CDS 141 PROJECT

Hendra - A nice exposition of KdV & central extensions. Eventually, one wants to see why the system is integrable in this setting.

On the Korteweg de-Vries equation

Hendra Adiwidjaja

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It has been almost one hundred and fifty years since the discovery of a self-preserving solitary nonlinear wave or soliton. However, there is still much activity going on on the subject nowadays among the physical and mathematical communities, mainly because of the many occurances of soliton in other branches of physics, for example quantum and plasma physics, and also because of the richness in its properties. One of the equation that was first known to possess this solitary wave type of solution is the Korteweg-de Vries or KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

(Note that there are some variance of the above equation especially in the coefficients and their signs, but we will stick with the most widely used one above.)

The main objective of this paper is to expose the Hamiltonian structure, and hence the symmetries, of the KdV equation using the Lie group and Lie algebra technique. A lot of work in this aspect has recently been done and much activity is still in progress. Here, our discussion will be more in an expository nature. However, since the recent work done by the previously mentioned authors involved a newly used technique, we need to provide some background on the history and development of the KdV equation and its soliton solution. The following two sections outline the exposition given in [2].

A brief history on Soliton and KdV equation

The occurance of a solitary wave was first noticed by J.S. Russel [2]. He described the wave as an elevation in the surface of the water in a narrow channel, which preserved its shape while moving along the channel. This observation or "discovery" led to a confusion among the science community at that period. It was widely believed that a solitary wave could not preserve its shape for a long period because of the forces of dispersion which tend to spread the wave. Hence, the discovery was not quite accepted and it wasn't until another discovery, the KdV equation, that the subject received more serious attention.

The existence of a shape-preserving solitary wave was established by Korteweg and de Vries, and it takes shape as a solution of an equation which bears their names. The shape preservation results from the balance between the dispersion term and the nonlinear term. This remarkable discovery provided some incentive few years later to the development of another main area of physics, namely wave mechanics, or quantum mechanics as is more widely known today. The locality of the wave was needed to describe the evolution of a fast moving particle with small mass, implied by the well-known principle of wave-particle duality. This local wave is given as a solution of another well-known equation, the Nonlinear Schrodinger (NLS) equation,

$$2iu_t + u_{xx} + 2u^2u^* = 0. \tag{2}$$

Besides the KdV and NLS equations, other equations of similar type, that is the ones that possess soliton solutions, like Kadomtsev-Petviashvili (KP), Boussinesq, and sine-Gordon equations, also received wide attention from mainly the mathematical community. Most of these equations are related to the water wave problems.

Derivation of the KdV equation

One of the interesting thing about the KdV equation is that it results as a constraint for other equations, namely the shallow-water wave and Fermi-Pasta-Ulam equations, which has to be satisfied in order for the uniform solutions to exist over long period of time. The KdV equation also comes out as an analog to the Euler equation in the appropriate Lie algebra. A brief discussion is given in the following paragraphs.

Consider a one-dimensional shallow-water-wave problem. The evolution of a small-amplitude long wave can be obtained from the equation of continuity of hydrodynamics and its boundary constraints at the upper and lower surfaces. After the scalings on the amplitude and length of the wave, and also on the shallowness of the water, the water wave equations are given as

$$
\begin{aligned}
\phi_{yy} + \varepsilon\phi_{xx} &= 0 \\
\phi_y &= 0, \qquad y = -h \\
\phi_t + \eta + \frac{1}{2}\mu\phi_x^2 + \frac{1}{2}\frac{\mu}{\varepsilon}\phi_y^2 &= 0, \quad y = 1 + \mu\eta \\
\eta_t + \mu\phi_x\eta_x &= \frac{1}{\varepsilon}\phi_y, \quad y = 1 + \mu\eta,
\end{aligned} \tag{3}
$$

where h is the constant depth of the channel, ϕ the velocity potential, η the pressure, x and y the horizontal and vertical coordinates, respectively, ε the ratio of the depth of the channel to the length of the wave, and μ the ratio of the amplitude of the wave and the depth of the channel. The solution of the above equation can be obtained by asymptotic power series expansion in y of the form

$$\phi(x, y, t) = F(x, t) - \frac{\varepsilon}{2}F_{xx}(y+h)^2 + \varepsilon^2\frac{1}{24}F_{xxxx}(y+h)^4 + \cdots . \tag{4}$$

