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Evelina Baesu

The falling cat. Ideas in control theory from geometrical mechanics.

Content:

« Ah bien, d'après Hertz, toutes les fois que nous imaginons une force, nous sommes dupe d'une illusion. »

H. Poincaré

§1 Introduction:

The geometrical concepts play an important role in unifying under the same umbrella phenomena which apparently have no relation and belong to completely different domains of science.

The phase shift is one of these examples which shows that complicating things, instead of accepting easy explanations can lead to the discovery of unbelievable invariant understructure (geometrical one). And this emphasizes once more the necessity of closed collaboration of different fields. There might be engineers saying that a complete and dynamical explanation of the falling cat phenomenon can be found in Kane and Scher's paper [1]. But then we wouldn't know that the cat has the same trajectory to a particle in Yang-Mills potentials (for whatever good it will do to the cat). So, geometry brings the qualitative ap.

On the other hand, the formalism which I'm trying to outline here helped building control theories used in space motions of satellites ..., possibly in medicine (small robots) etc. I will give more explanations later.

The central idea is that when one variable of the system moves in a periodic way (internal variable) then an overall motion is observed. Speaking about the falling cat example: it moves the joints creating an overall change in position of its body even if it moves under the constraint of <u>total angular ~~velocity~~ momentum being constant</u>.

There are many other examples as: the microorganisms (moving through fluids with high density), the insects' ~~fly~~, the fish ... this only to speak about the animal world.

§2. Geometrical phase. A bit of history.

According to Berry himself, one of the first published papers related to geometrical phase went out the press in 1941 (the year when he was born). The author is Vassily Vladimirskii. This was about the polarization of rotation of an outgoing ray relative to a parallel incident ray.

Another russian, Rytov came closer remarking that two circular polarizations have different phase velocities.

In 1956, in Bangalore, India, S. Pancharatnam, in investigating the interference patterns produced by plates of anisotropic crystal, found the existing theories inadequate to explain his observations. To calculate the phase change he represented states of polarization as points on the "Poincaré sphere". The poles representing left and right-handed circular polarization, equator ... linear polarizations .... seems clear that Pancharatnam discovered the geometric phase.

Unfortunately he died at 35 years old and the official discovery of the geometric phase remained unknown for 30 more years.

There were some other scientists, ~~but~~ touching the problem and have a glimpse once in a while in what it might have been but none of them came as close as the two physicists discussed above.

§ 3. The falling cat. The dynamical explanation.

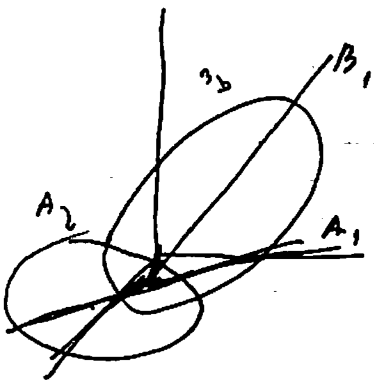
How can the cat actually succeed in falling on the feet? A simple dynamical reasoning leads ~~us~~ to the following argument:

- One thing the cat cannot change is zero angular momentum $\underline{\Pi} = 0$. But $\underline{\Pi} = I \underline{\Omega}$, and by changing the shape the cat changes ... I, the moment of inertia. So $\underline{\Omega}$ has to change consequently. The forces exerted by the muscles in changing the shape are internal to the system so Π remains 0.

- In their paper [1] Kane and Scher elaborate a model for the cat consisting from cylinders moving in some prescribed manner ~~around~~ joints. Probably the authors spent a lot of time watching cats, by the way in which they set up the constraints for the motion of the joints. The restrictions are meant to play the role of the cat's muscles, or control forces applied in the joints.

Approximating the cat by 2 ellipsoids and imposing the following 3 restrictions, the authors prove that the motion is possible by making use of the fact that the angular momentum is conserved.

The three restrictions are:



- the torso of the cat bends, but does not twist.
- At the instant of release, the spine is bent forward. Subsequent to this instant the spine is bent first to one side, then backward, then to the other side and finally forward again.
- The backward bend that occurs during the maneuver is far less pronounced than the initial and terminal forward bend.

