Algebraic Quantization

Harish Bhat

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Abstract

We describe algebraic formalism that allows structurally similar treatments of both classical and quantum mechanics, with $\hbar$ as a parameter that determines which regime we are in. Using this formalism, we find an identity that characterizes both classical and quantum mechanics. The identity in question leads naturally to the use of certain algebraic objects to model the space of quantum-mechanical observables: these objects are the self-adjoint parts of C^*-algebras. Thinking of the space of classical observables as a Poisson algebra, we find that we can easily formulate a natural definition of a quantization. We then give a quick sketch of one quantization scheme that is set in this formalism. We prove a theorem dealing with the classical (i.e. $\hbar \to 0$) limit of the quantized dynamics.

1 From Mechanics to C^*-Algebras

Our first goal is to describe an algebraic structure that is large enough to hold both quantum and classical mechanics. To motivate the use of general structures such as Jordan-Lie and C^*-algebras, we will first examine the differences between the Poisson structures of classical and quantum mechanics. After doing so, we will discover an identity that in some sense characterizes quantum and classical mechanics simultaneously. This identity will motivate the use of general structures, such as C^*-algebras, as a backbone for quantization theory.

1.1 Classical Poisson Structures

Let us examine a simple classical system: a single particle with phase space given by a Poisson manifold M. The Poisson bracket $\{\cdot,\cdot\}$ on M must be a map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

such that

1. Equipping the algebra of functions $C^\infty(M)$ with the bracket $\{\cdot,\cdot\}$ results in a Lie algebra, and

2. $\{\cdot,\cdot\}$ is a derivation over the algebra $C^\infty(M)$. That is, if we denote multiplication in the algebra $C^\infty(M)$ by $\star$, we must have

$$\{f \star g, h\} = f\{g,h\} + \{f,h\}g$$

for all f,g,h in $C^\infty(M)$.

This, however, is not the whole story. We can be more explicit when it comes to describing the algebra $C^\infty(M)$. This space, which consists of all smooth real-valued functions on M, is a real vector space under pointwise addition and multiplication by real numbers. It becomes an *algebra* when we equip it with a multiplication operation. That is, we define, for $f, g \in C^\infty(M)$,

$$(f \bullet g)(x) = f(x)g(x)$$

for all $x \in M$. This bilinear multiplication operation is clearly both commutative and associative. It turns out that these two properties of the multiplication operation together imply that $C^\infty(M)$ is a *Jordan algebra*, and for this reason we refer to this multiplication operation as the classical *Jordan product.*

Definition 1 *A* ***Jordan algebra*** $\mathcal{A}$ *is an algebra where the multiplication operation* $\bullet$ *satisfies the following two properties. (Here A and B are arbitrary elements of $\mathcal{A}$.)*

1. *Commutativity:* $A \bullet B = B \bullet A$

2. *Weak associativity:* $A \bullet (B \bullet (A \bullet A)) = (A \bullet B) \bullet (A \bullet A)$

Associativity clearly implies weak associativity. Although calling $C^\infty(M)$ a Jordan algebra is correct, there is a more precise term which applies:

Definition 2 *A **Poisson algebra** $\mathcal{A}$ is an algebra where the following two properties hold:*

1. *The multiplication operation is both commutative and associative, and*
2. *There is a Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{A}$.*

The Poisson algebra $C^\infty(M)$ has a physical interpretation: since this algebra consists of all real-valued functions on the phase space M, it is the space of all *observables* of our classical system.

Now let us switch gears and look at the situation on the quantum side. This time, we will start with a standard quantum mechanical framework and derive an expression for the Poisson bracket of two observables. Comparing the observable brackets of classical and quantum mechanics will reveal a simple algebraic property that characterizes both systems simultaneously.

1.2 Quantum Poisson Structures

The quantum setup is as follows[1]. We have a complex Hilbert space $\mathcal{H}$ with a Hermitian inner product $\langle\cdot,\cdot\rangle$. We choose the symplectic form

$$\Omega(\psi_1,\psi_2) = 2\hbar\,\mathrm{Im}\,\langle\psi_1,\psi_2\rangle,$$

for $\psi_1,\psi_2 \in \mathcal{H}$.

