HOMEWORK 3 SOLUTIONS

All questions are from Vector Calculus by Marsden and Tromba

Question 1: 2.2.12 Compute the following limits if they exist:

(a) $\lim_{x\to 0} \dfrac{\sin 2x - 2x}{x^3}$.

(b) $\lim_{(x,y)\to(0,0)} \dfrac{\sin 2x - 2x + y}{x^3 + y}$.

(c) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{2x^2 y \cos z}{x^2 + y^2}$.

Solution:

(a) By L'Hopital's rule,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x - 2x}{x^3} &= \lim_{x\to 0} \frac{2\cos 2x - 2}{3x^2} \\
&= \lim_{x\to 0} \frac{-4\sin 2x}{6x} \quad \text{(again by L'Hopital)} \\
&= -\left(\frac{4}{3}\right) \lim_{x\to 0} \frac{\sin 2x}{2x} = -\frac{4}{3}
\end{aligned}$$

since

$$\lim_{\theta\to 0} \frac{\sin \theta}{\theta} = 1.$$

(b) From (a), for $y = 0$

$$\frac{\sin 2x - 2x}{x^3} \quad \text{approaches} \quad -\frac{4}{3}$$

as $(x, 0)$ approaches $(0, 0)$. Setting $y = 2x$, we see that

$$\frac{\sin 2x - 2x + y}{x^3 + y} = \frac{\sin 2x}{x^3 + 2x} = \frac{\sin 2x}{2x}\left(\frac{2}{x^2 + 2}\right)$$

and hence

$$\lim_{(x,2x)\to 0} \frac{\sin 2x - 2x + y}{x^3 + y} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{x^2 + 2} = 1 \cdot 1 = 1.$$

Therefore,

$$\frac{\sin 2x - 2x + y}{x^3 + y}$$

has *two different* limits along two different rays approaching the origin $(0, 0)$, and consequently

$$\lim_{(x,y)\to(0,0)} \frac{\sin 2x - 2x + y}{x^3 + y}$$

does *not* exist.

Date: Math 1c Practical, 2008.

(c) We have $\lim_{(x,y,z)\to(0,0,0)} \dfrac{2x^2y\cos z}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} \times x = 0$ (Since $\frac{|2xy|}{x^2+y^2} \le 1$). $\Diamond$

Question 2: 2.2.24 Show that f is continuous at $\mathbf{x_0}$ if and only if

$$\lim_{\mathbf{x}\to\mathbf{x_0}} \|f(\mathbf{x}) - f(\mathbf{x_0})\| = 0$$

Solution:

By definition, f is continuous at (x_0) iff $\lim_{\mathbf{x}\to\mathbf{x_0}} f(\mathbf{x}) = f(\mathbf{x_0})$, since $\lim_{\mathbf{x}\to\mathbf{x_0}} f(\mathbf{x}) = f(\mathbf{x_0})$ is equivalent to $\lim_{\mathbf{x}\to\mathbf{x_0}} \|f(\mathbf{x})-f(\mathbf{x_0})\| = 0$, the consequence is then obvious. $\Diamond$

Question 3: 2.3.4d *Show that the following function is differentiable at each point in its domain. Determine if the function is C^1.*

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}.$$

Solution:

The domain of the function $f(x,y) = xy/\sqrt{x^2+y^2}$ is all points $(x,y) \neq (0,0)$. We have the partial derivatives

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\dfrac{\partial(xy)}{\partial x}\sqrt{x^2+y^2} - xy\cdot\dfrac{\partial}{\partial x}\sqrt{x^2+y^2}}{x^2+y^2} \\
&= \frac{y\sqrt{x^2+y^2} - x^2y(x^2+y^2)^{-1/2}}{x^2+y^2} \\
&= \frac{y^3}{(x^2+y^2)^{3/2}} \\
\frac{\partial f}{\partial y} &= \frac{x^3}{(x^2+y^2)^{3/2}}
\end{aligned}$$

Observe that these partial derivatives are all continuous in the domain of f. Therefore f is C^1. $\Diamond$

Question 4: 2.3.8c *Compute the matrix of partial derivatives of the function $f(x,y) = (x+y, x-y, xy)$.*

Solution:

$$Df(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ y & x \end{bmatrix}. \quad \Diamond$$

Question 5: 2.3.10 *Why should the graphs of $f(x,y) = x^2+y^2$, and $g(x,y) = -x^2-y^2+xy^3$ be called "tangent" at $(0,0)$?*

Solution:

At $(0,0)$,

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial g}{\partial x}(0,0)$$

and

$$\frac{\partial f}{\partial y}(0,0) = 0 = \frac{\partial g}{\partial y}(0,0).$$

Therefore, the graphs of both f and g have the same tangent plane at $(0,0,0)$, namely the plane $z = 0$; *i.e.,* the xy plane. Thus, it is reasonable to call the graphs tangent. $\Diamond$

Question 6: 2.4.18 *Suppose that a particle following the path*

$$\mathbf{c}(t) = (e^t, e^{-t}, \cos(t))$$

flies off on a tangent at $t_0 = 1$. Compute the position of the particle at time $t_1 = 2$.