By taking the limit as $\varepsilon \to 0$ and letting μ ($\ll 1$) finite, we obtain

$$F_{tt} - ((1+h)F_x)_x = -2\varepsilon F_x F_{xt} - \varepsilon F_t F_{xx} + \varepsilon\frac{(1+h)^2}{3}F_{xxxx}. \tag{5}$$

Further expansion of F as $F = f + \varepsilon F_1 + \cdots$ and imposition of the uniformity condition, that is the condition for the boundedness of solution at $x = \pm\infty$, result in the following equation,

$$2f_{\Theta X} + 3f_\Theta f_{\Theta\Theta} + \frac{1}{3}f_{\Theta\Theta\Theta\Theta} = 0, \tag{6}$$

which is the KdV equation after letting $f_\Theta = u$ and rescaling Θ and X. Another variance of the equation, the Perturbed KdV equation, is generated by slowly varying the depth of the channel. However, this results in a non-adiabatic variation, and the solitary wave doesn't strictly preserve its shape since a reflected wave is created. Clearly, the extent in which the shape of the solitary wave is modified depends on the extent of the depth variation.

Besides the water wave equations, the KdV equation also emerges from a mechanical system. This system was investigated by Fermi, Pasta, and Ulam around the middle of this century. It consists of an $N-1$ one-dimensional lattice of masses m connected by nonlinear springs. The method for solving the system is very similar to the one applied in the water wave equations.

A recent approach using Lie group method shows that the KdV equation is the analog of Euler equation for rigid body motion. Discussion about the application of this method to a special case of solution will be described later in the paper.

Conservation laws

We start this section by finding travelling wave solutions of KdV equation. Let $u(x,t) = \psi(x-ct)$ and substitute this to eqn. 1 to obtain

$$-c\psi' + 6\psi\psi' + \psi''' = 0. \tag{7}$$

After integrating the above equation, multiplication by $2\psi'$ yields

$$-c\psi^2 + 2\psi^3 + \psi'^2 = 0, \tag{8}$$

where the constants of integration vanish as a result of the boundary conditions $\psi(\pm\infty) = \psi'(\pm\infty) = \cdots = 0$. This equation has solutions of the form

$$u(x,t) = 2\eta^2 sech^2 \eta(x - ct), \tag{9}$$

where η is determined by initial condition. Since $sech(x)$ is an even function of x w.r.t. $x = 0$ and $\psi(x - ct)$ is a positive function, we see that this solution is a solitary wave travelling along the characteristic $x - ct$. Clearly the density of the wave is a conserved quantity. The arbitrariness of c results in an infinite number of solutions, all satisfying the KdV equation. From here, we can make a conjecture that suppose the initial wave consists of a (nonlinear) superposition of these travelling wave solutions, as $t \to \infty$ the wave will decompose into an infinite set of travelling waves, each with velocity c_i. If this conjecture is true, then this implies that after a collision between these travelling waves, the shape of the waves is retained, that is the faster wave is seen as taking over the slower wave. Computer simulation by Kruskal and Zabusky [2] showed that this is indeed the case. Besides the retention of shape, there is an after-collision phase shift, which is related to the Berry-Hannay geometric phase.

The existence of the geometric phase in the KdV solution is not the only relation between the KdV equation and geometric mechanics. In fact, the KdV equation is the Hamilton's equations of an infinite number of Hamiltonian, each corresponds to different velocities of the travelling waves which are constants of the motion. To obtain the Hamiltonian structure of the KdV equation, we go back to the original work made by Miura. First he observed that the modified

KdV (MKdV) equation

$$v_t + 6v^2 v_x + v_{xxx} = 0 \tag{10}$$

has an infinite number of solutions. Applying the transformation $u = v^2 - iv_x$, he obtained the relation between MKdV and KdV equations as

$$u_t + 6uu_x + u_{xxx} = \left(2v - i\frac{\partial}{\partial x}\right)(v_t + 6v^2 v_x + v_{xxx}). \tag{11}$$

Hence, if u satisfies the KdV equation, then v has to satisfy the MKdV equation. Linearizing the Miura transformation by $v = -i\phi_x/\phi$ yields

$$\phi_{xx} + u\phi = 0. \tag{12}$$

Here we notice that the KdV equation is invariant under Galilean translation. Indeed, by using transformation $u \to u + \lambda$ the KdV equation transforms to

$$u_t + 6(u + \lambda)u_x + u_{xxx} = 0. \tag{13}$$

Assuming the travelling wave solution, the above transformation only amounts to substracting the velocity by a constant 6λ. Therefore, the linearized Miura transformation is the stationary Schrödinger equation with eigenvalue λ and potential u,

$$\phi_{xx} = (\lambda + u)\phi = 0. \tag{14}$$

Obviously ϕ, which is the Hamiltonian, has to satisfy the transformed MKdV equation, but we will see below that in obtaining the Hamiltonian structure of the KdV equation, we will not have to worry about the transformed equation.