It is shown that the motion is possible if $A_1 B_1$ are centroidal principal axes of inertia, where A, B are the front and rear halves of the cat and $A_1 B_1$, the spin

Before showing how the falling cat is captured by the geometric framework, I will make an overview of the formalization of rigid body motions. This is because our cat will be treated as a bundle of rigid bodies interconnected by joints able to be operated by control devices.

§4. Rigid bodies.

The actual configuration of a rigid body is considered to be a mapping from the reference configuration. Each position of the rigid body is specified by a Euclidean motion +. Any rigid motion consists in a translation and a rotation.

- The translations are taking care off by expressing the motion ~~function~~ with respect to a frame passing through the center of mass.

Every rotation of the body corresponds to an orthogonal tensor $Q(t) \in SO_3$ (Q is proper orthogonal)

Remarks: It is to be remarked that besides a phylosophical difference and the forces which appear in one case and not in another between a change in observer and a superposed RB motion, another technical one is that $\boxed{\det Q = 1}$ in the case of RB motions, but can be $\det Q = \pm 1$ in a change in observer.

- Euler's equations for a RB are: $\boxed{\dot{\underline{\Pi}} = \underline{\Pi} \times \underline{\Omega}}$ where Π is the angular momentum (the lagrangian one) and Ω is the angular velocity. The kinetic energy: $K = \frac{1}{2} \Omega^T I \Omega$ where $\hat{I}$, the inertia tensor is 3×3 matrix (if the body is undeformed)
- Euler's equations in terms of Ω: $I \dot{\Omega} = I\Omega \times \Omega$.
- K is a quadratic form and its eigenvectors are the principal axes of inertia.
- For the RB, K is taken to be the Lagrangian.

- Making use of the Lie - Poisson structures for the RB, it can be shown that Euler's equations are Hamiltonian relative to the minus Lie - Poisson structure. The configuration space is $SO(3)$, the Lie - algebra g is $(\mathbb{R}^3, \times)$ and also $g \sim g^*$.
- Then the Lie - Poisson structure on $\mathbb{R}^3$ is the rigid body bracket:

$$\{f, g\}(\Pi) = -\Pi \cdot (\nabla f \times \nabla g).$$

The Poisson map ~~going from~~ carrying the canonical bracket into Lie - Poisson bracket is assured by Euler's angles.

- We have two parallel descriptions : one in terms of the Lagrangian on $TSO(3)$ and another one in terms of Hamilt. description on $T^*SO(3)$. (via Legendre transform.). The Hamiltonian. $\boxed{H = \frac{1}{2}\Pi \cdot (I^{-1}\Pi)}$ which is an ellipsoid. ———— ①
- But we have a constraint on this motion : the angular momentum, Π = constant
- In our Poisson formalism, a way of obtaining constants of motion are by Casimir's functions. On a Poisson manifold. Casimir's functions are the ones for which $\{C, f\} = 0$ for all f.
For the rigid body, $\boxed{C = |\Pi|^2}$ (for any Hamiltonian) —— ②
- In the space Π_1, Π_2, Π_3, C describes a sphere. The intersection between ② (this sphere) and the ellipsoid from ① gives the trajectories which are on the cover of the book.

- We will see later that the energy - momentum methods ~~and~~. Casimir's functions are used successfully by Bloch, Krishaprasad, Marsden and Sanchez in [3] in an attempt to study the stability of RB's with internal and external torques.

§5. Connections, Geometric phase. Reduction

The mathematical definition of a connection A is

- A - $\mathfrak{g}_\mu$-valued one-form on $J^{-1}(\mu) \subset P$
- $A_p \cdot \xi_P(p) = \xi$ for $\xi \in \mathfrak{g}_\mu$.
- $\chi_g^* A = Ad_g \circ A$ for $g \in G_\mu$. , G_μ principal bundle with a connection.

in §4] (Marsden J. Montgomery R and Ratiu T)

- The geometric phase is the holonomy of the path c_μ with respect to the connection A (parametrization independent)
- For the RB $P = T^* SO(3)$. The momentum map $J : P \to \mathbb{R}^3$
- The MW reduction of $T^* SO(3)$ by $SO(3)$ are the angular momenta (cotangent ones). The reduced spaces $J^{-1}(\mu)/G_\mu$ are spheres in the $\mathbb{R}^3$ of Euclidean space radius $\|\mu\|$. ($G_\mu \to$ rotations about μ axis).
- The trajectories of this reduced dynamic system are the ones obtained in §4.
- These are the strict definitions. But especially for connections (which is a cameleonic object) there is not only one point of view. They can be seen in terms of horizontal lift operators....