The observables in quantum mechanics are taken to be the self-adjoint complex linear operators on $\mathcal{H}$. We denote the space of such operators on $\mathcal{H}$ by $\mathfrak{U}$. Given an observable $H \in \mathfrak{U}$, we note that H naturally induces a map $\langle H\rangle \in C^\infty(\mathcal{H})$ given by

$$\langle H\rangle(\psi) = \langle H\psi,\psi\rangle$$

This we recognize as the *expectation value* of the observable H at ψ. The fact that $\langle H\rangle$ is real-valued follows directly from the self-adjointness of H.

We claim that the linear Hamiltonian vector field associated to H is

$$X_H = \frac{i}{\hbar}H.$$

Here follows a quick proof of this claim. Since H is self-adjoint, it is symmetric. Therefore, iX_H is symmetric, which means that

$$\langle iX_H\psi_1,\psi_2\rangle = \langle\psi_1, iX_H\psi_2\rangle.$$

[1] Here we follow [MR 99] rather slavishly. Treatments in other texts are similar.

Now taking the imaginary part of both sides, we arrive at Ω-skewness of X_H, which implies that X_H is Hamiltonian. Now note that

$$\frac{1}{2}\Omega(X_H\psi, \psi) = \hbar \operatorname{Im} \langle X_H\psi, \psi\rangle = -\hbar\langle iX_H\psi, \psi\rangle = \langle H\psi, \psi\rangle.$$

This shows that our construction for X_H is correct; the associated real-valued Hamiltonian function on $\mathcal{H}$ is precisely $\langle H\rangle$, as desired.

Now we can get to work on the Poisson structure of the space of observables. We will proceed in stages, first computing the bracket for those elements of $C^\infty(M)$ which arise as expectation values of observables, i.e. elements of $\mathfrak{U}$.

Given $A, B \in \mathfrak{U}$, by Ω-skewness of X_A and X_B,

$$\Omega(X_A\psi, X_B\psi) = -\Omega(\psi, X_A X_B\psi) = \Omega(X_A X_B\psi, \psi),$$

and

$$\Omega(X_A\psi, X_B\psi) = -\Omega(X_B X_A\psi, \psi).$$

Using the definition of the bracket along with these two facts,

$$\{\langle A\rangle, \langle B\rangle\}(\psi) = \Omega(X_A, X_B)(\psi) = \frac{1}{2}\left(\Omega(X_A X_B\psi, \psi) - \Omega(X_B X_A\psi, \psi)\right).$$

With our expressions for Ω, X_A, and X_B, we can rewrite this as

$$\{\langle A\rangle, \langle B\rangle\}(\psi) = -\frac{1}{\hbar}\operatorname{Im}\langle (AB - BA)\psi, \psi\rangle.$$

From the self-adjointness of A and B, a short computation[2] reveals that

$$\overline{\langle (AB - BA)\psi, \psi\rangle} = -\langle (AB - BA)\psi, \psi\rangle,$$

implying that $\langle (AB - BA)\psi, \psi\rangle$ is purely imaginary. Then

$$\{\langle A\rangle, \langle B\rangle\}(\psi) = -\frac{1}{i\hbar}\langle (AB - BA)\psi, \psi\rangle = \frac{i}{\hbar}\langle AB - BA\rangle(\psi),$$

where we have tacitly extended the definition of expectation value to operators such as $(AB - BA)$ that are not self-adjoint. Inspired by this bracket, we

[2]Here we explicitly note that there are issues with the domains of definition of the linear operators A and B that we are glossing over.

claim that a natural Poisson structure on the space of all quantum-mechanical *observables* is given by

$$\{A, B\}_\hbar = \frac{i}{\hbar}(AB - BA).$$

First, we note that because A and B are self-adjoint, then the right-hand side of the above expression is self-adjoint[3]. Here's the one-line proof:

$$\left[\frac{i}{\hbar}(AB - BA)\right]^* = -\frac{i}{\hbar}(B^*A^* - A^*B^*) = \frac{i}{\hbar}(AB - BA).$$