Following [3], suppose that we have an evolution equation of the form

$$u_t = K(u). \tag{15}$$

Let $\mathcal{A}$ be the function space (e.g., $L_2(-\infty, +\infty)$ or $L_2(\Omega)$ where Ω is a compact support) of $u(t)$ for $t \in \mathbf{R}$ and let $\mathcal{B}$ be the Hilbert space of self-adjoint functionals $L_{u(t)}$ on $\mathcal{A}$. If there exists an evolution of $L_{u(t)}$, corresponding to the evolution of $u(t)$ under eqn. 15, such that $L_{u(t)}$ is constant for all $t \in \mathbf{R}$, then this constant, which is an eigenvalue of $L_{u(t)}$, is a constant of the motion of eqn. 15. We can always find such $L_{u(t)}$ if $L_{u(t)}$ is unitarily equivalent as $u(t)$ evolves, that is

$$L(0) = U(t)^{-1}L(t)U(t). \tag{16}$$

Computing the t derivative of the above equation, we obtain

$$L_t = [B, L], \tag{17}$$

where $B = U_t U^*$ is a skew symmetric operator and $[\cdot, \cdot]$ is the commutator. Obviously, different $B(t)$ will give different functional K in eqn. 15. From the previous paragraph, we discover that the eigenvalues of eqn. 14 are invariant under the evolution of the KdV equation,

$$u_t = -6uu_x - u_{xxx}. \tag{18}$$

Hence, substituting $L = \partial^2 + u$ and $B = 4\partial^3 - 3(u\partial + \partial u)$ into eqn. 17 we re-obtain the KdV equation. At this stage the eigenfunctions of L are still irrelevant since L_t is just the multiplication operator u_t. However, since the eigenvalues of $L_{u(t)}$ are not discrete, it seems that we have to deal with an infinite number of first integrals. In fact, the first integrals of the KdV equation are given as [5]

$$\begin{aligned} I_{-1} &= \int u\,dx \\ I_0 &= \int u^2\,dx \\ I_1 &= \int \left(\frac{u'^2}{2} - u^3\right) dx \\ &\vdots \end{aligned} \tag{19}$$

They can be easily checked by taking the t derivative of the integrals and using the vanishing boundary conditions of $u, u_x, \cdots$. Besides being the first integrals, they are in fact the Hamiltonians [2] of the family of KdV equation, obtained for skew symmetric operators [3]

$$B_q = \partial^{2q+1} + \sum_{j=1}^{q} (b_j \partial^{2j-1} + \partial^{2j-1} b_j), \tag{20}$$

where q is an integer.

KdV equation as an analog of Euler equation

The discussion in the previous section about the existence of the Hamiltonian structure for the KdV equation suggests that all that was said can be developed further into a geometric formalism.

Before going further, we need some familiarity with the geometric aspect of the rigid body motion. Here we will follow the exposition given in Marsden [4]. Consider a rigid body in a space with no external field. If we take an inertial frame of reference with the center of mass of the body as the origin, then the motion of the body can be represented as a rotation around the origin. Let $R(t) \in SO(3)$ be a map of a reference configuration, $\mathcal{B} \in \mathbf{R}^3$, of the body to its configuration at time $t \in \mathbf{R}$. Hence, $x(t) = R^t(X)$ where $X \in \mathcal{B}$. The evolution of a point $x \in \mathbf{R}^3$ can then be described by

$$\dot{x} = \dot{R}R^{-1}x = \omega \times x, \tag{21}$$

where ω is the spatial angular velocity. Now define the body angular velocity as $\Omega = R^{-1}\omega$ and the moment of inertia operator I as

$$I : \mathbf{R}^3 \longrightarrow (\mathbf{R}^3)^*, \tag{22}$$

where * denotes the adjoint and I is symmetric. It follows that the Lagrangian on $TSO(3)$

