The following is a list of Theorems and Definitions referred to in the exercises.

Theorem 1 Let $\Delta(x, u^{(n)}) = 0$ be a nondegenerate system of differential equations. A connected local group of transformations G acting on an open subset $M \subset X \times U$ is a symmetry group of the system iff

$$pr^{(n)} v [\Delta(x, u^{(n)})] = 0 \quad \text{whenever} \quad \Delta(x, u^{(n)}) = 0$$

for every infinitesimal generator v of G.

** Corresponds to Theorem 2.71 in Olver **

Here $u^{(n)}$ refers to the space $(u, u_x, u_{xx}, \ldots)$

Definition Let $v = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u) \frac{\partial}{\partial u^\alpha}$

be a vector field defined on an open subset $M \subset X \times U$. The n^{th} prolongation of v is defined

$$pr^{(n)} v = v + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J (x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}$$ defined on the

corresponding jet space $M^{(n)} \in X \times U^{(n)}$, the second summation being over all multi indices $J = (j_1, \ldots j_k)$ with $1 \le j_k \le p$ $1 \le k \le n$. Here

$$\phi_\alpha^J(x, u^{(n)}) = D_J \left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha$$

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}.$$

** Corresponds to Theorem 2.36 in Olver **

Notation used let $x \in \mathbb{R}^n$, $u \in \mathbb{R}$

(1) $u^{(n)}$ refers to the space of all derivatives of u up to n^{th} order.

(2) In problem 2: $P_{\tilde{x}_i} = \frac{\partial P}{\partial \tilde{x}_i}$

$$Q_{\tilde{u}_{(1)i}} = \frac{\partial Q}{\partial \tilde{u}_{(1)i}} \quad \text{where} \quad \tilde{u}_{(1)i} = \frac{\partial \tilde{u}}{\partial \tilde{x}_i}$$

(3) Einstein summation convention is used

$$\text{eg. - } P_{x_i} Q_{u_i} = \frac{\partial P}{\partial x_1}\frac{\partial Q}{\partial u_{(1)}} + \frac{\partial P}{\partial x_2}\frac{\partial Q}{\partial u_{(2)}} + \dots \frac{\partial P}{\partial x_n}\frac{\partial Q}{\partial u_{(n)}}$$

Problem 1 Discuss the symmetry group of the Helmholtz Equation $\Delta u + \lambda u = 0$, λ a fixed constant $x \in \mathbb{R}^3$

sol: Define $F(x,y,z,u_{xx}\,u_{yy}\,u_{zz}) = u_{xx} + u_{yy} + u_{zz} + \lambda u = 0$.

We suppose $V = \alpha(x,y,z,u)\frac{\partial}{\partial x} + \beta(x,y,z,u)\frac{\partial}{\partial y} + \gamma(x,y,z,u)\frac{\partial}{\partial z}$
$+ \phi(x,y,z,u)\frac{\partial}{\partial u}$
is an infinitesimal generator for a 1-parameter symmetry group of $F(x,y,z,u_{xx}\,u_{yy}\,u_{zz}) = 0$.

Then it follows from Theorem 1 and Definition 1 that

$pr^{(2)}v\; F(x,y,z,u^{(2)}) = 0$ whenever $F(x,y,z,u^{(2)}) = 0$.
Here $pr^{(2)}v\; F = \phi^{xx} + \phi^{yy} + \phi^{zz} + \lambda\phi = 0$
where

$$\begin{aligned}
\phi^{xx} = {} & \phi_{xx} + (2\phi_{xu} - \alpha_{xx})u_x + (\phi_{uu} - 2\alpha_{xu})u_x^2 \\
& + (\phi_u - 2\alpha_x)u_{xx} - \alpha_{uu}u_x^3 - 3\alpha_u u_x u_{xx} \\
& - \beta_{xx}u_y - 2\beta_{ux}u_x u_y - 2\beta_x u_{xy} - 2\beta_u u_x u_{xy} \\
& - \beta_u u_y u_{xx} - \beta_{uu}u_y u_x^2 - \gamma_{xx}u_z - 2\gamma_{ux}u_x u_z \\
& - 2\gamma_x u_{xz} - 2\gamma_u u_x u_{xz} - \gamma_u u_z u_{xx} \\
& - \gamma_{uu}u_z u_x^2
\end{aligned}$$