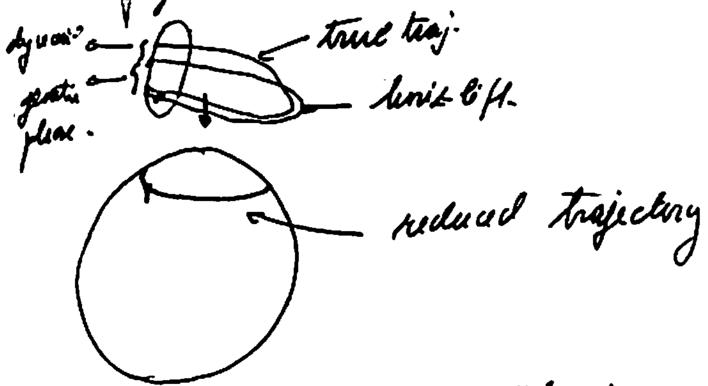
But explaining in simple words what a connection is I'm afraid it is not easy. As far as I understand a <u>connection is a prescription of horizontal directions</u> at each point. A rule with respect to which the space is divided into a vertical and a horizontal direction. As examples: Christoffel symbols are connections, parallel transport etc. Somehow similar to an ~~of~~ projector. Somehow. In the sense that the directions which project to zero are vertical directions. Also it can be thought as characterizing the curvature of the space.

The notion of parallel transport is also understood as a notion in the 'horizontal' direction defined by a connection.

· In the case of the sphere: the horizontal directions are given by the great circles.

· The general procedure of Marsden, Montgomery, Ratiu [4] is to reduce the equations of a dynamical system to the sphere (P_μ) solve it and then by reconstruction reach back the original trajectory.

· By ~~reconstruction~~ horizontal lifts, dynamical phase disappear. What remains is the geometric phase. I reproduce one of their figures here:



· Without going into details with the technicalities I ~~can~~ should say why is so important to work on reduced trajectories? Well, Because the path on the sphere can be controled (by suitable controls) and so, ~~but~~ by the procedure outlined above one can have the true trajectory modified at his will.

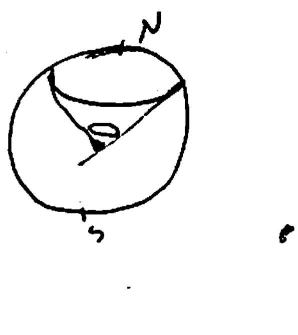
Coming back to the cat: controlling the trajectory on the sphere it's like moving the cat's tail and muscles.
And the geometric phase is the modified motion of the whole body.

In §4) as well as in §3) (only more general in §3) it is proven the following formula for $\Delta\theta$:

$$\boxed{\Delta\theta = -\lambda + \frac{2H_\mu T}{\|\mu\|} + 2k\pi} \;, \; \lambda = \qquad \text{—— (2)}$$

- As it can be seen from (2), the rotation angle $\Delta\theta$ of the RB about the axis μ is splited in 2 parts. One is dependent exclusive of the geometry (area), λ, and another is depended only on energy and period of motion, $\frac{2H_\mu T}{\|\mu\|}$.
- The first one, λ is the geometrical phase. The other one is the dynamic phase. Moreover, D being the area enclosed by the reduced trajectory: $\boxed{\lambda = \frac{\text{area } D}{\|\mu\|^2}}$
- ~~It is conceivable, I think, that λ depends only on the~~ area. So λ is the solid angle to the center of the sphere which encloses the trajectory
- I ~~want~~ to remark the following: the connections are very closed related to the constraints. Actually the connections are somehow determined by the constraints.
- In the cat's case: the constance of angular momentum determines the notion of "horizontal" (~~of~~ on the great circles)

- It is somewhat intuitively to think that the geometric phase λ depends only on the area because ~~it~~ the connection acts as a "projector" and changing the projection plane, the area changes!



