principal fiber bundle

In this section we characterize the Riemannian connection on the total space $\pi : Q \to Q/G$ in terms of the curvature of the mechanical connection on the bundle and in terms of the second fundamental form $\mathcal{S}$ of the immersion $G \cdot q \hookrightarrow Q$ (plus of course the Riemannian connections on the fiber and on the base). This is a reduction problem, as we want to reduce the dimensionality of the computation.

The next two subsections present an intrinsic presentation and local coordinate expressions.

4.1. Intrinsic treatment.

The tangent bundle T_qQ is spanned by a set of basic vector fields, say X^h and a set of infinitesimal generators ξ_Q. We start by describing the Riemannian connection ∇ on the horizontal subpace.

Lemma 1 (Integrability of the horizontal subbundle). *Let $X, Y \in T_{\pi(q)}B$*

$$\nabla_{X^h} Y^h = \left({}_B\nabla_X Y\right)^h - \frac{1}{2}\mathcal{B}(X^h, Y^h)_Q, \tag{4.1}$$

where ${}_B\nabla$ is the Riemannian connection on $(B, \langle\!\langle\cdot,\cdot\rangle\!\rangle_B)$ and the skew symmetric tensor $\mathcal{B} : \mathrm{hor}_q \times \mathrm{hor}_q \to \mathfrak{g}$ is the curvature of the mechanical connection.

Proof. Let us start by looking at the horizontal component. Since X^h and Y^h are invariant, so is their covariant derivative. Hence, since $\mathrm{hor}\, \nabla_{X^h} Y^h$ is both invariant and horizontal, it is the lift of a vector field on B. Recall now the two equations that determine the Riemannian connection ∇ on Q

$$[X^h, Y^h] = \nabla_{X^h} Y^h - \nabla_{Y^h} X^h,$$
$$\mathcal{L}_{X^h}\langle\!\langle Y^h, Z^h\rangle\!\rangle = \langle\!\langle \nabla_{X^h} Y^h, Z^h\rangle\!\rangle + \langle\!\langle Y^h, \nabla_{X^h} Z^h\rangle\!\rangle,$$

for all $X, Y, Z \in \mathfrak{X}(B)$. By applying the projection $T\pi$, these two equations naturally drop to the quotient manifold B with the induced metric. This proves

$$T\pi \cdot \nabla_{X^h} Y^h = {}_B\nabla_X Y.$$

▼

Consider now the vertical component: the map $(X^h, Y^h) \to \mathrm{ver}\, \nabla_{X^h} Y^h$ is a tensor, since $Y^h \notin \mathrm{ver}_q$. To compute it, we utilize the notion of geodesically invariant distribution, introduced in the previous subsection. The horizontal subspace hor_q is geodesically invariant, since each geodesic starting with horizontal velocity remains with horizontal velocity for all time. Hence this subspace is geodesically invariant, that is $\langle X^h : Y^h\rangle$ is horizontal. (An alternative proof is given by a detailed application of the Koszul formula.) Hence:

$$\begin{aligned}\mathrm{ver}\, \nabla_{X^h} Y^h &= \mathrm{ver}\left(\frac{1}{2}\langle X^h : Y^h\rangle + \frac{1}{2}[X^h, Y^h]\right)\\ &= \mathrm{ver}\,\frac{1}{2}[X^h, Y^h] = -\frac{1}{2}\mathcal{B}(X^h, Y^h)_Q\end{aligned}$$

by equation (2.6). ■

Now let us turn our attention to the case of vertical vectors. We compute the covariant derivative of η_Q with respect to ξ_Q. An instructive first question is:

- is the vertical subspace geodesically invariant?

(Note that this would also imply that the generic fiber $G \cdot q$ is totally geodesic, since it is a submanifold already.) The answer to this question is generally negative. Indeed one computes

$$\begin{aligned}
\langle\!\langle V, \nabla_{\xi_Q}\xi_Q \rangle\!\rangle &= -\langle\!\langle \xi_Q, \nabla_V \xi_Q \rangle\!\rangle && \text{Killing eq. (2.8)} \\
&= -\frac{1}{2}\mathcal{L}_V \langle\!\langle \xi_Q, \xi_Q \rangle\!\rangle && \text{eq. (2.2)} \\
&= -\frac{1}{2}\mathcal{L}_V \langle\!\langle \mathbb{I}(q)\xi, \xi \rangle\!\rangle = -\frac{1}{2}\left\langle (D\mathbb{I}(q)\cdot V)\xi, \xi \right\rangle, && (*)
\end{aligned}$$

which is generally nonzero [2]. An insightful interpretation of equation (*) can be given by introducing the notion of isometric immersion and of second fundamental form. We shall do so in the following, refering to (O'Neill 1966) for more details.

Fix $q \in Q$ and consider the immersion $G\cdot q = \pi^{-1}(q) \hookrightarrow Q$. The fiber $G\cdot q$ comes with the natural metric $\langle\!\langle \xi_Q, \eta_Q \rangle\!\rangle = \langle \mathbb{I}(q)\xi, \eta \rangle$, where $\mathrm{ver}_q = T(G\cdot q)$ is identified with $\mathfrak{g}\cdot q$. In other words, the immersion $G \cdot q = \pi^{-1}(q) \hookrightarrow Q$ is isometric and is therefore characterized by a symmetric tensor $\mathrm{ver}_q \times \mathrm{ver}_q \to \mathrm{hor}_q$ called the second fundamental form. This tensor describes the normal component of the covariant derivative of two normal vectors, where within our notation, a normal vector is vertical and a tangential vector is horizontal. For a complete discussion we refer to (Kobayashi & Nomizu 1963*b*, Chapter VII), here we state the main result:

Lemma 2 (Integrability of the vertical subbundle). *Let* $\xi, \eta \in \mathfrak{g}$

$$\nabla_{\xi_Q}\eta_Q \;=\; {}_{G\cdot q}\nabla_{\xi_Q}\eta_Q \;+\; \mathcal{S}(\xi_Q(q), \eta_Q(q)), \tag{4.2}$$

where ${}_{G\cdot q}\nabla$ *is the Riemannian connection on* $(G\cdot q, \mathbb{I}(q))$ *and the symmetric tensor* $\mathcal{S} : \mathrm{ver}_q \times \mathrm{ver}_q \to \mathrm{hor}_q$ *is the* second fundamental form *of the immersion* $G\cdot q \hookrightarrow Q$ *and satisfies*

$$\langle\!\langle X^h, \mathcal{S}(\xi_Q(q), \eta_Q(q)) \rangle\!\rangle = -\frac{1}{2}\left\langle (D\mathbb{I}(q)\cdot X^h)\xi, \eta \right\rangle,$$

for all $X^h \in \mathrm{hor}_q$.

Remark 1. In holonomic mechanics, the second fundamental form has the interpretation of the reaction force necessary to keep the geodesic equation on the constraint submanifold. Here $\mathcal{S}$ gives the horizontal directions generated by the covariant derivative of vertical vector fields.