$$\phi^{yy} = \phi_{yy} + (2\phi_{yu} - \beta_{yy})u_y + (\phi_{uu} - 2\beta_{yu})u_y^2$$
$$+ (\phi_u - 2\beta_y)u_{yy} - \beta_{yy}u_y^3 - 3\beta_u u_y u_{yy}$$
$$- \alpha_{yy}u_x - 2\alpha_{uy}u_x u_y - 2\alpha_y u_{xy} - 2\alpha_u u_y u_{xy}$$
$$- \alpha_u u_x u_{yy} - \alpha_{uu}u_x u_y^2 - \gamma_{yy}u_z - 2\gamma_{uy}u_y u_z$$
$$- 2\gamma_y u_{yz} - 2\gamma_u u_y u_{yz} - \gamma_u u_z u_{yy} - \gamma_{uu}u_z u_y^2$$

$$\phi^{zz} = \phi_{zz} + (2\phi_{zu} - \gamma_{zz})u_z + (\phi_{uu} - \gamma_{zu})u_z^2$$
$$+ (\phi_u - 2\gamma_z)u_{zz} - \gamma_{zz}u_z^3 - 3\gamma_u u_z u_{zz}$$
$$- \beta_{zz}u_y - 2\beta_{uz}u_{zy} - 2\beta_z u_{zy} - 2\beta_u u_z u_{zy}$$
$$- \beta_u u_y u_{zz} - \beta_{uu}u_y u_z^2 - \alpha_{zz}u_x - 2\alpha_{uz}u_x u_z$$
$$- 2\alpha_z u_{xz} - 2\alpha_u u_z u_{xz} - \alpha_u u_x u_{zz}$$
$$- \alpha_{uu}u_x u_z^2.$$

We must solve $\phi^{xx} + \phi^{yy} + \phi^{zz} + \lambda\phi = 0$ for α, β, γ and ϕ. Here $u(x,y,z)$ is some fixed solution to $\Delta u + \lambda u = 0 \quad x \in R^3$. Assuming u not to be the trivial solution, the products of different derivatives will be linearly independent. This gives us the following system of equations to solve,

Term : Equation

constant : $\phi_{xx} + \phi_{yy} + \phi_{zz} + \lambda\phi = 0$

Term	Equation
u_x	$2\phi_{xu} - \alpha_{xx} - \alpha_{yy} - \alpha_{zz} = 0$
u_y	$2\phi_{yu} - \beta_{yy} - \beta_{xx} - \beta_{zz} = 0$
u_z	$2\phi_{zu} - \gamma_{zz} - \gamma_{xx} - \gamma_{yy} = 0$
u_{xx}	$\phi_u - 2\alpha_x = 0$
u_{yy}	$\phi_u - 2\beta_y = 0$
u_{zz}	$\phi_u - 2\gamma_z = 0$
u_x^2	$\phi_{uu} - 2\alpha_{xu} = 0$
u_y^2	$\phi_{uu} - 2\beta_{yu} = 0$
u_z^2	$\phi_{uu} - 2\gamma_{zu} = 0$
u_x^3	$\alpha_{uu} = 0$
u_y^3	$\beta_{uu} = 0$
u_z^3	$\gamma_{uu} = 0$
$u_x u_{xx}$	$-3\alpha_u = 0$
$u_y u_{yy}$	$-3\beta_u = 0$
$u_z u_{zz}$	$-3\gamma_u = 0$
$u_x u_y$	$-2\beta_{ux} - 2\alpha_{uy} = 0$
$u_x u_z$	$-2\gamma_{ux} - 2\alpha_{uz} = 0$
$u_y u_z$	$-2\gamma_{uy} - 2\beta_{uz} = 0$
u_{xy}	$-2\beta_x - 2\alpha_y = 0$
u_{xz}	$-2\gamma_x - 2\alpha_z = 0$
u_{yz}	$-2\gamma_y - 2\beta_z = 0$
$u_y u_{xx}$	$-\beta_u = 0$
$u_x u_{yy}$	$-\alpha_u = 0$
$u_z u_{yy}$	$-\gamma_u = 0$

Term	Equation
$u_y u_{zz}$	$-\beta_u = 0$
$u_x u_{zz}$	$-\alpha_u = 0$
$u_z u_{xx}$	$-\gamma_u = 0$
$u_y u_x^2$	$-\beta_{uu} = 0$
$u_x u_y^2$	$-\alpha_{uu} = 0$
$u_z u_y^2$	$-\gamma_{uu} = 0$
$u_x u_z^2$	$-\alpha_{uu} = 0$
$u_y u_z^2$	$-\beta_{uu} = 0$
$u_z u_x^2$	$-\gamma_{uu} = 0$
$u_x u_{xy}$	$-2\beta_u = 0$
$u_x u_{xz}$	$-2\gamma_u = 0$
$u_y u_{yz}$	$-2\gamma_u = 0$
$u_y u_{xy}$	$-2\alpha_u = 0$
$u_z u_{yz}$	$-2\beta_u = 0$
$u_z u_{xz}$	$-2\alpha_u = 0$