$$L(R, \dot{R}) = \frac{1}{2}\int_{\mathcal{B}} \rho(X) \left\|\dot{R}X\right\|^2 d^3X, \tag{23}$$

can be reduced to the one on $so(3) \simeq \mathbf{R}^3$

$$L(\Omega) = \frac{1}{2}\int_{\mathcal{B}} \rho(X) \left\|\Omega \times X\right\|^2 d^3X, \tag{24}$$

or equivalently if written in terms of I

$$L(\Omega) = \frac{1}{2} < I\Omega, \Omega >, \tag{25}$$

where the inner product $< \cdot, \cdot >$ induces the left-invariant metric (that is, the metric is preserved under the left translation L_g) in $\mathbf{R}^3$. Applying the variational method with appropriate boundary conditions produces the Euler-Poincare equation

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = ad^*_\xi \frac{\delta l}{\delta \xi}, \tag{26}$$

which describes the evolution of the orbit of the co-adjoint representation of $SO(3)$ on $(\mathbf{R}^3)^*$ of the Euler equation $I\dot{\Omega} = I\Omega \times \Omega$. This equation of motion is the geodesic of the left-invariant metric on $SO(3)$ since it satisfies the principle of least action. The Hamiltonian of the reduced Lagrangian above is $H(M) =< M, I^{-1}M >$, where $M = I\Omega$.

Now we go back to the KdV equation and we will shortly see that it possesses a structure very similar to the one described above [1]. Here, we consider periodic solutions to the KdV equation, that is, solutions with a compact support. Without loosing generality, we can consider the support as $[0, 2\pi]$. Solutions with infinite support can always be approximated by scaling the $[0, 2\pi]$-support. Since S^1 is one-dimensional, the vector fields on the circle can be represented by 2π-periodic L_2 functions. The Jacobi-Lie bracket of the Lie algebra of vector fields is given as

$$[u, v] = uv' - u'v, \tag{27}$$

where $u, v \in \mathcal{X}(S^1)$. At this moment we need to build a new Lie algebra such that its left action is Hamiltonian. This is the case if the momentum map (in this case the map from the algebra to its dual) is infinitesimally equivariant. If the original Lie algebra is symplectic and connected (which is it is in our case), then we can always built such a map by enlarging the algebra to its unique central extension defined by the two-cocycle $\Sigma(u, v)$ of the algebra [4]. For $\mathcal{X}(S^1)$, the cocycle is given as

$$\Sigma(u, v) = \gamma \int_0^{2\pi} u'(x)v''(x)\, dx, \tag{28}$$

where $\gamma \in \mathbf{R}$ is a constant. In our case, the new Lie algebra is termed the Virasoro algebra

$$\mathbf{v} = \{(u,a) \mid u \in \mathcal{X}(S^1), a \in \mathbf{R}\} \tag{29}$$

and the new Lie algebra bracket becomes

$$[(u,a),(v,b)] = \left(-uv' + u'v, \gamma \int_0^{2\pi} u'(x)v''(x)\,dx\right), \tag{30}$$

since the Lie algebra bracket $[u,v] = -[u,v]_{JL}$ for a left Lie algebra action. The inner product on this algebra is given as

$$\langle (u,a),(v,b)\rangle = ab + \int_0^{2\pi} u(x)v(x)\,dx, \tag{31}$$

for $\mathbf{v}^* = \mathbf{v}$.

It is commonly known that the fluid systems are right invariant systems. Therefore, the KdV system which also belongs to the fluid systems has a right invariant property. As a consequence, we need to use the plus Lie- Poisson bracket

$$\{f,h\}(u,a) = \left\langle (u,a), \left[\frac{\delta f}{\delta(u,a)}, \frac{\delta h}{\delta(u,a)}\right]\right\rangle, \tag{32}$$

where $f,g \in \mathcal{F}(\mathbf{v})$ with values in $\mathbf{R}$. Moreover, the functional derivative of $f(u,a)$ is given as

$$\frac{\delta f}{\delta(u,a)} = \left(\frac{\delta f}{\delta u}, \frac{\delta f}{\delta a}\right), \tag{33}$$

where the L_2-functional derivative is defined as

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[F(\mathbf{v}+\varepsilon\delta\mathbf{v}) - F(\mathbf{v})] = \int_\Omega \frac{\delta F}{\delta \mathbf{v}} \cdot \delta\mathbf{v}\,d^3x. \tag{34}$$