Remark 2. In the vast literature on the stability of relative equilibria for mechanical systems with symmetries, a similar tensor is defined as

$$\begin{aligned}
\mathrm{ident} :\ & \mathfrak{g} \times TQ \to \mathfrak{g}^* \\
& (\xi, \delta q) \mapsto \mathrm{ident}_\xi(q)\cdot\delta q \triangleq -(D\mathbb{I}(q)\cdot\delta q)\xi.
\end{aligned}$$

Remark 3. In the two previous lemma, the mechanical curvature $\mathcal{B} : \mathrm{hor}_q \times \mathrm{hor}_q \to \mathrm{ver}_q$ and the second fundamental form $\mathcal{S} : \mathrm{ver}_q \times \mathrm{ver}_q \to \mathrm{hor}_q$ play a similar role. Note that they are both tensor and they characterize the integrability of the covariant derivative ∇ on the two subspaces hor_q and ver_q.

Recall now the decomposition of ∇ into its symmetric and skew components. Since $\mathcal{B}$ is skew, it is the Lie bracket of two horizontal vectors that generates vertical

[2] When this latter term is zero, the motion $\exp(t\xi)\cdot q$ is a *relative equilibium*. See (Simo, Lewis & Marsden 1991) or Kobayashi & Nomizu (1963*a*, Proposition 5.7) for a treatment within the realm of Riemannian geometry.

directions, while the symmetric product is a closed operator on hor_q. Viceversa, since $\mathcal{S}$ is symmetric, the symmetric product of two vertical vector fields has an horizontal component and the Lie bracket is closed in ver_q.

Proof. Let us start by computing the horizontal component of $\nabla_{\xi_Q}\eta_Q$. Note that $[\xi_Q, \eta_Q] = -[\xi, \eta]_{\mathfrak{g}Q}$ is vertical, hence we only need to compute the symmetric part:

$$\begin{aligned}\langle\langle X^h, \nabla_{\xi_Q}\eta_Q + \nabla_{\eta_Q}\xi_Q\rangle\rangle &= -\langle\langle \xi_Q, \nabla_{X^h}\eta_Q\rangle\rangle - \langle\langle \eta_Q, \nabla_{X^h}\xi_Q\rangle\rangle \qquad \text{Killing eq. (2.8)}\\ &= -\mathcal{L}_{X^h}\langle\langle \xi_Q, \eta_Q\rangle\rangle = -\mathcal{L}_{X^h}\langle \mathbb{I}(q)\xi, \eta\rangle.\end{aligned}$$

▼

For the vertical component, we use the same reasoning as in the previous lemma's proof to show that

$$\mathrm{ver}\,\nabla_{\xi_Q}\eta_Q \;=\; {}_{G\cdot q}\nabla_{\xi_Q}\eta_Q.$$

In particular, following the same previous steps we have

$$\begin{aligned}\langle\langle \zeta_Q, \nabla_{\xi_Q}\eta_Q\rangle\rangle &= -\left\langle \left(D\mathbb{I}(q)\cdot\zeta_Q\right)\xi, \eta\right\rangle\\ &= \langle\langle \zeta_Q, \left(\langle \xi : \eta\rangle_{\mathfrak{g},\mathbb{I}}\right)_Q\rangle\rangle.\end{aligned}$$

More details are contained in the Appendix A. ■

We complete the description of ∇ by considering the cross cases $\nabla_{X^h}\xi_Q$ and $\nabla_{\xi_Q}X^h$. Since X^h is invariant under the group action

$$[\xi_Q, X^h] = 0, \qquad \forall \xi \in \mathfrak{g}.$$

Hence $\nabla_{X^h}\xi_Q = \nabla_{\xi_Q}X^h$. Hence we only need to compute one of them. By following the same argument we have in both cases

$$\begin{aligned}\langle\langle \eta_Q, \nabla_{\xi_Q}X^h\rangle\rangle &= -\langle\langle \nabla_{\xi_Q}\eta_Q, X^h\rangle\rangle\\ &= -\langle\langle \mathcal{S}(\xi_Q, \eta_Q), X^h\rangle\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle\langle Y^h, \nabla_{X^h}\xi_Q\rangle\rangle &= -\langle\langle \nabla_{X^h}Y^h, \xi_Q\rangle\rangle\\ &= -(-\frac{1}{2})\langle\langle \mathcal{B}(X^h, Y^h), \xi_Q\rangle\rangle.\end{aligned}$$

(The $-1/2$ factor is due to the asymmetric definitions of $\mathcal{B}$ and $\mathcal{S}$ in the two previous lemmas.) We can therefore summarize the two previous equalities in the following formula, sometime called Weingarten's formula:

$$\langle\langle V, \nabla_{X^h}\xi_Q\rangle\rangle \;=\; -\,\langle\langle \mathcal{S}(\xi_Q, \mathrm{ver}\,V), X^h\rangle\rangle + \frac{1}{2}\langle\langle \mathcal{B}(X^h, \mathrm{hor}\,V), \xi_Q\rangle\rangle. \qquad (4.3)$$

Note the tensorial dependence.

4.2. Local coordinate expressions. Recall the definition of $\mathcal{B}_{loc} : T_rB \times T_rB \to \mathfrak{g}$ in equation (2.7) and define $\mathcal{S}_{loc} : \mathfrak{g} \times \mathfrak{g} \to T_rB$ as

$$\mathcal{S}_{loc}(\xi,\eta) \triangleq T\pi\, \mathcal{S}(\xi_Q(q), \eta_Q(q))$$

for all $X \in T_rB$ and $\xi, \eta \in \mathfrak{g}$. From the lemmas in the previous subsection we easily obtain:

Lemma 3 (Integrability of horizontal and vertical subbundles). *Let* $X, Y \in T_{\pi(q)}B$ *and* $\xi, \eta \in \mathfrak{g}$. *Then*

$$\nabla_{X^h} Y^h = \left({}_B\nabla_X Y\right)^h - \frac{1}{2}\mathcal{B}_{loc}(X,Y)^\ell \tag{4.4}$$

$$\nabla_{\xi^\ell}\eta^\ell = {}_{\alpha_q}\nabla_{\xi^\ell}\eta^\ell + \mathcal{S}_{loc}(\xi,\eta)^h, \tag{4.5}$$

where $\mathcal{S}_{loc}$ *can be computed as*

$$\langle\!\langle X, \mathcal{S}_{loc}(\xi,\eta)\rangle\!\rangle_B = -\left\langle I({}_I\nabla_\xi \eta), \mathcal{A}_{loc}\cdot X\right\rangle - \frac{1}{2}\left\langle (DI(r)\cdot X)\xi, \eta\right\rangle.$$

Proof. Regarding the horizontal subbubdle, equation (4.4) is equivalent to the one proven in the previous subsection. Instead the computation on the vertical subbundle requires a little bit of algebra. Indeed, since vertical invariant vector fields don't enjoy anymore the useful Killing property, the proof here is based on a detailed application of the Koszul formula (2.3). ■

As in the previous section, we now look at cross covariant derivatives. Using Koszul formula, for all $V_q \in T_qQ$

$$\begin{aligned}\langle\!\langle V_q, \nabla_{\xi^\ell} X^h\rangle\!\rangle &= \mathcal{L}_{X^h}\langle\!\langle V_q, \xi^\ell\rangle\!\rangle + \langle\!\langle \xi^\ell, [V_q, X^h]\rangle\!\rangle + \langle\!\langle V_q, [\xi^\ell, X^h]\rangle\!\rangle \\ &= \mathcal{L}_{X^h}\langle\!\langle \operatorname{ver} V_q, \xi^\ell\rangle\!\rangle + \langle\!\langle \xi^\ell, [\operatorname{ver} V_q + \operatorname{hor} V_q, X^h]\rangle\!\rangle + \langle\!\langle \operatorname{ver} V_q, [\xi^\ell, X^h]\rangle\!\rangle.\end{aligned}$$