Many of these Equations are redundant. The last 18 Equations tell us that

$$\boxed{\alpha, \beta, \gamma \text{ are independent of } u}$$

Collecting what is left over without any redundancies we have

(1)
$$-\beta_x = \alpha_y$$
$$-\gamma_x = \alpha_z$$
$$-\gamma_y = \beta_z$$

(2)
$$\phi_u = 2\alpha_x$$
$$\phi_u = 2\beta_y$$
$$\phi_u = 2\gamma_z$$

(3)
$$2\phi_{xu} = \Delta\alpha$$
$$2\phi_{yu} = \Delta\beta$$
$$2\phi_{zu} = \Delta\gamma$$

(4)
$$\phi_{uu} = 0$$

(5)
$$\Delta\phi + \lambda\phi = 0$$

Equations (4) & (5) tell us that ϕ is (at most) linear in u and that it satisfies Helmholtz Equation.

Differentiating (2) and using this in (3) gives the following

(*)
$$-3\alpha_{xx} + \alpha_{yy} + \alpha_{zz} = 0$$
$$-3\beta_{yy} + \beta_{xx} + \beta_{zz} = 0$$
$$-3\gamma_{zz} + \gamma_{xx} + \gamma_{yy} = 0$$

Next we differentiate (1) and (2), comparing like terms and using this in (*) we have the result

$$\alpha_{xx} = \alpha_{yy} = \alpha_{zz} = 0$$

$$B_{yy} = B_{xx} = B_{zz} = 0$$

$$\gamma_{zz} = \gamma_{xx} = \gamma_{yy} = 0$$

$\boxed{\alpha, B, \gamma \text{ are at most linear in } x, y, z}$

With this result we have from (3) that

$\boxed{\Phi \text{ is independent of } x, y, z}$

Putting it all together we have

$$\alpha = a_1 x + a_2 y + a_3 z + a_4$$
$$\beta = b_1 x + b_2 y + b_3 z + b_4$$
$$\gamma = c_1 x + c_2 y + c_3 z + c_4$$
$$\phi = Au + B$$

We can get rid of some of the constants by using this general form in (1) & (2). The result is

$$a_2 = -b_1$$
$$a_3 = -c_1$$
$$c_2 = -b_3$$
$$a_1 = b_2 = c_3 = \tfrac{1}{2}A$$

We are left with 7 infinitesimal generators

$$v_1 = \frac{\partial}{\partial x}$$
$$v_2 = \frac{\partial}{\partial y}$$
$$v_3 = \frac{\partial}{\partial z}$$
$$v_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$
$$v_5 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$
$$v_6 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$
$$v_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{1}{2}u\frac{\partial}{\partial u}$$

The generators $v_1 - v_3$ correspond to translations in the x, y, z directions respectively while $v_4 - v_6$ generate rotations in the x-y, x-z and y-z plane respectively. The last vector field generates dilations in (x, y, z, u) space.

Hence given that $F(x, y, z, u)$ is a solution to $\Delta F + \lambda F = 0$

(1) $F(x+\epsilon, y, z, u)$

(2) $F(x, y+\epsilon, z, u)$

(3) $F(x, y, z+\epsilon, u)$

(4) $F(x\cos\epsilon + y\sin\epsilon, -x\sin\epsilon + y\cos\epsilon, z, u) = 0$

(5) $F(x\cos\epsilon + z\sin\epsilon, y, -x\sin\epsilon + z\cos\epsilon, u) = 0$

(6) $F(x, y\cos\epsilon + z\sin\epsilon, -y\sin\epsilon + z\cos\epsilon, u) = 0$

(7) $e^{\frac{1}{2}\epsilon} F(e^{-\epsilon}x, e^{-\epsilon}y, e^{-\epsilon}z)$

are also solutions.