Solution:

The velocity vector is $(e^t, -e^{-t}, -\sin t)$, which at $t_0 = 1$ is the vector $(e, -e^{-1}, -\sin 1)$. The particle is at $(e, e^{-1}, \cos 1)$ at $t_0 = 1$. Hence the tangent line placing the particle at its "take off" point at $t = 1$ is

$$\ell(t) = (e, e^{-1}, \cos 1) + (t-1)(e, -e^{-1}, -\sin 1).$$

At $t = 2$, the position of the particle is on the line and is at

$$\ell(2) = (e, e^{-1}, \cos 1) + (e, -e^{-1}, -\sin 1) = (2e, 0, \cos 1 - \sin 1). \quad \Diamond$$

Question 7: 2.5.8 *Suppose that a function is given in terms of rectangular coordinates by $u = f(x,y,z)$. If $x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi$, express $\partial u/\partial\rho, \partial u/\partial\theta$, and $\partial u/\partial\phi$ in terms of $\partial u/\partial x, \partial u/\partial y$, and $\partial u/\partial z$.*

Solution:

By the chain rule,

$$\begin{aligned}
\frac{\partial u}{\partial \rho} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \rho} \\
&= \frac{\partial u}{\partial x}\cdot\cos\theta\sin\phi + \frac{\partial u}{\partial y}\cdot\sin\theta\sin\phi + \frac{\partial u}{\partial z}\cdot\cos\phi \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\cdot(-\rho\sin\theta\sin\phi) + \frac{\partial u}{\partial y}\cdot\rho\cos\theta\sin\phi + \frac{\partial u}{\partial z}\cdot 0 \\
&= -\sin\theta\sin\phi\rho\frac{\partial u}{\partial x} + \cos\theta\sin\phi\rho\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial \phi} &= \frac{\partial u}{\partial x}\cdot\rho\cos\theta\cos\phi + \frac{\partial u}{\partial y}\cdot\rho\sin\theta\cos\phi + \frac{\partial u}{\partial z}\cdot(-\rho\sin\phi) \\
&= \rho\cos\theta\cos\phi\frac{\partial u}{\partial x} + \rho\sin\theta\cos\phi\frac{\partial u}{\partial y} - \rho\sin\phi\frac{\partial u}{\partial z}. \quad \Diamond
\end{aligned}$$

Question 8: 2.5.12 *Suppose that the temperature at the point (x,y,z) in space is $T(x,y,z) = x^2 + y^2 + z^2$. Let a particle follow the right circular helix $\sigma(t) = (\cos t, \sin t, t)$ and let $T(t)$ be its temperature at time t.*

(a) *What is $T'(t)$?*

(b) *Find an approximate value for the temperature at $t = (\pi/2) + 0.01$.*

Solution:

(a) $T(t) = T(\sigma(t)) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$. $T'(t) = 2t$.

(b) By the linear approximation, an approximate value is

$$T\left(\frac{\pi}{2}\right)+T'\left(\frac{\pi}{2}\right)\cdot\left(\frac{\pi}{2}+0.01-\frac{\pi}{2}\right)=1+\left(\frac{\pi}{2}\right)^2+2\cdot\frac{\pi}{2}\cdot 0.01\approx 3.4988 \quad \Diamond$$

Question 9: 2.6.15 Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r=\|\mathbf{r}\|$. Prove that:

$$\nabla(\frac{1}{r})=-\frac{\mathbf{r}}{r^3}$$

Solution:
$\nabla(\frac{1}{r})=\frac{\partial(\frac{1}{r})}{\partial x}\mathbf{i}+\frac{\partial(\frac{1}{r})}{\partial y}\mathbf{j}+\frac{\partial(\frac{1}{r})}{\partial z}\mathbf{k}=-\frac{x}{r^3}\mathbf{i}-\frac{y}{r^3}\mathbf{j}-\frac{z}{r^3}\mathbf{k}=-\frac{\mathbf{r}}{r^3}.\quad\Diamond$

Question 10: 2.6.16 *Captain Ralph is in trouble near the sunny side of Mercury. The temperature of the ship's hull when he is at location (x,y,z) will be given by $T=e^{-x^2-2y^2-3z^2}$, where x, y, and z are measured in meters. He is currently at $(1,1,1)$.*

(a) In what direction should he proceed in order to decrease the temperature most rapidly?

(b) If the ship travels at e^8 meters per second, how fast will be the temperature decrease if he proceeds in that direction?

(c) Unfortunately, the metal of the hull will crack if cooled at a rate grater than $\sqrt{14}e^2$ degrees per second. Describe the set of possible directions in which he may proceed to bring the temperature down at no more than that rate.

Solution:
(a) In order to cool the fastest, the captain should proceed in the direction in which T is decreasing the fastest; that is, in the direction of the *negative* gradient at the point $(1,1,1)$, namely $-\nabla T(1,1,1)$.Since

$$\nabla T=\frac{\partial T}{\partial x}\mathbf{i}+\frac{\partial T}{\partial y}\mathbf{j}+\frac{\partial T}{\partial z}\mathbf{k}=-2xT\mathbf{i}-4yT\mathbf{j}-6zT\mathbf{k},$$

we have

$$-\nabla T(1,1,1)=2e^{-6}\mathbf{i}+4e^{-6}\mathbf{j}+6e^{-6}\mathbf{k},$$

which is the direction required.

(b) The cool down speed is (degrees per second)

$$v\times\|-\nabla T(1,1,1)\|=\sqrt{56}e^2$$

(c) The possible direction $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ (assume it is a norm 1 vector) should satisfy the following:

1) The temperature will decrease if proceed in that direction (Otherwise the hull will be melt).

2) The cool down speed is no greater than $\sqrt{14}e^2$ (Otherwise the metal of the hull will be cracked).

Now, condition 1 equivalents to $-2xe^{-6}-4ye^{-6}-6ze^{-6}<0$, condition 2 equivalents to $v\times|\nabla T\cdot(x,y,z)|=e^8\times|-2xe^{-6}-4ye^{-6}-6ze^{-6}|\leq\sqrt{14}e^2$, we get the following set: $\{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}: x^2+y^2+z^2=1, 0<x+2y+3z\leq\frac{\sqrt{14}}{2}\}\quad\Diamond$