We can now check that $\{\cdot, \cdot\}_\hbar$ satisfies the properties required of a Poisson bracket. The only nontrivial properties are the Jacobi and Leibniz identities. The Jacobi identity follows from a direct calculation:

$$\{\{A, B\}_\hbar, C\}_\hbar = \left\{\frac{i}{\hbar}(AB - BA), C\right\}_\hbar = -\frac{1}{\hbar^2}[(AB - BA)C - C(AB - BA)],$$

and

$$\begin{aligned}[ABC - BAC - CAB + CBA] + [BCA - CBA - ABC + ACB]\\ + [CAB - ACB - BCA + BAC] = 0.\end{aligned}$$

The Leibniz property can be proved simply as well:

$$\begin{aligned}\{AB, C\}_\hbar &= \frac{i}{\hbar}(ABC - CAB) = \frac{i}{\hbar}(ABC - ACB + ACB - CAB)\\ &= A\frac{i}{\hbar}(BC - CB) + \frac{i}{\hbar}(AC - CA)B = A\{B, C\}_\hbar + \{A, C\}_\hbar B\end{aligned}$$

Following the line of reasoning in the classical case, we now turn to a closer analysis of the space of observables $\mathfrak{U}$. This time, in order to develop $\mathfrak{U}$ as a Jordan algebra, we introduce the symmetric product

where does this idea come from? Landsman?

$$A \bullet B = \frac{1}{2}(AB + BA),$$

defined for all $A, B \in \mathfrak{U}$. Notice that if $\mathfrak{U}$ consists of operators that commute, this quantum Jordan product essentially reduces to the classical Jordan product considered above. To show conclusively that $\bullet$ makes $\mathfrak{U}$ into a Jordan algebra, we need to prove the following:

[3] Again, we are skirting some issues relating to the domains of definition of A and B.

Proposition 1 *The product $\bullet$ satisfies weak associativity.*

Proof: Since $A \bullet A = AA = A^2$, we need to show

$$A \bullet (B \bullet A^2) = (A \bullet B) \bullet A^2.$$

The left-hand side is

$$\begin{aligned} A \bullet (B \bullet A^2) &= \frac{1}{2}[A \bullet (BA^2 + A^2B)] \\ &= \frac{1}{4}[ABA^2 + A^3B + BA^3 + A^2BA], \end{aligned}$$

while the right-hand side is

$$\begin{aligned} (A \bullet B) \bullet A^2 &= \frac{1}{2}(AB + BA) \bullet A^2 \\ &= \frac{1}{4}[ABA^2 + BA^3 + A^3B + A^2BA]. \end{aligned}$$

Both sides are equal, finishing the proof. ■

A quick check shows that $\bullet$ is not associative. Thus $\mathfrak{U}$, equipped with the quantum Jordan product $\bullet$ and the quantum Poisson bracket $\{\cdot,\cdot\}_\hbar$, is not a Poisson algebra. We will see in the next section how to properly describe $\mathfrak{U}$ when it is armed with $\bullet$ and $\{\cdot,\cdot\}_\hbar$.

1.3 The Associator Identity

Armed with both Jordan and Poisson structures on $\mathfrak{U}$, we state the following *associator identity*:

$$(A \bullet B) \bullet C - A \bullet (B \bullet C) = \frac{\hbar^2}{4}\{\{A, C\}_\hbar, B\}_\hbar.$$

Proof: Using definitions, we expand both sides separately:

$$\begin{aligned} \{\{A, C\}_\hbar, B\}_\hbar &= \{-\frac{i}{\hbar}(AC - CA), B\}_\hbar \\ &= -\frac{1}{\hbar^2}[(AC - CA)B - B(AC - CA)] \\ &= \frac{1}{\hbar^2}[-ACB + CAB + BAC - BCA], \end{aligned}$$

and

$$
\begin{aligned}
(A \bullet B) \bullet C - A \bullet (B \bullet C) &= \left(\frac{1}{2}(AB + BA)\right) \cdot C - A \cdot \left(\frac{1}{2}(BC + CB)\right) \\
&= \frac{1}{4}(ABC + CAB + BAC + CBA) \\
&\quad - \frac{1}{4}(ABC + BCA + ACB + CBA) \\
&= \frac{1}{4}(CAB - ACB + BAC - BCA).
\end{aligned}
$$