Problem 2 Prove that a differential equation $P(x, u^{(n)}) = 0$ is equivalent to a linear differential equation $\Delta(\tilde{u}) = F(\tilde{x})$ under a change of variables $x = E(\tilde{x}, \tilde{u})$ $u = \phi(\tilde{x}, \tilde{u})$ iff it admits an infinite dimensional symmetry group with the generators of the form

$$v = \rho(E, \phi)\left\{\frac{\partial E}{\partial \tilde{u}}\frac{\partial}{\partial \tilde{x}} + \frac{\partial \phi}{\partial \tilde{u}}\frac{\partial}{\partial \tilde{u}}\right\} \quad (*) \quad \text{where}$$

$\rho(x, u)$ is an arbitrary solution to a linear differential equation. Take $x \in R^n$, $u \in R$.

PF Suppose $P(x, u^{(n)}) = 0$ admits a generator of the form $(*)$. We define the transformation of the space $(\tilde{x}, \tilde{u}^n)$ to $(x, u^{(n)})$ under the change of variables $x = E(\tilde{x}, \tilde{u})$, $u = \phi(\tilde{x}, \tilde{u})$ by $T: (\tilde{x}, \tilde{u}^{(n)}) \to (x, u^{(n)})$. Then $P(x, u^{(n)})$ is transformed to $T^{-1}P(x, u^{(n)}) = Q(\tilde{x}, \tilde{u}^n)$ and the operator v is transformed to $T^{-1} v T = \hat{v}$.
If $v P(x, u^{(n)}) = 0$ whenever $P(x, u^{(n)}) = 0$ then we have
$$\hat{v}\, Q(\tilde{x}, \tilde{u}^{(n)}) = T^{-1} v\, T\, T^{-1} P(x, u^{(n)})$$
$$= T(v\, P(x, u^{(n)})) = 0 \quad \text{whenever } P(x, u^n) = 0.$$
Next noting that the generator $\hat{v} = \rho(\tilde{x}) \frac{\partial}{\partial \tilde{u}}$

transforms like
$$T \tilde{v} T^{-1} = T \rho(\tilde{x}) \frac{\partial}{\partial \tilde{u}} T^{-1}$$

$$= \rho(T(\tilde{x}, 0))\, T \frac{\partial}{\partial \tilde{u}} T^{-1}$$

$$= \rho(E(\tilde{x}, \tilde{u}), \phi(\tilde{x}, \tilde{u}))\left(\frac{\partial x}{\partial \tilde{u}}\frac{\partial}{\partial x} + \frac{\partial u}{\partial \tilde{u}}\frac{\partial}{\partial u}\right)T^{-1}$$

$$\Rightarrow \quad T\tilde{V}T^{-1} = \rho(\xi, \phi)\left\{\frac{dx}{d\tilde{u}}\frac{d}{d\tilde{x}} + \frac{du}{d\tilde{u}}\frac{d}{d\tilde{u}}\right\},$$

we conclude that $Q(\hat{x}, \hat{u}^{(n)})$ has a generator of the form $\tilde{V} = \rho(\tilde{x})\frac{d}{d\tilde{u}}$. Hence $\rho(\tilde{x})\frac{d}{d\tilde{u}} Q(\tilde{x}, \tilde{u}^{(n)}) = 0$.

To get an idea $\rho(\tilde{x})\frac{d}{d\tilde{u}} Q(\tilde{x}, \tilde{u}^{(n)})$

looks like we consider the special case $Q(\tilde{x}, \tilde{u}^{(n)}) = Q(\tilde{x}, \tilde{u}, \tilde{u}_i)$. In this case we have

$$\rho(\tilde{x})\frac{d}{d\tilde{u}} Q(\tilde{x}, \tilde{u}, \tilde{u}_{(i)}) = \frac{d}{d\tilde{u}}\left(\rho(\tilde{x}) Q(\tilde{x}, \tilde{u}, \tilde{u}_{(i)})\right)$$

$$= \rho(\tilde{x}) Q_{\tilde{u}} + \frac{d\rho(\tilde{x})}{d\tilde{x}_i}\frac{dQ}{d\tilde{u}_{(i)}}$$

$$= \rho(\tilde{x}) Q_{\tilde{u}} + \rho_{\tilde{x}_i} Q_{\tilde{u}_{(i)}}$$

Generalizing to the n-derivative case we have

$$(1) \quad \rho(\tilde{x})\frac{d}{d\tilde{u}} Q(\tilde{x}, \tilde{u}^{(n)}) = \rho Q_{\tilde{u}} + \rho_{\tilde{x}_{i_1}} Q_{\tilde{u}_{(i)}} + \rho_{\tilde{x}_{i_1} x_{i_2}} Q_{\tilde{u}_{(i_1 i_2)}} \cdots + \rho_{\tilde{x}_{i_1} \ldots \tilde{x}_{i_n}} Q_{\tilde{u}_{(i_1 \ldots i_n)}} = 0.$$