Setting $V_q = \eta^\ell$ we compute the vertical component. Recall that

$$[\eta^\ell, X^h(r)] = -(\operatorname{ad}_\eta \mathcal{A}_{loc}(r)\cdot X)^\ell, \tag{4.6}$$

so that

$$\begin{aligned}\langle\!\langle \eta^\ell, \nabla_{\xi^\ell} X^h\rangle\!\rangle &= \left\langle (DI(r)\cdot X)\eta, \xi\right\rangle - \langle I\xi, \operatorname{ad}_\eta(\mathcal{A}_{loc}\cdot X)\rangle - \langle I\eta, \operatorname{ad}_\xi(\mathcal{A}_{loc}\cdot X)\rangle \\ &= \left\langle (DI(r)\cdot X)\eta, \xi\right\rangle + \left\langle (I\operatorname{ad}_{\mathcal{A}_{loc}\cdot X} - \operatorname{ad}^*_{\mathcal{A}_{loc}\cdot X} I)\xi, \eta\right\rangle.\end{aligned}$$

Setting $V_q = Y^h$ we compute the horizontal component:

$$\begin{aligned}\langle\!\langle Y^h, \nabla_{\xi^\ell} X^h\rangle\!\rangle = \langle\!\langle \xi^\ell, [Y^h, X^h]\rangle\!\rangle &= -\frac{1}{2}\langle\!\langle \xi^\ell, \mathcal{B}(Y^h, X^h)\rangle\!\rangle \\ &= -\frac{1}{2}\langle I\xi, \mathcal{B}_{loc}(Y,X)\rangle.\end{aligned}$$

Finally, we don't need to repeat all this algebra to compute $\nabla_{X^h}\xi^\ell$ since it sufficies to change a $-$ sign in the previous equations to obtain the right answer.

5. The reduced Euler-Lagrange equations

We here present a global and a local version of the reduced Euler-Lagrange equations. To split the geodesic equation on Q into two set of equations, we need a parametrization of the vertical subspace V_qQ. For the case of a non-trivial bundle, the notion of *locked spatial velocity* $\omega \triangleq \mathcal{A}(\dot{q})$ allows us to write global equations. When the bundle is trivial (or locally for the nontrivial case), the *locked body velocity* Ω appears to be a more convenient choice.

The standard reference for this material is (Marsden & Scheurle 1993) and indeed in Appendix B we show the equivalence between the standard equations and the ones presented in the following.

5.1. Simple mechanical systems with symmetry. A simple mechanical systems, whose kinetic energy is given by $\langle\langle\cdot,\cdot\rangle\rangle$, can be described in terms of ∇ by the following intrinsic equation:

$$\nabla_{\dot{q}}\dot{q} = 0, \tag{5.1}$$

or in local coordinates

$$\ddot{q}^i Y_i + \dot{q}^j\dot{q}^k \nabla_{Y_j} Y_k = 0, \tag{5.2}$$

where $\{Y_i\}$ is a basis for T_qQ.

Assume now that $Q \to B$ is a principal fiber bundle with structure group G and that the metric $\langle\langle\cdot,\cdot\rangle\rangle$ is invariant under the group action. We start our analysis of the equation of motion (5.1) by recovering the classic conservation law: the notion of infinitesimal isometry and Killing equation (2.8) are perfectly suited for this purpose.

Lemma 4 (Conservation of momentum). *Let $J : TQ \to \mathfrak{g}^*$ be the momentum map defined by $\langle J(\dot{q}), \xi_Q\rangle \triangleq \langle\langle \dot{q}, \xi_Q\rangle\rangle$. Then $J(\dot{q}(t))$ is a constant of motion.*

Proof. Since the connection ∇ is compatible with the metric

$$\begin{aligned}\frac{d}{dt}\langle\langle \dot{q}, \xi_Q\rangle\rangle &= \langle\langle \nabla_{\dot{q}}\dot{q}, \xi_Q\rangle\rangle + \langle\langle \dot{q}, \nabla_{\dot{q}}\xi_Q\rangle\rangle \\ &= \langle\langle \dot{q}, \nabla_{\dot{q}}\xi_Q\rangle\rangle = 0,\end{aligned}$$

where the last equality holds since ξ_Q is the infinitesimal isometry associated with the group action and threfore the Killing equation (2.8) holds. ∎

5.2. Lagrangian reduction on a non-trivial bundle. To split the equation of motion (5.1) into two sets of equations, we decompose the velocity $\dot{q}$ in its horizontal and vertical components as follows

$$\dot{q} = \dot{r}^\alpha Y_\alpha^h + \omega^a e_a^r, \tag{5.3}$$

where $\{X_\alpha\}$ are a basis for $T_{\pi(q)}B$, the $\{e_a\}$ span the Lie algebra $\mathfrak{g}$ and, as defined previously, $e_a^r = (e_a)_Q$. Note that a more natural parametrization of $\dot{q}$ might consider the conserved quantity J as variable. Here we go along a more complicated path with the goal of performing Lagrangian reduction.

Proposition 1 (Reduced Euler-Lagrange equations). *Consider a simple mechanical system with symmetry. For a curve $q(t) \in Q$, define the* shape $r = \pi(q) \in Q/G = B$ *and the* locked spatial velocity $\omega = \mathcal{A}(\dot{q}) \in \mathfrak{g}$. *Then the following are equivalent:*

i) $q(t)$ satisfies the Euler-Lagrange equations (5.2) *on Q;*

ii) the reduced Euler-Lagrange equations hold:

$$\langle\langle \delta r, {}_B\nabla_{\dot r}\dot r\rangle\rangle = \langle \mathbb{I}(q)\omega, \mathcal{B}(\dot r, \delta r)\rangle + \frac{1}{2}\left\langle (D\mathbb{I}(q)\cdot \delta r^h)\omega, \omega\right\rangle \tag{5.4}$$

$$\dot\omega + \nabla^r_\omega \omega = -\mathbb{I}(q)^{-1}\left(D\mathbb{I}(q)\cdot \dot r^h\right)\omega. \tag{5.5}$$

Proof. Corresponding to the decomposition (5.3), we write the equation of motion as

$$\ddot r^\alpha X^h_\alpha + \dot\omega^a e^r_a \quad + \quad \dot r^\alpha \dot r^\beta \nabla_{X^h_\alpha} X^h_\beta + \omega^a\omega^b \nabla_{e^r_a} e^r_b$$
$$= -\dot r^\alpha \omega^a (\nabla_{e^r_a} X^h_\alpha + \nabla_{X^h_\alpha} e^r_a), \tag{5.6}$$

where for the cross terms on the right hand side it holds $\nabla_{e^r_a} X^h_\alpha + \nabla_{X^h_\alpha} e^r_a = 2\nabla_{e^r_a} X^h_\alpha$, as explained in Subsection 4.1.