Comparing these two expansions proves the identity. ■

Before discussing the significance of this identity, we must first explain the parameter $\hbar$. In numerous physical contexts, it is convenient to choose units in which both the speed of light c and Planck's constant $\hbar$ are equal to 1. Now, for a single particle system, we can use these units to describe the characteristic length scale l of our problem. If we find that l is comparable with $\hbar = 1$, then we are clearly in the quantum regime: the dynamics of our particle will be quantum dynamics. But if instead $l \gg 1$, we find ourselves in the classical regime: the dynamics will be classical.

Now we will switch the roles of the length scale and $\hbar$. That is, choose the same units, and fix a length scale l. Now as we vary $\hbar$ from l to 0, we pass gradually from the quantum to the classical regimes. Indeed, when $\hbar = 0$, we should completely recover classical mechanics: in the system of units we have chosen, *any* length scale $l > 0$ is now infinitely larger than Planck's constant.

Keeping this in mind, we return to the associator identity. Our quantum Poisson bracket $\{\cdot,\cdot\}_\hbar$ on $\mathfrak{U}$ is defined for any $\hbar > 0$. For $\hbar = 0$, we have the classical Poisson bracket

$$\{\cdot,\cdot\}_\hbar = \{\cdot,\cdot\},$$

defined on $C^\infty(M) \times C^\infty(M)$. Furthermore, in the classical case, observables commute, so the quantum Jordan product is equivalent to the classical Jordan product. Putting everything together, we see that the associator identity

$$(A \bullet B) \bullet C - A \bullet (B \bullet C) = \frac{\hbar^2}{4}\{\{A, C\}_\hbar, B\}_\hbar$$

holds for *both* quantum and classical observables A, B, and C. We have already seen that it holds for quantum observables. In the classical case, we have $\hbar = 0$, and

$$(A \bullet B) \bullet C - A \bullet (B \bullet C) = 0.$$

Another way of saying this is: *the associativity of classical observables under the Jordan product is an algebraic characterization of classical mechanics.*

Finally, we return to the problem of describing $\mathfrak{U}$ equipped with $\bullet$ and $\{\cdot, \cdot\}_\hbar$.

Definition 3 *An algebra $\mathcal{A}$ is a Jordan-Lie algebra if the following three properties hold:*

1. *Denoting multiplication in the algebra by $\bullet$, $(\mathcal{A}, \bullet)$ is a Jordan algebra.*

2. *There is a Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{A}$.*

3. *The associator identity, as stated above, holds for $\bullet$ and $\{\cdot, \cdot\}$.*

Thus we have already shown that $(\mathfrak{U}, \bullet, \{\cdot, \cdot\}_\hbar)$ is a Jordan-Lie algebra.

1.4 Unifying Algebraic Structures

What we have seen so far is that by combining the Jordan and Poisson structures of the quantum and classical algebras of observables, we obtain a simple algebraic characterization of both quantum and classical mechanics. Both algebras in question are Jordan-Lie algebras. As we will see, the quantum Jordan structure is compatible with a vector space norm in just the right way so that $\mathfrak{U}$ is a Jordan-Lie-Banach algebra.

Definition 4 *A* ***Jordan-Lie-Banach algebra*** *$\mathfrak{U}$ is a Jordan-Lie algebra and a Banach space such that for all $A, B \in \mathfrak{U}$, we have*

$$||A \bullet B|| \leq ||A||\,||B||$$

and

$$||A||^2 \leq ||A^2 + B^2||,$$

where $||\cdot||$ and $\bullet$ denote the norm and Jordan product in $\mathfrak{U}$, respectively.

For our quantum-mechanical setup, the norm $||\cdot||$ arises via

$$||A||^2 = \langle A, A \rangle.$$

Then it is easy to show that both of the above properties are satisfied. Now we get our first glimpse of a C^*-algebra. Intuitively, such an object is both a complex Banach space and an algebra, equipped not only with a multiplication operation but also an involution. The involution map encodes the adjoint operation that we are used to in the Hilbert space context.