By hypothesis $\rho(\tilde{x})$ satisfies some linear differential equation

$$(2) \quad a\rho + a^{i_1}\rho_{\tilde{x}_{i_1}} + \ldots a^{i_1 \ldots i_n}\rho_{\tilde{x}_{i_1} \ldots \tilde{x}_{i_n}} = 0$$

Using (1) & (2) to solve for $p(\tilde{x})$ we have

$$0 = (a Q_{\tilde{u}_{(i_1)}} - a^{i_1} Q_{\tilde{u}}) p_{x_{i_1}} + \ldots (a Q_{\tilde{u}_{(i_1 \ldots i_n)}} - a^{i_1 \ldots i_n} Q_{\tilde{u}}) p_{x_{i_1} \ldots x_{i_n}}$$

Therefore since $p(\tilde{x})$ is an arbitrary solution to $\Delta p = 0$ we have the condition

$$a Q_{\tilde{u}_{(i_1)}} = a^{i} Q_{\tilde{u}}$$

$$\vdots \qquad\qquad \vdots$$

$$a Q_{\tilde{u}_{(i_1 \ldots i_n)}} = a^{i_1 \ldots i_n} Q_{\tilde{u}} .$$

We may rewrite this as

$$\frac{\partial Q}{a^{i} \partial \tilde{u}_{(i)}} = \frac{\partial Q}{a \partial \tilde{u}} = \ldots = \frac{\partial Q}{a^{i_1 \ldots i_n} \partial \tilde{u}_{(i_1 \ldots i_n)}}$$

which implies that $Q(\tilde{x}, \tilde{u}^{(n)}) = \tilde{Q}(a\tilde{u} + a^{i}\tilde{u}_{i} + \ldots a^{i_1 \ldots i_n}\tilde{u}_{i_1 \ldots i_n}, x)$ for some function $\tilde{Q}$. Assuming $Q(\tilde{x}, \tilde{u}^{(n)}) = 0$ we have $\tilde{Q}(a\tilde{u} + a^{i}\tilde{u}_{i} + \ldots a^{i_1 \ldots i_n}\tilde{u}_{i_1 \ldots i_n}, x) = 0$ which may be rewritten

$$(\ast\ast) \qquad a\tilde{u} + a^{i}\tilde{u}_{i} + \ldots a^{i_1 \ldots i_n}\tilde{u}_{i_1 \ldots i_n} = F(\tilde{x}) .$$

$\boxed{\text{Conclusion: The change of variables defined above maps } P(x, u^{(n)}) \text{ to the linear equation } (\ast\ast)}$

Next we assume that the change of variables $x = E(\hat{x}, \hat{u})$, $u = \Phi(\hat{x}, \hat{u})$ maps $P(x, u^{(n)}) = 0$ to some linear equation $\Delta \hat{u} = F(\hat{x})$. This linear equation admits an infinitesimal generator of the form $\hat{v} = \rho(\hat{x}) \frac{\partial}{\partial \hat{u}}$ where $\Delta \rho = 0$. Therefore since $\hat{v}$ transform like

$$v = T \hat{v} T^{-1} = \rho(E, \Phi) \left\{ \frac{\partial x}{\partial \hat{u}} \frac{\partial}{\partial \hat{x}} + \frac{\partial u}{\partial \hat{u}} \frac{\partial}{\partial \hat{u}} \right\}$$

we have the desired result.

Problem 3 Use the above result to linearize the Thomas Eq.

$$F(x,t,u) = u_{xt} + \alpha u_t + \beta u_x + \gamma u_x u_t = 0 \quad (*)$$

Sol

We begin by looking for an infinite dimensional symmetry group of $(*)$. Let

$$v = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$$

be an infinitesimal generator for the 1-parameter symmetry group of $(*)$. Again using Theorem 1 and Definition 1 we have

$$pr^{(2)}v\, F(x,t,u) = 0 \quad \text{when} \quad F(x,t,u) = 0$$

or

$$\phi^x \beta + \phi^t \alpha + \phi^t \gamma u_x + \phi^x \gamma u_t + \phi^{xt} = \quad (**)$$

$$(\beta + \gamma u_t)\phi^x + (\alpha + \gamma u_x)\phi^t + \phi^{xt} = 0.$$

Here

$$\phi^x = \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t$$

$$\phi^t = \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2$$

$$\phi^{xt} = D_x D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xxt} + \tau u_{xtt}$$