We now project the previous equation to the base and, to simplify the notation, multiply both sides with the tangent vector $\delta r \in T_{\pi(q)}B$ to obtain:

$$\langle\langle \delta r, \ddot r^\alpha X_\alpha + \dot r^\alpha \dot r^\beta {}_B\nabla_{X_\alpha} X_\beta\rangle\rangle \;=\; -2\; \dot r^\alpha \omega^a \langle\langle \delta r^h, \nabla_{e^r_a} X^h_\alpha\rangle\rangle \;-\; \omega^a\omega^b \langle\langle \delta r^h, \nabla_{e^r_a} e^r_b\rangle\rangle.$$

Equation (5.4) follows from the results in Subsection 4.1.

The same steps lead to the equation (5.5) along the vertical directions. By applying the mechanical connection one form $\mathcal{A}$ to equation (5.6) we obtain

$$\dot\omega + \omega^a\omega^b \mathcal{A}(\nabla_{e^r_a} e^r_b) = -\omega^a \dot r^\alpha \mathcal{A}(\nabla_{e^r_a} {X_\alpha}^h) - \dot r^\alpha \dot r^\beta \mathcal{A}(\nabla_{X^h_\alpha} {X_\beta}^h).$$

As before, we evaluate the right hand side using the equalities obtained in Subsection 4.1 ∎

5.3. Lagrangian reduction on a trivial bundle. Here we rely on the notation introduced in Subsection 4.2. On $Q = B\times G$, define the *locked body velocity* $\Omega \triangleq g^{-1}\dot g + \mathcal{A}_{loc}(r)\cdot \dot r$ and decompose the velocity $\dot q$ in its horizontal and vertical components as

$$\dot q = \dot r^\alpha Y^h_\alpha + \Omega^a e^\ell_a, \tag{5.7}$$

where, as before, $\{X_\alpha\}$ are a basis for T_rB, $\{e_a\}$ span the Lie algebra $\mathfrak{g}$ and $e^\ell_a = (0, T_eL_g(e_a))$.

Proposition 2 (Reduced Euler-Lagrange equations on a trivial bundle). *Consider a simple mechanical system with symmetry. For a curve* $q(t) = (r(t), g(t)) \in Q = B\times G$, *let* $\Omega = g^{-1}\dot g + A_{loc}(r)\cdot \dot r \in \mathfrak{g}$ *be the* locked spatial velocity. *Then the following are equivalent:*

i) $q(t)$ *satisfies the Euler-Lagrange equations* (5.2) *on* Q*;*

ii) the reduced Euler-Lagrange equations hold:

$$\langle\langle \delta r, {}_B\nabla_{\dot r}\dot r\rangle\rangle = \langle I(r)\Omega, \mathcal{B}_{loc}(\delta r, \dot r)\rangle + \langle I({}_I\nabla_\Omega\Omega), \mathcal{A}_{loc}\cdot\delta r\rangle + \frac{1}{2}\langle (DI(r)\cdot\delta r)\Omega, \Omega\rangle, \tag{5.8}$$

$$\dot\Omega + \nabla^\ell_\Omega \Omega = -I(r)^{-1}\Big(DI(r)\cdot \dot r\Big)\Omega + I(r)^{-1}\,\mathrm{ad}^*_{\mathcal{A}_{loc}(r)\cdot\dot r}(I(r)\Omega). \tag{5.9}$$

Proof. We follow the same steps as in the previous proof. Corresponding to the decomposition (5.7), we write the equation of motion as

$$\ddot{r}^\alpha X_\alpha^h + \dot{\Omega}^a e_a^\ell \quad + \quad \dot{r}^\alpha \dot{r}^\beta \nabla_{X_\alpha^h} X_\beta^h + \Omega^a \Omega^b \nabla_{e_a^\ell} e_b^\ell = -\dot{r}^\alpha \Omega^a (\nabla_{e_a^\ell} X_\alpha^h + \nabla_{X_\alpha^h} e_a^\ell)$$
$$= -\dot{r}^\alpha \Omega^a \left\langle e_a^\ell : X_\alpha^h \right\rangle. \quad (5.10)$$

We now project the previous equation to the base and, to simplify the notation, multiply both sides with the tangent vector $\delta r \in T_{\pi(q)}B$ to obtain:

$$\langle\!\langle \delta r, \ddot{r}^\alpha X_\alpha + \dot{r}^\alpha \dot{r}^\beta {}_B\nabla_{X_\alpha} X_\beta \rangle\!\rangle =$$
$$- \ \dot{r}^\alpha \Omega^a \langle\!\langle \delta r^h, \langle e_a^\ell : X_\alpha^h \rangle \rangle\!\rangle - \Omega^a \Omega^b \langle\!\langle \delta r^h, \nabla_{e_a^\ell} e_b^\ell \rangle\!\rangle.$$

We compute the right hand side using some equalities proven in Subsection 4.2:

$$-\dot{r}^\alpha \Omega^a \langle\!\langle \delta r^h, \langle e_a^\ell : X_\alpha^h \rangle \rangle\!\rangle = -\dot{r}^\alpha \Omega^a \left\langle I(r) e_a \ , \ \mathcal{B}_{loc}(X_a^h, \delta r^h) \right\rangle$$
$$= - \left\langle I(r)\Omega \ , \ \mathcal{B}_{loc}(\dot{r}^h, \delta r^h) \right\rangle$$

$$-\Omega^a \Omega^b \langle\!\langle \delta r^h, \nabla_{e_a^\ell} e_b^\ell \rangle\!\rangle = -\Omega^a \Omega^b \langle\!\langle \delta r, \mathcal{S}_{loc}(e_a, e_b) \rangle\!\rangle_B$$
$$= -\langle\!\langle \delta r, \mathcal{S}_{loc}(\Omega, \Omega) \rangle\!\rangle_B$$
$$= \langle I(_{I}\nabla_\Omega \Omega), \mathcal{A}_{loc} \cdot \delta r \rangle + \frac{1}{2} \left\langle (DI(r) \cdot \delta r)\Omega, \Omega \right\rangle,$$

and equation (5.8) follows.

The same steps lead to the equation (5.9) along the vertical directions: let $\eta \in \mathfrak{g}$ and multiply equation (5.10) by η^ℓ to obtain

$$\left\langle I(r)\eta, \dot{\Omega} \right\rangle + \Omega^a \Omega^b \langle\!\langle \eta^\ell, \nabla_{e_a^\ell} e_b^\ell \rangle\!\rangle = -\Omega^a \dot{r}^\alpha \langle\!\langle \eta^\ell, \langle e_a^\ell : X_\alpha^h \rangle \rangle\!\rangle - \dot{r}^\alpha \dot{r}^\beta \langle\!\langle \eta^\ell, \nabla_{X_\alpha^h} X_\beta{}^h \rangle\!\rangle.$$

As before, we evaluate the right hand side using the equalities obtaines in Subsection 4.2:

$$\Omega^a \Omega^b \langle\!\langle \eta^\ell, \nabla_{e_a^\ell} e_b^\ell \rangle\!\rangle = \Omega^a \Omega^b \langle I(r)\eta, {}_I\nabla_{e_a} e_b \rangle = \langle I(r)\eta, {}_I\nabla_\Omega \Omega \rangle$$

$$\Omega^a \dot{r}^\alpha \langle\!\langle \eta^\ell, \langle e_a^\ell : X_\alpha^h \rangle \rangle\!\rangle = \Omega^a \dot{r}^\alpha \left\langle (DI(r) \cdot X_\alpha) e_a, \eta \right\rangle + \langle I(r) e_a \ , \ \mathrm{ad}_\eta \mathcal{A}_{loc}(r) \cdot X_\alpha \rangle$$
$$= \left\langle (DI(r) \cdot \dot{r})\Omega, \eta \right\rangle + \langle I(r)\Omega \ , \ \mathrm{ad}_\eta \mathcal{A}_{loc}(r) \cdot \dot{r} \rangle$$

$$\dot{r}^\alpha \dot{r}^\beta \langle\!\langle \eta^\ell, \nabla_{X_\alpha^h} X_\beta{}^h \rangle\!\rangle = -\frac{1}{2} \dot{r}^\alpha \dot{r}^\beta \langle\!\langle \eta^\ell, \mathcal{B}_{loc}(X_\alpha^h, X_\beta^h)^\ell \rangle\!\rangle$$
$$= -\frac{1}{2} \langle\!\langle I(r)\eta, \mathcal{B}_{loc}(\dot{r}, \dot{r}) \rangle\!\rangle = 0.$$

The result now follows by eliding η from the equations. ∎

6. Controllability

We refer to Lewis & Murray (1996) and to the previous report (Bullo & Lewis 1996*a*) for part of the notation in this section.