Definition 5 *The algebra* $\mathfrak{C}$ *is a* C^***-algebra** *if the following properties hold:*

1. *The multiplication operation in the algebra* $\mathfrak{C}$ *is associative.*
2. *There exists* $* : \mathfrak{C} \to \mathfrak{C}$, *the involution map, such that*
   (a) $(A^*)^* = A$,
   (b) $(AB)^* = B^*A^*$, *and*
   (c) $(\lambda A)^* = \bar{\lambda}A^*$,

   for all $A, B \in \mathfrak{C}$ *and all* $\lambda \in \mathbb{C}$.
3. $\mathfrak{C}$ *is a complex Banach space with norm* $|| \cdot ||$ *such that*
   (a) $||AB|| \leq ||A||\,||B||$ *and*
   (b) $||A^*A|| = ||A||^2$,

   where A *and* B *are arbitrary elements of* $\mathfrak{C}$.

The following pair of theorems relate C^*-algebras to the JLB-algebras that we dealt with in our algebraic treatment of quantum mechanical observables.

Theorem 1 *If* $\mathfrak{U}$ *is a* C^**-algebra, and* $\hbar$ *is a nonzero real, we define the operations*

$$A \bullet B = \frac{1}{2}(AB + BA)$$

and

$$\{A, B\}_\hbar = \frac{i}{\hbar}(AB - BA)$$

on the self-adjoint part $\mathfrak{U}_\mathbb{R}$ *of* $\mathfrak{U}$. *Together with the norm inherited from* $\mathfrak{U}$, *these operations turn* $\mathfrak{U}_\mathbb{R}$ *into a JLB-algebra.*

Theorem 2 *If $\mathfrak{U}_{\mathbb{R}}$ is a JLB-algebra with $\hbar^2 \geq 0$, the complexification $\mathfrak{U}$ is a C^*-algebra when equipped with*

$$\begin{aligned} AB &= A \bullet B - \frac{1}{2} i\hbar \{A, B\}_{\hbar}, \\ (A + iB)^* &= A - iB, \quad \textit{and} \\ ||A|| &= ||A^* A||^{1/2}. \end{aligned}$$

The proofs are unimportant and somewhat too involved[4] for our purposes, so we omit them. We simply note the consequences in terms of a general quantum-mechanical setup. For one, we can rid ourselves of the Hilbert space $\mathcal{H}$ entirely! All we need for quantum mechanics now is a C^*-algebra $\mathfrak{C}$ from which we extract the self-adjoint part $\mathfrak{U}$. This algebraic object turns out to be isomorphic to a JLB-algebra, the bracket on which generates the "equation of motion" for any quantum-mechanical observable. The reader may object to the introduction of what may seem to be needless formalism. But in fact, this kind of generalization—specifically, the progression from specific geometric structures to general geometric structures—is exactly what characterizes the modern approach to classical mechanics.

In the beginning, we start with systems of particles in ordinary Euclidean space $\mathbb{R}^3$. The equations of motion can be written in Hamiltonian form; then, a geometric point of view shifts our focus from the differential equations to Hamiltonian vector fields. It turns out that, on $\mathbb{R}^n$, these vector fields can be defined quite simply relative to the canonical symplectic form

$$\Omega = \sum_{i=1}^{n} dq^i \wedge dp_i$$

This symplectic formalism turns out to be quite general itself—in fact we can speak of Hamiltonian systems on arbitrary symplectic manifolds, manifolds with nondegenerate closed two-forms that may look very, very different from the canonical form that we have written above. A closer look reveals that we can generalize the symplectic manifold to a Poisson manifold and still retain enough structure to define Hamiltonian vector fields and the subsequent equations of motion. At the end of the day, our concept of a classical

[4]In particular, for the first theorem, it is not easy to prove the second JLB norm-inequality; for the second theorem, proving that the purported norm is legitimate is a bit complicated when the complexification $\mathfrak{U}$ is noncommutative. For both proofs, see [Landsman 98].

mechanical system has transformed, mathematically, into a Poisson manifold $(P, \{\cdot, \cdot\})$ and a Hamiltonian function $H : P \to \mathbb{R}$. There are too many advantages of this approach to list here, but any short list would have to include:

- The formalism handles fields and particles equally well.
- Extremely general conservation laws are easy to derive.
- The full power of the mathematical theories of symplectic and Poisson geometry can be brought to bear on mechanical problems.
- We can express symmetry properties of the system very easily, and use those symmetries to derive a reduced system whose phase space is of smaller dimension than the original one.