$$= \phi_{xt} + \phi_{tu} u_x - \xi_{tu} u_x^2 - \xi_{tx} u_x + \phi_{ux} u_t$$

$$+ \phi_u u_{xt} + \phi_{uu} u_t u_x - \tau_{tx} u_t - \tau_{tu} u_t u_x - \tau_t u_{tx}$$
$$- \xi_{ux} u_x u_t - \xi_t u_{xx} - \xi_{uu} u_x^2 u_t - \xi_u u_{xx} u_t - \xi_u u_x u_{tx}$$
$$- \tau_{ux} u_t^2 - \tau_{uu} u_t^2 u_x - 2\tau_u u_t u_{xt} - \xi_x u_{xt}$$
$$- \xi_u u_{xt} u_x - \tau_x u_{tt} - \tau_u u_{tt} u_x .$$

Using these identities in (**) gives

$$\beta \phi_x + \beta \phi_u u_x^{(1)} - \beta \xi_x u_x - \beta \tau_x u_t - \beta \xi_u u_x^{2\,(3)}$$
$$- \beta \tau_u u_x u_t^{(2)} + \alpha \phi_t - \alpha \xi_t u_x + \alpha \phi_u u_t^{(1)}$$
$$- \alpha \tau_t u_t - \alpha \xi_u u_x u_t^{(3)} - \alpha \tau_u u_t^{2\,(2)} + \gamma \phi_x u_t$$
$$+ \gamma \phi_u u_x u_t^{(1)} - \gamma \xi_x u_x u_t + \gamma \phi_t u_x - \gamma \xi_t u_x^2$$
$$+ \gamma \phi_u u_t u_x - \gamma \tau_t u_t u_x - \gamma \xi_u u_x^2 u_t^{(3)} - \gamma \tau_u u_t^2 u_x^{(2)}$$
$$+ \phi_{xt} + \phi_{tu} u_x - \xi_{tu} u_x^2 - \xi_{tx} u_x + \phi_{ux} u_t$$
$$+ \phi_u u_{xt}^{(1)} + \phi_{uu} u_t u_x - \tau_{tx} u_t - \tau_{tu} u_{tx}$$
$$- \tau_t u_{xt} - \xi_{ux} u_x u_t - \xi_{uu} u_x^2 u_t - \xi_u u_{xx} u_t - \xi_u u_x u_{tx}^{(3)}$$
$$- \tau_{ux} u_t^2 - \tau_{uu} u_t^2 u_x - \tau_u u_t u_{xt}^{(2)} - \xi_x u_{xt} - \xi_u u_{xt} u_x$$
$$- \tau_x u_{tt} - \tau_u u_{tt} u_x - \tau_u u_t u_{xt} - \xi_t u_{xx}$$

$$= 0$$

which we must solve for $\alpha, \beta, \gamma, \phi$ with u some fixed solution to (*).

The terms circled in red and numbered are terms which vanish when $u(x,t)$ is a solution to (*).

They are

(1) $\phi_u (u_{xt} + \alpha u_t + \beta u_x + \gamma u_x u_t) = 0$
(2) $\tau_u u_t (u_{xt} + \alpha u_t + \beta u_x + \gamma u_x u_t) = 0$
(3) $\xi_u u_x (u_{xt} + \alpha u_t + \beta u_x + \gamma u_x u_t) = 0$.

Removing these terms and again assuming u not to be the trivial solution the linear independence of the products of different derivatives gives us the following system of equations.

Term	:	Equation
constant	:	$\beta \phi_x + \alpha \phi_t + \phi_{xt} = 0$
u_x	:	$-\beta \xi_x - \alpha \xi_t + \gamma \phi_t + \phi_{tu} - \xi_{tx} = 0$
u_t	:	$-\beta \tau_x - \alpha \tau_t + \gamma \phi_x + \phi_{ux} - \tau_{tx} = 0$
$u_x u_t$	:	$-\gamma \xi_x - \gamma \tau_t + \gamma \phi_u + \phi_{uu} - \xi_{ux} - \tau_{tu}$
u_x^2	:	$-\xi_t \gamma - \xi_{tu} = 0$
u_t^2	:	$-\tau_x \gamma - \tau_{ux} = 0$
$u_x u_t^2$	:	$-\tau_{uu} - \gamma \tau_u = 0$
$u_t u_x^2$	:	$-\gamma \xi_u - \xi_{uu} = 0$
u_{tx}	:	$-\tau_t - \xi_x = 0$
$u_x u_{xt}$	:	$-\xi_u = 0$
$u_t u_{xt}$	:	$-\tau_u = 0$
$u_{xx} u_t$	:	$-\xi_u = 0$
u_{tt}	:	$-\tau_x = 0$
$u_{tt} u_x$	:	$-\tau_u = 0$
u_{xx}	:	$-\xi_t = 0$

