CDS 140a Problem Set 5

November 6, 2009

8) The linearization at the origin is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

with

$$A = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}.$$

The eigenvalues of A are $\lambda_{\pm} = \mu \pm i$. When $\mu < 0$, the origin is a stable spiral in the linearized system, in agreement with the full nonlinear phase portrait. When $\mu > 0$, the origin is an unstable spiral in the linearized system, again in agreement with the nonlinear phase portrait. When $\mu = 0$, the linearization has a center at the origin. In contrast, the nonlinear system has a stable spiral at the origin for $\mu = 0$. Note that this does not violate Liapunov's Theorem, since the eigenvalues of A all have zero real part when $\mu = 0$.

10) Let $U \subseteq \mathbb{R}^n$ be an open set containing the periodic orbit $\gamma([0, \tau])$ and let $\mathcal{D}_X$ denote the set of all $(x, t) \in U \times \mathbb{R}$ for which there is an integral curve $c : I \to U$ through x with $c(0) = x$ and $t \in I$. Since any point on the periodic orbit has an infinite solution lifetime, $\gamma([0, \tau]) \times \mathbb{R} \subset \mathcal{D}_X$. Moreover, $\mathcal{D}_X$ is open in $U \times \mathbb{R}$ by Proposition 1.3.10(ii). Fix a number $T > 0$. Then for any $t \in [0, \tau]$, there exists a finite number $b(t) > 0$ that depends continuously on the parameter t such that $B_{b(t)}(\gamma(t)) \times \{T\} \subset \mathcal{D}_X$, where $B_{b(t)}(\gamma(t))$ denotes the open ball of radius $b(t)$ centered at $\gamma(t)$. Define $\varepsilon = \inf_{t \in [0,\tau]} b(t)$. Since $[0, \tau]$ is a compact interval, $b(t)$ achieves its infimum on $[0, \tau]$, so ε must be nonzero. This proves that there is a positive number ε such that any point lying within a distance ε from the periodic orbit has a solution lifetime of at least T.