A *simple mechanical control system with symmetries* is defined by a Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ on a configuration manifold Q (defining the kinetic energy), a function V on Q (defining the potential energy), and m one-forms, $F^1,\ldots,F^m$, on Q (defining the inputs). Additionally a Lie group G acts freely and properly on Q by isometries and both the potential energy and the input one forms are invariant under the action.

Denote with $q(t)\in Q$ the configuration of the system and with $\dot q(t)\in T_qQ$ its velocity. The equation of motion are then

$$\nabla_{\dot q(t)}\dot q(t) = dV^{\sharp}(q(t)) + u^a(t)Y_a(q(t)), \tag{6.1}$$

where ∇ is the Riemannian connection associated with $\langle\langle\cdot,\cdot\rangle\rangle$ and $Y_a = (F^a)^{\sharp}$ are the input vector fields. Additionally, let $\mathcal{Y} = \{Y_1,\ldots,Y_m\}$.

Theorem 6.1 (Lewis-Murray). *The system* (6.1) *is*

i) locally configuration accessible at $q\in Q$ *if* $\mathrm{rank}(\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{Y}))(q)) = \dim(Q)$,

ii) STLCC at $q\in Q$ *if it is locally configuration accessible at* q *and if every bad symmetric product can be written as a linear combination of good symmetric products of lower order at* q, *and*

iii) equilibrium controllable if it is STLCC at each $q\in Q$.

Based on the general condition above, we split our analysis into three subcases, depending on whether the mechanical system has some shape variables or not, and on whether the control inputs preserve the momentum or not. We call horizontal forces, the input vector fields that preserve the momentum. Note that for the pure Lie group case, when no shape is present, forces cannot be only horizontal.

6.1. The pure Lie group case. This class includes rigid bodies with external forces and torques. We refer to the previous report (Bullo & Lewis 1996*a*) for a precise description of this case.

6.2. Bundle case with pure horizontal forces. This class includes rigid bodies with internal momentum wheels and more generally shape variables.

Assume the mechanical system with symmetry has only horizontal forces, that is the input vectors Y_i are horizontal.

Lemma 5. *The symmetric closure of a family of horizontal vector fields is horizontal.*

Proof. For all $\xi\in\mathfrak{g}$

$$\langle\langle Y_i,\xi_Q\rangle\rangle = 0.$$

Taking Lie derivative along Y_i:

$$0 = \mathcal{L}_{Y_i}\langle\langle Y_i,\xi_Q\rangle\rangle = \langle\langle \nabla_{Y_i}Y_i,\xi_Q\rangle\rangle + \langle\langle Y_i,\nabla_{Y_i}\xi_Q\rangle\rangle.$$

Recall that ξ_Q is a Killing vectorf field and therefore $\langle\langle\nabla_a\xi_Q,a\rangle\rangle = 0$ for all vector fields a. Hence

$$\langle\langle\nabla_{Y_i}Y_i,\xi_Q\rangle\rangle = \langle\langle\langle Y_i : Y_i\rangle,\xi_Q\rangle\rangle = 0.$$

The proof is complete by noting that, thanks to the linearity of ∇:

$$\langle Y_i : Y_j \rangle = \langle Y_i + Y_j : Y_i + Y_j \rangle - \langle Y_i : Y_i \rangle - \langle Y_j : Y_j \rangle .$$

■

Note that this simple result is a direct consequence of the geometric properties of the horizontal subbundle described in the previous sections: hor_q is indeed geodesically invariant, since every geodesic starting with a horizontal velocity maintains a horizontal velocity for all time.

Given this conservation law (symmetric product of horizontal remains horizontal), the accessibility computations simplify and drop down to the base space. Since $\mathcal{Y}$ is invariant and horizontal, we can drop it down to the base space. Let $\mathcal{Y}_B = T\pi \cdot \mathcal{Y}$. Then the previous lemma allows us to compute:

$$\overline{\text{Lie}}_Q(\overline{\text{Sym}}_Q(\mathcal{Y})) = \overline{\text{Lie}}_Q(\overline{\text{Sym}}_B(\mathcal{Y}_B)^h), \tag{6.2}$$

where *symclosB*$\cdot$ means symmetric closure on the base space.

Remark 4 (Comparison with previous results). Within the framework for mechanical controllability, this result is in agreement with the Ambrose-Singer theorem, see (Kobayashi & Nomizu 1963*a*). This leads to the application of the techniques described in Scott's work (Kelly & Murray 1995).

But it is a stronger result in that it allows for the base dynamics to be not fully actuated. In this latter case, the controllability check allows for new possible directions generated by the symmetric product on the base space. As an example we examine the case of three coupled planar rigid body with only one actuated joint.

6.3. **Bundle case with general forces.** Assume now that the mechanical system with symmetry has general forces which do not need preserve the momentum. Then the formalism in Section 4 applies, maybe !?

This case is difficult since it is not "natural" with respect to the tool employed in this report. First off, forces are not natural and cannot be decomposed into horizontal and vertical components before the kinetic energy is given. Hence constructing $\mathcal{Y}$, there is no structure to the generic vector field Y_i. Instead it might be more appropriate to express all the computations and geometric concepts associated with them on the cotangent bundle side.

Indeed, going back to the reduced Euler-Lagrange equations, these can be written in a much simpler form than the one presented in the previous subsections, if we were to allow the use of the momentum as variable.

7. Examples

We classify the examples mentioned in the previous section and depicte some of them in Figure 3.

Lie group case: rigid bodies with external forces and torques.

Bundle case with horizontal inputs: rigid bodies with momentum wheels or oscillators, coupled rigid bodies with internal torques and forces (e.g. planar body with arms). Swimming animals and falling cats/. (Also: symmetric satellite with inputs that preserve the right invariance of the kinetic energy).

Bundle case with general inputs: Classic examples are the pendulum on a cart and the robotic leg. More generally, coupled bodies with complete actuation at one point and planar rotating chains as studied in (Baillieul 1987). Underwater vehicle on $SE(3)$ has inertia matrix invariant under translation, but not rotations, hence its configuration manifold can be thought of as a bundle; also its input do not preserve the momentum.