The last six equations tell us that

$$\boxed{\begin{array}{l} \xi,\ \tau \text{ are independent of } u \\ \xi \text{ is independent of } t \\ \tau \text{ is independent of } x \end{array}}$$

Using this we are left with the following defining equations

(1) $$-\tau_t = \xi_x$$

(2) $$-\gamma(\xi_x + \tau_t) + \gamma\phi_u + \phi_{uu} = 0$$

(3) $$-\alpha\tau_t + \gamma\phi_x + \phi_{ux} = 0$$

(4) $$-\beta\xi_x + \gamma\phi_t + \phi_{tu} = 0$$

(5) $$\beta\phi_x + \alpha\phi_t + \phi_{xt} = 0 .$$

Using (1) in (2) gives

$\gamma\phi_u = -\phi_{uu}$ so $\phi(x,t,u)$ has the form

$$\boxed{\phi(x,t,u) = A\,F(x,t)\,e^{-\gamma u} + g(x,t)}$$

with F, g to be determined, A a constant.

We can extract additional information from (1). We note since ξ, τ are independent of u and $\xi = \xi(x)$, $\tau = \tau(t)$

$\tau_t = -\xi_x$ implies that ξ, τ have the form

$$\boxed{\begin{aligned} \tau(t) &= a_1 t + a_2 \\ \xi(x) &= a_3 x + a_4 \end{aligned}} \qquad a_1 = -a_3$$

The remaining equations to be used are

$$\left.\begin{aligned} +\alpha a_1 + \gamma \phi_x + \phi_{ux} &= 0 \\ -\beta a_1 + \gamma \phi_t + \phi_{tu} &= 0 \\ \beta \phi_x + \alpha \phi_t + \phi_{xt} &= 0 \end{aligned}\right\} (***)$$

To find the transformation which maps (*) into a linear equation we do not have to actually solve (***). We have that

$\phi(x,t,u) = A e^{-\gamma u} F(x,t) + g(x,t)$ with

$F(x,t)$ and $g(x,t)$ satisfying (***). Therefore there is an infinite dimensional symmetry group G to (*) which has as a generator

$$v = F(x,t) e^{-\gamma u} \frac{\partial}{\partial u}.$$

Using this and problem 2
we let $\tilde{u} = e^{+\gamma u}$. Then $\tilde{u}$ satisfies

$$\tilde{u}_x = \gamma u_x e^{\gamma u}$$
$$\tilde{u}_t = \gamma u_t e^{\gamma u}$$
$$\hat{u}_{xt} = \gamma u_{xt} e^{\gamma u} + \gamma^2 u_x u_t e^{\gamma u}$$

or

$$\boxed{\begin{gathered} B\tilde{u}_x + \alpha \tilde{u}_t + \tilde{u}_{xt} = 0 \\ \hat{u} = e^{\gamma u} \end{gathered}}$$

<u>Conclusion</u> The mapping $T: R^3 \to R^3$ defined by

$$(x, t, u) \longrightarrow (x, t, e^{\gamma u})$$

maps (*) to the linear equation

$$B\tilde{u}_x + \alpha \tilde{u}_t + \tilde{u}_{xt} = 0 .$$

Problem 4 Use problem 2 to linearize the nonlinear Heat Equation

$$F(x,t,u) = \frac{u_{xx}}{u_x^2} - u_t = 0 \qquad (1)$$

sol Again we begin looking for an infinite dimensional symmetry group G of (1). We suppose

$$v = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$$

is an infinitesimal generator of G. Then in view of Theorem 1 and Definition 2 we have

$$pr^{(2)} v\, F = \phi^t + 2\phi^x \frac{u_{xx}}{u_x^3} - \frac{\phi^{xx}}{u_x^2} = 0 \qquad (2) \text{ with}$$

$$\phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt}$$
$$= \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2$$

$$\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt}$$
$$= \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t$$

$$\phi^{xx} = \phi_{xx} + 2(\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3$$
$$- \tau_{uu}u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx}$$
$$- 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}$$