But in the same time and surprisingly enough it doesn't matter which cap I chose the North pole or the South one because (2) is modulo (2π) and the solid angles at the center of the sphere add up to $\boxed{4\pi}$.

• Also another remark will be that periodical ~~to~~ reduced trajectory does <u>not</u> mean periodical true trajectory. On the sphere, all the trajectories are periodical except 2: the <u>homoclinic</u> <u>orbits</u>. This is another way of saying that the 2 orbits (reduced and true) differ by something ($\delta\theta$)

<u>~~Reduction~~</u> <u>Reconstruction</u>: The reduced trajectory is something like ~~a~~ an „Eulerian" description of the motion. Some supplementary forces are coming in which are not real forces. That's why it is needed a procedure to obtain back the true trajectory among all possible one.

Revenons à nos moutons (or to our cats) in the cat's problem. The <u>reduction</u> <u>space</u> is the shape space, which means the totality of the shapes of its body, conceived as a bunch of ... cylinders ~~so~~ having some fixed angles one to each other. Of course the original space is the shape space modulo a rototranslation.

In other words: <u>reduction takes advantages of the symmetries to simplify the problem (reduce the number of unknowns eventually) but then, the solution inherits the simplifications and by „reconstruction" methods it is cleaned of the arbitrariness</u>.

§6: Optimizations and controls:

I wish to present in few words Montgomery's approach to the cat's problem.

Now we are not interested anymore in the possibility of accomplish a such a motion, but, due to the new geometrical insight gained we can ask ourselves (worried about cat's energy) which is the optimal way in which she can do it.

So, in general, this problem is stated as follows:

„Given a deformable body in free-fall with initial angular momentum zero, find the most efficient way to deform it so that to achieve a desired re-orientation."

- Killing the translations → set G, the center of mass at zero: (this is a problem when there is friction!)
- As we've said before, the angular momentum being conserved (=0) caracterizes a connection. A tangent vector would be one with the angular momentum conserved is zero.

The conclusion is that: "A curve q in Q is extremal for the cat problem $\Leftrightarrow$ $\exists\, p(t)$, smooth covector s.t. $(q(t), p(t))$ satisfies Hamilton's differential equation for H_0. Where H_0 is the ~~vertical~~ horizontal kinetic energy. $H_0 = \frac{1}{2}\|h_q^* p\|^2$, $p \in T^*Q$

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One can realize that the phenomenon of the phase shift raises questions ~~about~~ to control them. The idea that it can be possible to change the trajectory of a body (in space let's say) without influencing the mechanical laws (conservation of linear momentum) it is quite surprising.

The most important application ~~that it~~ for this phenomena is at the stability of RB's.

• In [3], the authors study the idea of stability of coupled rigid and flexible bodies. The method used is energy-Casimir method. They show that three internal rotors can realise any internal torque feedback for the rigid body.

One of the conclusions is that mechanical systems subject to ~~...~~ some external forces determined by ~~...~~ some feedback laws, can be modelled ~~by~~ as ~~Hamiltonian~~ systems. These are good news because all the arsenal of battle on energy-momentum, Casimir function etc can be used to design it.

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Finally, not insisting anymore on the applications of the geometric phase (which are a lot, this is clear) I just want to briefly acknowledge the paper of Koiller. J [1992] [6] in which he treats non-holonomic constraints.

The new thing about such a problem is the fact that the forces which assure the constraint break the symmetry of zero angular-momentum.

And now, in the final I should explain the motto of the paper. It was said by H. Poincare in his review of Hertz's book, as a reaction at his ideas that one can replace the idea of force by equivalent velocity constraints. More exactly he stated that the geometric curvature of the path is always a minimum, subjected to the constraints.

References:

[1] Kane T.R & Scher M.P. : "A dynamical explanation of the falling cat phenomenon". [1969]

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[6] Koiller J. "Reduction of some classical non-holonomic systems with symmetry". [1992]

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[9] Abraham & Marsden : "The foundation of mechanics".