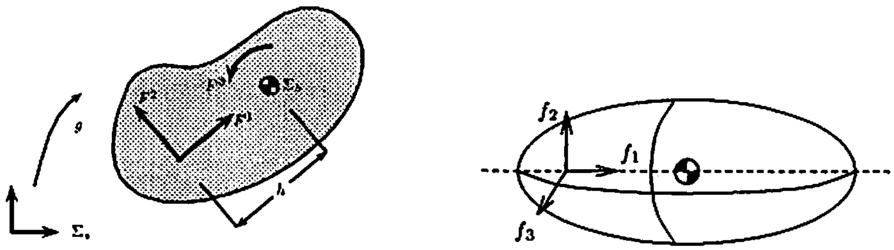

FIGURE 1. **Lie group case:** Single rigid bodies with external torques and forces.

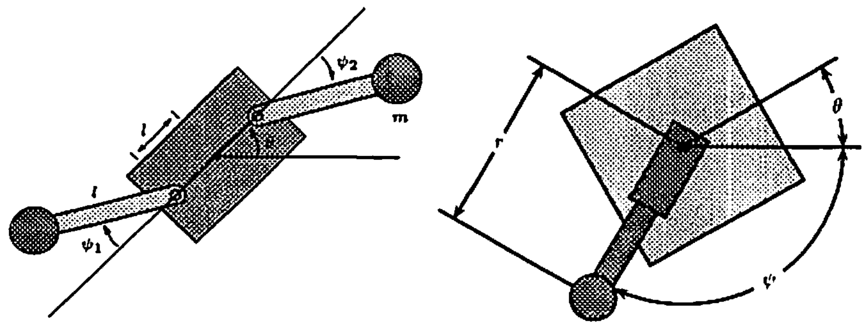

FIGURE 2. **Bundle case with horizontal forces:** Planar bodies with arms, with torques applied on ϕ_1 and ϕ_2. Hopping robot with torque applied on $\phi - \theta$ and r directions.

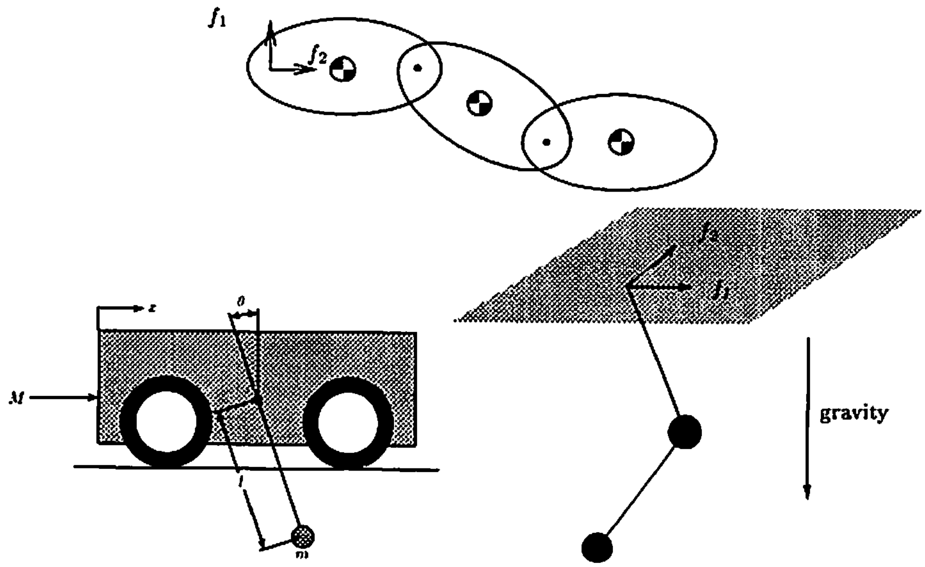

FIGURE 3. **General case:** Coupled planar bodies with external forces at one end. Pendulum on a cart with actuation on the abelian group $\mathbb{R}$. Double spherical pendulum with actuation at the top on the group $\mathbb{R}^2$.

8. Conclusions

With this project I feel that I have accomplished a deeper understanding in the area of mechanical systems with symmetries. More precisely the theories of

1. Riemannian geometry of fiber bundles and
2. Lagrangian reduction

are now familiar to me. In terms of research done in the area of nonlinear control of mechanical systems, I was able to unify previous results in the following two areas:

3. mechanical systems on Lie groups and
4. mechanical systems with symmetries and horizontal forces (i.e. internal forces that preserve the momentum).

Work still remains to be done for the general case of

- mechanical systems with symmetries and generic forces.

In a more general view, this work is interesting in its implications for other research areas, like nonholonomic systems with symmetry, and I believe that it is instructive in its relationships to different problems but for the same class of systems, like the energy momentum method.

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Appendix A. Riemannian connections on a Lie group

In this appendix we describe the Riemannian connection of a Lie group endowed with a left invariant metric and of the fiber $G \cdot q$, where only right invariant vector fields are globally defined $\xi^r = \xi_Q$ and where the locked inertia tensor $\mathbb{I}(q)$ is equivariant. Work in this area (motivated by very different interests), can be found in Arnold (1966), Bloch & Crouch (1995), Bullo & Lewis (1996*b*). Additionally, some links can be found with the work in (Simo et al. 1991).

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Given $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{g}}$ an an inner product on $\mathfrak{g}$, we obtain a metric structure on TG by left-translation. Such a Riemannian metric is by construction *left-invariant*, as it is preserved by all left translations L_g.

We use the decomposition (2.4)

$$\nabla_X Y = \frac{1}{2} \langle X : Y \rangle + \frac{1}{2} [X, Y]$$

to concentrate only on the symmetric product, since the equalities

$$[\xi^\ell, \eta^\ell] = [\xi, \eta]_{\mathfrak{g}}^\ell \tag{A.1}$$

$$[\xi^\ell, \eta^r] = 0 \tag{A.2}$$

$$[\xi^r, \eta^r] = -[\xi, \eta]_{\mathfrak{g}}^r, \tag{A.3}$$

always hold. Recall that by $[\cdot, \cdot]_{\mathfrak{g}}$ we mean the Lie bracket on the Lie algebra and that

$$\xi^\ell \triangleq T_e L_g \cdot \xi \qquad \text{and} \qquad \xi^r \triangleq T_e R_g \cdot \xi.$$

Equation (A.2) is proven showing that the flow of the two vector fields commute. To perform the various computations, Koszul formula (2.3) in the form

$$\begin{aligned}\langle\!\langle X\,,\ \langle Y : Z\rangle\rangle\!\rangle = \mathcal{L}_Z\langle\!\langle X, Y\rangle\!\rangle + \mathcal{L}_Y\langle\!\langle X, Z\rangle\!\rangle - \mathcal{L}_X\langle\!\langle Y, Z\rangle\!\rangle \\ + \langle\!\langle Z, [X, Y]\rangle\!\rangle + \langle\!\langle Y, [X, Z]\rangle\!\rangle\end{aligned} \tag{A.4}$$

reveals itself very useful. Denote with I the invariant metric of G and its restriction to $\mathfrak{g}$: $I : \mathfrak{g} \to \mathfrak{g}^*$. There exists a natural bilinear form on $(\mathfrak{g}, I)$ given by

$$\langle \xi : \eta \rangle_{\mathfrak{g},I} \triangleq -I^{-1}\left(\mathrm{ad}_\xi^* I\eta + \mathrm{ad}_\eta^* I\xi\right). \tag{A.5}$$

We state our results in the following lemmas. Note that the first one also proves that the Riemannian connection ∇ on G is left invariant, as defined in equation (2.9).