Analogous advantages are realized when we switch from a quantum-mechanical setup founded on Hilbert spaces to one founded on C^*-algebras. On the quantum side, instead of upgrading geometric structures, we are updating algebraic ones. Specifically,

- The C^*-algebra approach allows us to deal effectively with quantum systems involving infinite numbers of particles [BR 87], and it allows us to make progress with a general theory of quantum fields as in [Haag 91].
- There is a vast amount of literature on C^*- and other Banach algebras that can be used to our advantage.
- The C^*-algebraic approach to quantization theory shows conclusively that the Poisson algebras on the classical side find their natural counterpart on the quantum side in C^*-algebras.
- The C^*-algebra formalism leads to a symmetry-reduction-like process called *induction* by [Landsman 98].

To keep things brief, we will not provide examples of all of these aspects of the theory. We now focus solely on an example of C^*-algebraic quantization theory.

2 Quantization

2.1 Definitions and Motivations

Recall that, roughly speaking, a quantization is supposed to be a way to obtain a quantum system from a classical one. The following definition makes this precise:

Definition 6 *A* ***strict quantization of a Poisson algebra*** $\tilde{\mathfrak{U}}^0_{\mathbb{R}}$ *consists of*

1. *A collection of points* $I_0 \subset \mathbb{R}$ *with accumulation point* $0 \notin I_0$*,*
2. *A collection of* C^**-algebras* $\{\mathfrak{U}^\hbar\}_{\hbar\in I}$*, with self-adjoint parts denoted by* $\mathfrak{U}^\hbar_{\mathbb{R}}$*, and*
3. *A collection of maps* $\{Q_\hbar : \tilde{\mathfrak{U}}^0_{\mathbb{R}} \to \mathfrak{U}^\hbar_{\mathbb{R}}\}$

such that the following properties hold (here $I = I \cup \{0\}$*):*

1. *The function* $\hbar \mapsto ||Q_\hbar(f)||$ *is continuous on* I *for all* $f \in \tilde{\mathfrak{U}}^0_{\mathbb{R}}$*, and, in particular,*
$$\lim_{\hbar\to 0} ||Q_\hbar(f)|| = ||f||;$$
2. *For all* $f, g \in \tilde{\mathfrak{U}}^0_{\mathbb{R}}$*,*
$$\lim_{\hbar\to 0} ||Q_\hbar(f) \bullet Q_\hbar(g) - Q_\hbar(fg)|| = 0;$$
3. *For all* $f, g \in \tilde{\mathfrak{U}}^0_{\mathbb{R}}$*,*
$$\lim_{\hbar\to 0} ||\{Q_\hbar(f), Q_\hbar(g)\}_\hbar - Q_\hbar(\{f, g\})|| = 0;$$
4. *For each* $\hbar \in I$*, the set*
$$\{Q_\hbar(f) | f \in \tilde{\mathfrak{U}}^0_{\mathbb{R}}\}$$
is dense in $\mathfrak{U}^\hbar_{\mathbb{R}}$*.*

(The Poisson algebra $\tilde{\mathfrak{U}}^0_{\mathbb{R}}$ *is required to be a dense subset of the self-adjoint part* $\mathfrak{U}^0_{\mathbb{R}}$ *of a commutative* C^**-algebra* $\mathfrak{U}^0$*.)*

Although this definition is lengthy, it is quite natural. The first half of the definition has at its core a collection of maps $Q_\hbar$ which take classical observables as inputs and yield quantum observables as outputs. The second half of the definition makes sure that these maps obey four properties. In words, we can restate the first three of these four properties as follows: in the $\hbar \to 0$ limit,

1. the quantized classical observable should converge to the original classical observable,
2. the quantum Jordan product should converge to the classical Jordan product, and
3. the quantum Poisson bracket should converge to the classical Poisson bracket.