Putting this all in to (2) gives

$$0 = \phi_t u_x^3 - \xi_t u_x^4 + (\phi_u - \tau_t) u_t u_x^3 - \xi_u u_x^4 u_t^{(1)}$$
$$- \tau_u u_t^2 u_x^3{}^{(2)} + 2\phi_x u_{xx} + 2(\phi_u - \xi_x) u_x u_{xx}$$
$$- 2\tau_x u_t u_{xx} - 2\xi_u u_x^2 u_{xx} - 2\tau_u u_x u_t u_{xx}$$
$$- \phi_{xx} u_x - 2(\phi_{xu} - \xi_{xx}) u_x^2 + \tau_{xx} u_x u_t$$
$$- (\phi_{uu} - 2\xi_{xu}) u_x^3 + 2\tau_{xu} u_x^2 u_t + \xi_{uu} u_x^4$$
$$+ \tau_{uu} u_x^3 u_t - (\phi_u - 2\xi_x) u_x u_{xx} + 2\tau_x u_{xt} u_x$$
$$+ 3\xi_u u_x^2 u_{xx}{}^{(1)} + \tau_u u_t u_x u_{xx}{}^{(2)} - 2\tau_u u_x^2 u_{xt}$$
$$+ 2\xi_u u_x^2 u_{xx}{}^{(1)} + 2\xi_u u_x^2 u_{xx}.$$

Again we must solve for τ, ξ, ϕ with $u(x,t)$ some fixed solution to $\frac{u_{xx}}{u_x^2} = u_t$. With this in mind

we remove the terms circled in red. They are

(1) $\xi_u (u_x^2 u_{xx} - u_x^4 u_t) = 0$

(2) $\tau_u (u_t u_x u_{xx} - u_t^2 u_x^3) = 0$.

Then assuming the linear independence of the products of different derivatives we set the coefficients equal to zero. The result is the following set of differential equations

Term : Equation

u_x : $-\phi_{xx} = 0$

Term	Equations
u_{xx}	$2\phi_x = 0$
u_x^2	$-2(\phi_{xu} - \xi_{xx}) = 0$
u_x^3	$\phi_t - \phi_{uu} + 2\xi_{xu} = 0$
u_x^4	$-\xi_t + \xi_{uu} = 0$
$u_x u_{xx}$	$2(\phi_u - \xi_x) - \phi_u + 2\xi_x = \phi_u = 0$
$u_t u_x^3$	$\phi_u - \tau_t + \tau_{uu} = 0$
$u_t u_{xx}$	$-2\tau_x = 0$
$u_t u_x$	$\tau_{xx} = 0$
$u_t u_x^2$	$2\tau_{xu} = 0$
$u_x u_t u_{xx}$	$-2\tau_u = 0$
$u_{xt} u_x$	$2\tau_x = 0$
$u_x^2 u_{xt}$	$-2\tau_u = 0$

The last six equations tell us that

$$\boxed{\tau = \tau(t) \text{ is independent of } x, u}$$

The 1st and 5th equations tell us that

$$\boxed{\phi = \phi(t) \text{ is independent of } x, u}$$

With this in mind we have the following defining equations left

(1) $\xi_{xx} = 0$

(2) $\xi_t = \xi_{uu}$

(3) $\phi_t = -2\xi_{xu}$

(4) $\tau_t = \tau_{uu} = \tau_t = 0$

At this point we note that

$$\boxed{\begin{array}{ll} \xi(x,t,u) & \text{is linear in } x \text{ (at most)} \\ \xi(x,t,u) & \text{satisfies } \xi_t = \xi_{uu} \end{array}}$$

Letting $\gamma(t,u)$ be an arbitrary solution to the heat equation $\gamma_t = \gamma_{uu}$ we have that

$$V = \gamma(t,u)\frac{\partial}{\partial x} \text{ is an infinitesimal generator}$$

for the symmetry group G.

Using problem 2 as motivation we consider the change of variables

$$x = \tilde{u}, \quad t = t, \quad u = \tilde{x}.$$

Under this change of variables

$$V \to \hat{V} = \gamma(t, \tilde{x}) \frac{\partial}{\partial \hat{u}}$$

which is an infinitesimal generator for the infinite dimensional symmetry group of

$$\tilde{u}_{\tilde{x}\hat{x}} = \tilde{u}_t .$$

Therefore the results of problem 2 tell us that the change of variables

$$\boxed{\begin{array}{l} x = \hat{u}, \quad t = t, \quad u = \tilde{x} \qquad \text{maps} \\ u_x^{-2} u_{xx} = u_t \qquad \text{to the linear heat equation} \\ \tilde{u}_{\tilde{x}\hat{x}} = \hat{u}_t \end{array}}$$