Lemma 6 (Symmetric product of invariant vector fields).

$$\langle \xi^\ell : \eta^\ell \rangle = \left(\langle \xi : \eta\rangle_{\mathfrak{g},I}\right)^\ell \tag{A.6}$$

Proof. For all $\zeta \in \mathfrak{g}$, using (A.4)

$$\begin{aligned}\langle\!\langle \zeta^\ell, \langle \xi^\ell : \eta^\ell\rangle\rangle\!\rangle &= 0 + 0 - 0 + \langle\!\langle \xi^\ell, [\zeta^\ell, \eta^\ell]\rangle\!\rangle + \langle\!\langle \eta^\ell, [\zeta^\ell, \xi^\ell]\rangle\!\rangle \\ &= \langle\!\langle \xi, [\zeta, \eta]_{\mathfrak{g}}\rangle\!\rangle + \langle\!\langle \eta, [\zeta, \xi]_{\mathfrak{g}}\rangle\!\rangle.\end{aligned}$$

■

Lemma 7 (Symmetric product of vector fields).

$$\langle \xi^r : \eta^\ell\rangle(g) = \left([\mathrm{Ad}_{g^{-1}}\xi, \eta]_{\mathfrak{g}} + \langle \mathrm{Ad}_{g^{-1}}\xi : \eta\rangle_{\mathfrak{g},I}\right)^\ell$$

$$\langle \xi^r : \eta^r\rangle(g) = \left(\langle \mathrm{Ad}_{g^{-1}}\xi : \mathrm{Ad}_{g^{-1}}\eta\rangle_{\mathfrak{g},I}\right)^\ell$$

Proof. Recall the following basic equalities in Marsden & Ratiu (1994):

$$\mathcal{L}_{\xi^\ell}\,\mathrm{Ad}_{g^{-1}}\eta = [\mathrm{Ad}_{g^{-1}}\eta, \xi]_{\mathfrak{g}}$$

$$\mathcal{L}_{\xi^r}\,\mathrm{Ad}_{g^{-1}}\eta = \mathrm{Ad}_{g^{-1}}[\eta, \xi]_{\mathfrak{g}}.$$

For the case of two right invariant vector fields we have, as it was proven above,

$$\begin{aligned}\langle\!\langle \zeta^\ell, \langle \xi^r : \eta^r\rangle\rangle\!\rangle &= -\mathcal{L}_{\zeta^\ell}\langle\!\langle \xi^r, \eta^r\rangle\!\rangle = -\mathcal{L}_{\zeta^\ell}\langle I\xi, \eta\rangle \\ &= -\left\langle I\left[\mathrm{Ad}_{g^{-1}}\eta, \zeta\right]_{\mathfrak{g}}, \mathrm{Ad}_{g^{-1}}\xi\right\rangle - \left\langle I\left[\mathrm{Ad}_{g^{-1}}\xi, \zeta\right]_{\mathfrak{g}}, \mathrm{Ad}_{g^{-1}}\eta\right\rangle \\ &= \left(\langle \mathrm{Ad}_{g^{-1}}\xi : \mathrm{Ad}_{g^{-1}}\eta\rangle_{\mathfrak{g},I}\right)^\ell.\end{aligned}$$

And a similar proof works for the mixed case. ■

Regarding the case of right invariant vector fields, we can define the tensor $\mathbb{I}(g) : \mathfrak{g} \to \mathfrak{g}^*$ as $\mathbb{I}(g) = \mathrm{Ad}_g^* I\,\mathrm{Ad}_g$ and define the operator $\langle\cdot : \cdot\rangle_{\mathfrak{g},\mathbb{I}}$ as in equation (A.5).

Lemma 8 (Symmetric product of right invariant vector fields).

$$\langle \xi^r : \eta^r \rangle = \left(\langle \xi : \eta \rangle_{\mathfrak{g}, \mathbb{I}} \right)^r. \tag{A.7}$$

Proof. The proof takes advantage of the Killing equation (2.8) which characterizes infinitesimal isometries. For all $\zeta \in \mathfrak{g}$,

$$\begin{aligned}
\langle\!\langle \zeta^r, \langle \xi^r : \eta^r \rangle \rangle\!\rangle &= \langle\!\langle \zeta^r, \nabla_{\xi^r} \eta^r \rangle\!\rangle + \langle\!\langle \zeta^r, \nabla_{\eta^r} \xi^r \rangle\!\rangle \\
&= -\langle\!\langle \xi^r, \nabla_{\zeta^r} \eta^r \rangle\!\rangle - \langle\!\langle \eta^r, \nabla_{\zeta^r} \xi^r \rangle\!\rangle && \text{by Killing eq.} \\
&= -\mathcal{L}_{\zeta^r} \langle\!\langle \xi^r, \eta^r \rangle\!\rangle && \text{Riemannian connection} \\
&= -\langle\!\langle \xi^r, \mathcal{L}_{\zeta^r} \eta^r \rangle\!\rangle - \langle\!\langle \eta^r, \mathcal{L}_{\zeta^r} \xi^r \rangle\!\rangle && \text{infinitesimal isometry} \\
&= -\langle\!\langle \xi^r, [\zeta^r, \eta^r] \rangle\!\rangle - \langle\!\langle \eta^r, [\zeta^r, \xi^r] \rangle\!\rangle \\
&= -\left\langle \xi, \mathbb{I} \, [\zeta, \eta]_{\mathfrak{g}} \right\rangle - \left\langle \eta, \mathbb{I} \, [\zeta, \xi]_{\mathfrak{g}} \right\rangle.
\end{aligned}$$

■

A.1. **The fiber of a principal bundle.** Consider now the "affine group" $G \cdot q$, generic fiber of a principal bundle $\pi : Q \to B$. We describe the Riemannian connection ${}_{G \cdot q}\nabla$ on $G \cdot q$, by looking at Lie brackets and symmetric products.

Regarding Lie brackets, only equation (A.3) is well-defined:

$$[\xi_Q, \eta_Q] = [\xi, \eta]_{\mathfrak{g}Q}.$$

Regarding the symmetric product, the last lemma easily generalizes (but not the second to last!):

$$\langle \xi_Q : \eta_Q \rangle (g) = \left(\langle \xi : \eta \rangle_{\mathfrak{g}, \mathbb{I}} \right)_Q. \tag{A.8}$$

Remark 5 (Derivatives of locked inertia tensor). The steps used in the previous derivation are the same as the ones in Simo et al. (1991, Proposition 2.3). Indeed the result there is that:

$$D\mathbb{I}(q) \cdot \eta_Q = -\operatorname{ad}_\eta^* \mathbb{I}(q) - \mathbb{I}(q) \operatorname{ad}_\eta. \tag{A.9}$$

Additionally,

$$D\mathbb{I}(r, e) \cdot X^h = DI(r) \cdot X + \left(\operatorname{ad}_\zeta^* I(r) + I(r) \operatorname{ad}_\zeta \right) \Big|_{\zeta = \mathcal{A}_{loc}(r) \cdot X} \tag{A.10}$$