(Here all convergence is norm convergence, as in the definition itself.) The fourth property simply requires that any quantum observable can be represented as the limit (in some sense) of a sequence of quantized classical observables.

Not obvious at all is the fact that this concept of strict quantization is actually an algebraically intrinsic concept. That is, under very mild conditions, a strict quantization of a Poisson algebra automatically yields an algebraic object called a *continuous field of C^*-algebras.* This object is intrinsic to the theory of C^*-algebras: in words, it is an algebraically natural assignment of a C^*-algebra $\mathfrak{U}_x$ to each "point" x of a fixed C^*-algebra $\mathfrak{C}$. The space of sections of such a field may be identified with $\mathfrak{C}$ itself. For details, we refer to §II.1.2 of [Landsman 98].

2.2 Berezin Quantization

We now describe one specific example of a strict quantization, the Berezin quantization. Our classical system consists of the cotangent bundle $T^*\mathbb{R}^n$ equipped with the canonical symplectic form and hence the canonical Poisson bracket as well. We choose the space of observables

$$\tilde{\mathfrak{U}}_{\mathbb{R}}^0 = C_c^\infty(T^*\mathbb{R}^n),$$

given by elements of $C^\infty(T^*\mathbb{R}^n)$ with compact support. It is a Poisson algebra under the inherited bracket, and it is furthermore a dense subspace of the space of smooth *bounded* functions on the cotangent bundle, using the supremum norm.

Now we take $I_0 = \mathbb{R} \setminus \{0\}$, i.e. $I = \mathbb{R}$, and define, for any $\hbar \in I_0$,

$$Q_\hbar^B : \tilde{\mathfrak{U}}_\mathbb{R}^0 \to \mathfrak{U}_\mathbb{R}^\hbar$$

given by

$$Q_\hbar^B(f) = \int_{T^*\mathbb{R}^n} \frac{d^n p d^n q}{(2\pi\hbar)^n} f(p,q)[\Psi_\hbar^{(p,q)}].$$

Here $[\Psi_\hbar^{(p,q)}]$ denotes the projection onto the one-dimensional subspace of $L^2(\mathbb{R}^n)$ whose image in $\mathbb{P}L^2(\mathbb{R}^n)$ is

$$\Psi_\hbar^{(p,q)}(x) = (\pi\hbar)^{-n/4} \exp(-ipq/(2\hbar)) \exp(ipx/\hbar) \exp(-(x-q)^2/(2\hbar)).$$

Then $\mathfrak{U}^\hbar$ is the C^*-algebra of compact operators on the Hilbert space $L^2(\mathbb{R}^n)$. The motivation for the Berezin quantization is quite complicated, but it rests primarily on analytical ideas involving well-behaved $\hbar \to 0$ convergence of functionals involving transition probabilities. The following theorem guarantees that the collection of C^*-algebras and maps defined above actually works:

Theorem 3 *The Berezin quantization ($I_0 = \mathbb{R} \setminus \{0\}$ with $Q_\hbar^B$ and $\mathfrak{U}^\hbar$ as aboove) is a strict quantization. Furthermore, it satisfies the nondegeneracy condition for each $\hbar$:*

$$Q_\hbar(f) = 0 \iff f = 0.$$

The proof involves Taylor expanding the identities that need to be proved and estimating the derivatives of the classical compactly-supported observables that figure in these expansions. We will not involve ourselves with the technical details of this proof. For our purposes, what matters most is that once we know that the Berezin quantization is bona fide, it becomes very easy to prove statements concerning the so-called classical limit.

