Charged Particles and Paradox in the Equivalence Principle

by

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By the principle of equivalence a stationary particle with charge q in a static gravitational field is comparable with a charged particle in free (no gravity) space with acceleration g. But, it is commonly known that if we accelerate a charged particle it will emit radiation, and that charged particles stationary or moving with constant velocity on the Earth do not emit radiation. These facts seem to contradict the principle of equivalence but the reason is not immediately apparent. In the following discussion we will denote the frame of reference for the stationary charge in the gravitational field as **G**, and the accelerating frame as **R.**

As a small aside, when I was researching this I discussed the problem with a number of Berkeley professors in order to come to understand the present outlook on this problem. Unfortunately, their comments were not very enlightening. Prof. Shuggart thought the answers may be in an E&M text book. *E&M* textbook.[1] Prof. Price thought that there wasn't a paradox because perhaps charged particles on the Earth do emit radiation and that the radiation is too weak to detect because the acceleration g is so small.[2] Prof. Sachs seemed to see the problem right away and said that most likely the principle of equivalence

[1] Jackson, *Classical Electrodynamics*

[2] It seems that his reasoning leaves us with the question of *where* the particles are getting the energy to radiate.

wasn't being broken because gravity is a local phenomena and acceleration is a global phenomena.

The equation for the rate at which a dipole emits radiation is

$$P_R = -\frac{d\mathbf{W}}{dt} = \int_S \mathbf{S}\cdot\mathbf{n}\,da = \frac{\ddot{\mathbf{p}}^2}{16\pi^2\varepsilon_0 c^3}\int\frac{\sin^2\theta}{r^3}\mathbf{r}\cdot\frac{\mathbf{r}}{r}r^2\sin\theta\,d\theta\,d\phi, \tag{1}$$

where $\mathbf{p}$ is the dipole moment. This gives us

$$P_R = -d\frac{\mathbf{W}}{dt} = \frac{q^2}{4\pi\varepsilon_0}\frac{2}{3}\frac{\ddot{\mathbf{p}}^2}{c^3}. \tag{2}$$

And since

$$\dot{\mathbf{p}} = q\dot{\mathbf{r}} = q\mathbf{v}, \text{ and } \ddot{\mathbf{p}} = q\dot{\mathbf{v}}$$

we get

$$P_R = \frac{q^2}{4\pi\varepsilon_0}\frac{2}{3}\frac{\dot{\mathbf{v}}^2}{c^3} = A\,\dot{\mathbf{v}}^2 \tag{3}$$

for $v \ll c$ and A the vector potential.

Equation (3) is only correct for slowly moving charges, but it doesn't give instantaneous rate of loss of energy since $\frac{d\mathbf{W}}{dt} = -\mathbf{F}\cdot\mathbf{v}$ would hold and dividing by v when the velocity is zero but the acceleration is not zero would give us an infinite reaction force, which is obviously incorrect.[3] Why should we be worried about velocities near c? Prof. Shuggart thought that a classical book on *E&M* would be sufficient. The problem comes from that fact that with *any* constant acceleration, v approaches c. Once we wrote down $\dot{\mathbf{v}}$ we accepted all the effects of special relativity.

Integrating (3) by parts gives us[4]

$$\frac{d\mathbf{W}}{dt} = -A\,\mathbf{v}\ddot{\mathbf{v}} + A\frac{d}{dt}\mathbf{v}\dot{\mathbf{v}} = -\frac{q^2}{4\pi\varepsilon_0}\frac{2}{3c}\mathbf{v}\ddot{\mathbf{v}} + \frac{q^2}{4\pi\varepsilon_0}\frac{2}{3c}\frac{d}{dt}\mathbf{v}\dot{\mathbf{v}} \tag{4}$$

And we have to use the Lorentz transformation from G to F as

$$x = \alpha,$$
$$y = \beta,$$
$$z = \zeta\cosh\tau,$$

[3] An oscillating spring for example.

[4] I have to thank Brian Whitus for help with the next few equations.

$$t = \frac{\zeta}{c}\sinh\tau,$$

and setting $\gamma = 1 - \left[\frac{v}{c}\right]^2$.

By apply this Lorentz transformation we can get an equation for any velocity, thus

$$\mathbf{F} = A\left[\frac{\ddot{\mathbf{v}}}{\gamma^2} + \frac{3\mathbf{v}\dot{\mathbf{v}}^2}{c^2\gamma^3}\right] \tag{6}$$

$$d\frac{\mathbf{W}}{dt} = A\frac{\dot{\mathbf{v}}^2}{\gamma^2} \tag{7}$$

$$\frac{d\mathbf{W}}{dt} = -\mathbf{F}v + \frac{d}{dt}A\frac{\mathbf{v}\dot{\mathbf{v}}}{\gamma^2} \tag{8}$$

The last term of equation (8) vanishes when v is zero and hence motion is periodic. But radiation is emitted by stationary particles.[5] A charged particle iwth constant acceleration is stationary at ζ, and the line element is

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 = \zeta^2\,d\tau^2 - d\alpha^2 - d\beta^2 - d\zeta^2 \tag{9}$$

It seems that radiation is emitted in G but that it is emitted into a part of space-time that is *unreachable* in any normal sense. Or in other words, as we look upon the charged particle in G from infinity

And Fulton and Rohrlich briefly summerizes "First, within the framework of the Maxwell-Lorentz equations, a charge in uniformly accelerated motion radiates at a constant and finite rate. Secondly, this radiation ... is Lorentz invariant but not conformally invariant.[6] Thirdly, there is no radiation reaction, but there is energy conservation, *provided* one accepts some equation of motion (Abraham-Dirac); one is then also forced to accept the physical picture emerging from this equation. Otherwise, i.e., without accepting an equation of motion no answer concerning energy conservation or radiation reaction can be given. Finally, there is no contradiction with the principle of equivalence.

[5] Fulton, Rohrlich, *Ann. of Phys.*, April, 1960.

[6] ???