The (+) Lie-Poisson bracket for functions on S^1 can be written as

$$
\begin{aligned}
\{f,h\}(u,a) &= \left\langle (u,a), \left[\frac{\delta f}{\delta(u,a)}, \frac{\delta h}{\delta(u,a)}\right]\right\rangle \\
&= \left\langle (u,a), \left[\left(\frac{\delta f}{\delta u}, \frac{\delta f}{\delta a}\right), \left(\frac{\delta h}{\delta u}, \frac{\delta h}{\delta a}\right)\right]\right\rangle \qquad (35)\\
&= \int_0^{2\pi} \left[u\left(\left(\frac{\delta f}{\delta u}\right)' \frac{\delta h}{\delta u} - \frac{\delta f}{\delta u}\left(\frac{\delta h}{\delta u}\right)'\right) + a\gamma \left(\frac{\delta f}{\delta u}\right)' \left(\frac{\delta h}{\delta u}\right)''\right] dx \\
&= \left[u\frac{\delta f}{\delta u}\frac{\delta h}{\delta u} + a\gamma \frac{\delta f}{\delta u}\left(\frac{\delta h}{\delta u}\right)''\right]_0^{2\pi} \\
&\quad - \int_0^{2\pi} \left(\frac{\delta f}{\delta u} u' \frac{\delta h}{\delta u} + 2u\frac{\delta f}{\delta u}\left(\frac{\delta h}{\delta u}\right)' + a\gamma \frac{\delta f}{\delta u}\left(\frac{\delta h}{\delta u}\right)'''\right) dx \\
&= -\int_0^{2\pi} \left(\frac{\delta f}{\delta u} u' \frac{\delta h}{\delta u} + 2u\frac{\delta f}{\delta u}\left(\frac{\delta h}{\delta u}\right)' + a\gamma \frac{\delta f}{\delta u}\left(\frac{\delta h}{\delta u}\right)'''\right) dx,
\end{aligned}
$$

where we have used the fact that functions on S^1 are periodic. By definition, the Lie-Poisson equation $\dot{f} = \{f, h\}$ is equivalent to

$$
\dot{f} = \{f,h\}(u,a) = \left\langle (\dot{u},\dot{a}), \left(\frac{\delta f}{\delta u}, \frac{\delta f}{\delta a}\right)\right\rangle. \qquad (36)
$$

Thus, we obtain for $f = a$ and $f = u$

$$
\begin{aligned}
\dot{a} &= 0 \qquad (37)\\
\dot{u} &= -u'\frac{\delta h}{\delta u} - 2u\left(\frac{\delta h}{\delta u}\right)' - a\gamma\left(\frac{\delta h}{\delta u}\right)'''.
\end{aligned}
$$

The above equations are just the Hamilton equations for Hamiltonian $h(u, a)$. Now consider the moment of inertia operator on $\mathbf{v}$, defined as

$$
\begin{aligned}
I: \quad \mathbf{v} &\longrightarrow (\mathbf{v})^* = \mathbf{v} \qquad (38)\\
(u,a) &\longmapsto (u,a).
\end{aligned}
$$

Then the Hamiltonian $H(M) = \langle M, I^{-1}M\rangle = \langle M, M\rangle$ is given for $M = (u, a)$

as

$$h(u,a) = \frac{1}{2}a^2 + \frac{1}{2}\int_0^{2\pi} u^2(x)\,dx. \tag{39}$$

Substituting this $h(u,a)$ into the previously-obtained Hamilton equations, we obtain

$$u_t + 3uu_x + a\gamma u''' = 0. \tag{40}$$

This equation can be easily transformed into the KdV equation

$$v_\tau + 6vv_x + v_{xxx} = 0, \tag{41}$$

by letting $u(t,x) = v(\tau(t),x)$ and $a = 1/2\gamma$. Hence, since the Hamilton equations are just the Euler equation for the Moment of Inertia operator and Hamiltonian given above, the KdV equation is the Euler equation and hence the geodesic on the Virasoro algebra **v**.

Comment

Even though KdV equation is an old equation, we see that it is still attracting a lot of attention because of its ubiquity and its remarkable properties. Furthermore, the geometric properties of the KdV equation has been reasonably well-developed [6] and further study has been pursued on the equations associated with it (e.g., the super-KdV eqn).