□

We finish by summarizing the results obtained and introducing some additional notation. On a Lie group

$$\nabla_{\xi^\ell} \eta^\ell = \left({}_l\nabla_\xi \eta \right)^\ell, \tag{A.11}$$

and on the fiber of a principal bundle

$${}_{G \cdot q}\nabla_{\xi_Q} \eta_Q = \left(\nabla_\xi \eta \right)_Q. \tag{A.12}$$

Appendix B. Verification of the reduced equations

It is instructive to verify that the equations presented in the previos section coincide with the standard ones in (Marsden & Scheurle 1993). We start with a left invariant Lagrangian $L : TQ \to \mathbb{R}$ which is only kinetic energy. Introcing a potential energy term is only an additional burden in notation. Following the steps described in (Ostrowski 1995, page 43), let $(r,g) \in Q$, let $\eta \triangleq g^{-q}\dot{g}$ be the angular velocity and define $l(r,\dot{r},\eta) = L(r,\dot{r},g,\dot{g})$. In local coordinates one can show that

$$l(r,\dot{r},\eta) = \frac{1}{2}\begin{bmatrix}\dot{r}\\ \eta\end{bmatrix}^T \begin{bmatrix} m(r) & \mathcal{A}_{loc}(r)^T I(r) \\ I(r)\mathcal{A}_{loc}(r) & I(r)\end{bmatrix}\begin{bmatrix}\dot{r}\\ \eta\end{bmatrix}.$$

Now we implement the shift in velocity and define the locked angular velocity as $\Omega = \eta + \mathcal{A}_{loc}(r)\dot{r}$. As it is well known, the kinetic energy expressed in these new coordinates $l_{lock}(r,\dot{r},\Omega) \triangleq l(r,\dot{r},\Omega - \mathcal{A}_{loc}(r)\dot{r})$ has the diagonal form

$$l_{lock}(r,\dot{r},\Omega) = \frac{1}{2}\begin{bmatrix}\dot{r}\\ \Omega\end{bmatrix}^T \begin{bmatrix} m(r) - \mathcal{A}_{loc}(r)^T I(r)\mathcal{A}_{loc}(r) & 0 \\ 0 & I(r)\end{bmatrix}\begin{bmatrix}\dot{r}\\ \Omega\end{bmatrix}.$$

Marsden & Scheurle (1993, page 17) write the reduced Euler-Lagrange equations as

$$\frac{d}{dt}\frac{\partial l_{lock}}{\partial \dot{r}^\alpha} - \frac{\partial l_{lock}}{\partial r^\alpha} = c^d_{ab} A^a_\alpha \frac{\partial l_{lock}}{\partial \dot{\Omega}^d}\Omega^b + B^d_{\alpha\beta}\frac{\partial l_{lock}}{\partial \dot{\Omega}^d}\dot{r}^\beta \qquad \text{(E.L. on B)}$$

$$\frac{d}{dt}\frac{\partial l_{lock}}{\partial \dot{\Omega}^b} = x^a_{db}\frac{\partial l_{lock}}{\partial \dot{\Omega}^a}\Omega^d - c^a_{db}A^d_\alpha \frac{\partial l_{lock}}{\partial \dot{\Omega}^a}\dot{r}^\alpha, \qquad \text{(E.P. on } \mathfrak{g})$$

where we denote with A^a_α and $B^a_{\alpha\beta}$ the local components of the connection one form $\mathcal{A}_{loc}$ and its curvature. We state the equivalence between the two approaches as follows.

Corollary 1. *The equations presented in Proposition 2 in Section 5 coincide with the reduced Euler-Lagrange equations* (E.L. on B) *and* (E.P. on $\mathfrak{g}$).

Proof. Let $M(r) = m(r) - \mathcal{A}_{loc}(r)^T I(r)\mathcal{A}_{loc}(r)$ and compute

$$\begin{aligned}\frac{d}{dt}\frac{\partial l_{lock}}{\partial \dot{r}^\alpha} - \frac{\partial l_{lock}}{\partial r^\alpha} &= (\frac{d}{dt}\frac{\partial}{\partial \dot{r}^\alpha} - \frac{\partial}{\partial r^\alpha})\Big(\frac{1}{2}\dot{r}^T M\dot{r} + \frac{1}{2}\Omega^T I\Omega\Big) \\ &= (\frac{d}{dt}\frac{\partial}{\partial \dot{r}^\alpha} - \frac{\partial}{\partial r^\alpha})\Big(\frac{1}{2}\dot{r}^T M\dot{r}\Big) + \frac{1}{2}\Omega^T\frac{\partial I}{\partial r^\alpha}\Omega \\ &= \Big(M\,{}_B\nabla_{\dot{r}}\dot{r}\Big)_\alpha + \frac{1}{2}\Omega^T\frac{\partial I}{\partial r^\alpha}\Omega.\end{aligned}$$

We then have

$$\frac{\partial l_{lock}}{\partial \dot{\Omega}^d} = (I\Omega)_d,$$

$$\frac{d}{dt}\frac{\partial l_{lock}}{\partial \dot{\Omega}^d} = \frac{\partial I_{ad}}{\partial r^\alpha}\dot{r}^\alpha\Omega^d + I\dot{\Omega}^d.$$

Eventually, let $\delta r \in TB$ and multiply (E.L. on B) with δr^α:

$$\begin{aligned}\langle\langle \delta r, {}_B\nabla_{\dot{r}}\dot{r}\rangle\rangle &= -\frac{1}{2}\Omega^T\frac{\partial I}{\partial r^\alpha}\Omega\delta r^\alpha + c^d_{ab}(\mathcal{A}_{loc}\cdot\delta r)^a(I\Omega)_d\Omega^b + B^d_{\alpha\beta}(I\Omega)_d\dot{r}^\beta\delta r^\alpha \\ &= -\frac{1}{2}\Omega^T(DI(r)\cdot\delta r)\Omega + (I(r)\Omega)^T[\mathcal{A}_{loc}\cdot\delta r,\Omega] + (I(r)\Omega)^T\mathcal{B}_{loc}(\delta r,\dot{r}) \\ &= -\frac{1}{2}\langle\Omega,(DI(r)\cdot\delta r)\Omega\rangle + \langle I(r)\Omega,[\mathcal{A}_{loc}\cdot\delta r,\Omega] + \mathcal{B}_{loc}(\delta r,\dot{r})\rangle.\end{aligned}$$

Now we look at the Euler-Poincarè part: let $\eta \in \mathfrak{g}$. Multiplying (E.P. on $\mathfrak{g}$) with η^b we have

$$\begin{aligned}\left\langle I(r)\dot{\Omega}, \eta \right\rangle &= -\left\langle (DI(r) \cdot \dot{r})\Omega, \eta \right\rangle + \left(I(r)\Omega\right)_a c^a_{db} \Omega^d \eta^b - (I(r)\Omega)_a c^a_{db} (\mathcal{A}_{loc} \cdot \dot{r})^d \eta^b \\ &= -\left\langle (DI(r) \cdot \dot{r})\Omega, \eta \right\rangle + \left\langle I(r)\Omega, [\Omega - \mathcal{A}_{loc} \cdot \dot{r}, \eta]_{\mathfrak{g}} \right\rangle\end{aligned}$$

and eliding η

$$I(r)\dot{\Omega} = \mathrm{ad}^*_{\Omega}(I(r)\Omega) - (DI(r) \cdot \dot{r})\Omega - \mathrm{ad}^*_{\mathcal{A}_{loc} \cdot \dot{r}}(I(r)\Omega).$$

■

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