2.3 The Classical Limit

Now we will see that the definitions have all been set up correctly: in the $\hbar \to 0$ limit, we recover (in a certain sense) the classical dynamics of a quantized system. Furthermore, it is easy to prove this fact. In order to set up the appropriate theorem, we start with a Hamiltonian function

$$h \in \tilde{\mathfrak{U}}_\mathbb{R}^0.$$

In words, h is a compactly supported smooth real-valued function on the phase space $T^*\mathbb{R}^n$. This implies that the Hamiltonian vector field X_h must be bounded and therefore complete, by a standard theorem in dynamical systems. Hence, using the flow φ^t of h, we obtain a one-parameter family of linear maps α_t^0 defined on $\tilde{\mathfrak{U}}_{\mathbb{R}}^0$:

$$\alpha_t^0(f)(x) = f \circ \varphi^t(x).$$

On the quantum side, we can analogously define a one-parameter automorphism group $\alpha_t^\hbar$ on $\mathfrak{U}^\hbar$ via

$$\alpha_t^\hbar(A) = \exp(itQ_\hbar^B(h)/\hbar)A\exp(-itQ_\hbar^B(h)/\hbar)$$

Now we have

Theorem 4 *For all fixed t,*

$$\lim_{\hbar\to 0} ||Q_\hbar^B(\alpha_t^0(f)) - \alpha_t^\hbar(Q_\hbar^B(f))|| = 0$$

Proof: Using the fundamental theorem of calculus, we write

$$Q_\hbar^B(\alpha_t^0(f)) - \alpha_t^\hbar(Q_\hbar^B(f)) = \int_0^t ds \frac{d}{ds}\alpha_{t-s}^\hbar(Q_\hbar^B(\alpha_s^0(f)))$$

Now since

$$\alpha_s^0(f) \in \tilde{\mathfrak{U}}_{\mathbb{R}}^0,$$

the Poisson bracket with the Hamiltonian gives us the time-evolution of the observable:

$$\frac{d}{ds}\alpha_s^0(f) = \{h, \alpha_s^0(f)\}.$$

Furthermore, we have through a bit of calculation

$$\frac{d}{ds}\alpha_{t-s}^\hbar(Q_\hbar(g)) = -\frac{i}{\hbar}(Q_\hbar^B(h)Q_\hbar^B(g) - Q_\hbar^B(g)Q_\hbar^B(h)).$$

Putting these facts together, we write

$$Q_\hbar^B(\alpha_t^0(f)) - \alpha_t^\hbar(Q_\hbar^B(f)) =$$
$$\int_0^t ds\alpha_{t-s}^\hbar\left(Q_\hbar^B(\{h, \alpha_s^0(f)\}) - \{Q_\hbar^B(h), Q_\hbar^B(\alpha_s^0(f))\}_\hbar\right)$$

Now using the norm-preservation of the $\alpha_t^\hbar$ group, we have

$$||Q_\hbar^B(\alpha_t^0(f)) - \alpha_t^\hbar(Q_\hbar^B(f))|| \leq \int_0^t ds Q_\hbar^B(\{h, \alpha_s^0(f)\}) - \{Q_\hbar^B(h), Q_\hbar^B(\alpha_s^0(f))\}_\hbar.$$

Now, as $\hbar \to 0$, by the third property (Poisson bracket norm-convergence) of strict quantizations, the norm under the integral vanishes. ■.

The interpretation behind this theorem is as follows: in norm, we have equivariance of the quantization map under dynamical evolution. On the classical side, once we fix the Hamiltonian h and hence the flow φ, we can describe the observable $\alpha_t^0(f)$ as a measurement of f after the phase space has evolved for time t.

On the quantum side, we have the operator $\alpha_t^\hbar$ which effectively conjugates an observable A by the time-evolution operator. The overall effect is to create a time t-advanced observable $\alpha_t^\hbar(A)$, just as in the classical case.

What the theorem says, therefore, is that the quantization of the t-time-elapsed classical observable converges in norm to the t-time-elapsed quantized classical observable. More roughly, the quantum dynamics converge in norm, for all t, to the classical dynamics, in the $\hbar \to 0$ limit.

We started this section with a classical system, and after quantizing it and taking the classical limit, we are presented with the original classical dynamics. All of the machinery that has been introduced has made clear and precise the often vaguely formulated "correspondence principle" between classical and quantum mechanics, at least in the case of a simple Euclidean phase space.

We have presented only one scheme of quantization. Other schemes exist, some with better equivariance properties. Furthermore, quantization commutes with reduction in a certain sense. Suppose we start with a classical system with symmetries. It turns out that reducing the quantized classical system and quantizing the reduced classical system yield the same results. These concepts, like the others mentioned at the end of §1, are topics for future reports.

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